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LECTURE 1

Fix $n \geq 0$. Let $\mathbb{R}^n_+ := \{(t_1, \dots, t_n) \in \mathbb{R}^n : t_n \geq 0\}$ the upper half-space.
Note: $\mathbb{R}^0 = \mathrm{Map}(\emptyset, \mathbb{R}) = \{\emptyset \to \mathbb{R}\} = \{pt\}$ and $\mathbb{R}^0_+ = \emptyset$.

§ Top

Def. A topological n-manifold is a paracompact Hausdorff topological space N that is locally homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+$,
meaning: $\forall x \in N \;\; \exists U \subseteq N$ open, $x \in U$, $U \approx \mathbb{R}^n$ or $U \approx \mathbb{R}^n_+$.
Then the interior of N is $\mathrm{Int}\, N := \{x \in N : \exists U \subseteq N, x \in U, U \approx \mathbb{R}^n\}$
and the boundary of N is $\partial N := N \setminus \mathrm{Int}\, N$.
If $\partial N = \emptyset$ we say N is a manifold without boundary.
If N is compact and without boundary, we say it is a closed manifold.
non-compact ——"—— ——"—— an open manifold.

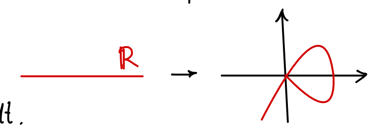
Denote by $\mathcal{T}op$ the category whose objects are top. manifolds, and morphisms are cts maps.
note: $\mathrm{Aut}\, N = \mathrm{Homeo}(N)$.

Recall: - top. space is Hausdorff if any two points have disjoint open nbhds.
- top. space is paracompact if any open cover has a locally finite refinement.

Fact: Hausdorff + paracompact $\Rightarrow$ Every open cover has a subordinate partition of unity.
- a map $f: X \to Y$ of top. spaces is a homeomorphism if it is continuous and has a continuous inverse $f^{-1}: Y \to X$. We write $X \approx Y$.
- a map $f: X \to Y$ of top. spaces is a top. embedding if $f: X \to f(X)$ is a homeomorphism. We write $f: X \xhookrightarrow{\mathcal{T}op} Y$.

Examples. $\emptyset, \mathbb{R}^n, S^n, \mathbb{RP}^n, \mathbb{CP}^n, \mathbb{R}^n_+, \mathbb{D}^n$ surfaces, products, knot complements
open subset of a manifold, e.g. $GL(n, \mathbb{R})$

Thm [see Munkres]
Every topological n-manifold embeds into $\mathbb{R}^{n'}$ for some n'. (use part. of unity)
In fact, $n' = 2n$ suffices. (hard!)

Fact. Any continuous injective map $M \to N$ from a compact to any manifold is a top. embedding.
note: Not true in general, for example $\mathbb{R} \to$ 
However, we have the following fundamental result.

Thm [Brouwer 1910] – Invariance of Domain – (see Hatcher 2B.)
If $U \subseteq \mathbb{R}^n$ is open and $f: U \to \mathbb{R}^n$ continuous and injective,
then $f(U) \subseteq \mathbb{R}^n$ is open. Moreover, f is a top. embedding.

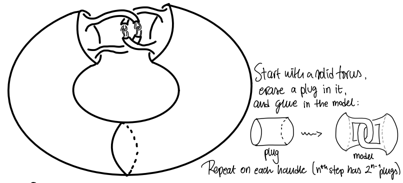
Cor. If N is a top. n-manifold, then ∂N is a top. $(n-1)$-manifold without boundary.

note: Inv. of Domain follows from the following fundamental result:

Thm [Brouwer 1910] – Jordan-Brouwer Separation Thm –
If $f: S^{n-1} \to S^n$ is continuous and injective then $S^n \setminus f(S^{n-1})$ has two components.

Q: Are closures of both of these components homeomorphic to the n-disc $\mathbb{D}^n$?

Thm [Schönflies] For $n=2$: yes.



Counterexample for $n=3$ [Alexander Horned Sphere, 1924]
This is an embedding $S^2 \overset{f}{\hookrightarrow} S^3$ such that $S^3 \setminus f(S^2) \approx \mathring{\mathbb{D}}^3 \sqcup \mathcal{G}$
where $\mathcal{G}$ has infinitely generated fundamental group and $\overline{\mathcal{G}} \subseteq S^3$ is not a manifold.

key **Thm** [Brown 1960, Mazur 1959 + Morse 1960] – Top Schönflies Thm –
For any $n \geq 1$ and a locally flat embedding $S^{n-1} \hookrightarrow S^n$,
the closure of each component of the complement is homeomorphic to $\mathbb{D}^n$.

note: We will define loc. flat embeddings later on.
This is a natural condition to avoid wild phenomena (like Alexander Horned Sphere).
It implies that each closure is a top. manifold.
Another natural additional structure that eliminates wildness: smooth.

§ Diff

def. A smooth n-manifold is a paracompact Hausdorff top. space N together with the data of a smooth structure, defined as a maximal collection $\{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$ of pairwise smoothly compatible charts that cover N.

CHART: $(U_\alpha, \varphi_\alpha)$ where $U_i \subseteq N$ open and $\varphi_i : U_i \hookrightarrow \mathbb{R}^n$ or $\mathbb{R}^n_+$ top. embedding
$(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) SMOOTHLY COMPATIBLE if
$U_\alpha \cap U_\beta \neq \emptyset \Rightarrow \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta)^{\subseteq \mathbb{R}^n} \longrightarrow U_\alpha \cap U_\beta \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)^{\subseteq \mathbb{R}^n}$ is smooth
(recall: smooth = infinitely differentiable = C^∞, and $\mathbb{R}^n_+ \xrightarrow{sm.} \mathbb{R}^n_+$ means locally a restriction of $\mathbb{R}^n \xrightarrow{sm.} \mathbb{R}^n$
MAXIMAL: if (V, ψ) smoothly compatible with every $(U_\alpha, \varphi_\alpha)$, then $\exists \alpha \in I$ $(V, \psi) = (U_\alpha, \varphi_\alpha)$.

Remarks.
- $\varphi_\beta \circ \varphi_\alpha^{-1}$ are called transition maps
- Clearly, N is a smooth n-manifold $\Rightarrow$ N is a top. n-manifold.
- There are analogous definitions of C^k n-manifolds: replace smoothly by C^k-compatible, transition maps are C^k-differentiable.
- However, by a theorem of Whitney every C^k-structure for $k \geq 1$ contains a smooth structure. We will study only smooth ones.

Exercise. Check that in the above list all examples have a smooth structure.

Exercise. The boundary of a smooth n-manifold is a smooth (n-1)-manifold.

def. A map $f: M \to N$ between smooth manifolds is smooth if $\forall \alpha, \beta$ s.t. $f(U_\alpha) \subseteq V_\beta$ we have $\varphi_\alpha(U_\alpha) \xrightarrow{\varphi_\alpha^{-1}} U_\alpha \xrightarrow{f} V_\beta \xrightarrow{\psi_\beta} \psi_\beta(V_\beta)$ is smooth.
- If additionally f has a smooth inverse, we call it a diffeomorphism $f: M \xrightarrow{\cong} N$.
- A top. embedding $f: M \hookrightarrow N$ of smooth manifolds which at every point $x \in M$ has injective derivative is called a smooth embedding.

def. Denote by Diff the category of smooth manifolds with morphisms smooth maps.
note: $\mathrm{Aut}\,N = \mathrm{Diff}(N)$

key Thm [Cor of Smale 1962] – Diff Schönflies Thm –
For any $n \geq 1$, $n \neq 4$ and a smooth embedding $S^{n-1} \hookrightarrow S^n$, the closure of each component of the complement is diffeomorphic to D^n.

– 4D Schönflies Conjecture – Diff Schönflies holds for $n = 4$. still open!

note: the first step in the proof of Diff Schönflies is to show that any of the two closures, call it A, is a smooth manifold, that is homotopy equivalent to D^n. We say A is a homotopy D^n.
Strategy: $A \cup_\partial D^n$ is a homotopy sphere. Is it diffeomorphic to S^n?
If yes, we would be done by Palais' Thm [1960].

Q: Is every homotopy S^n (smooth n-manifold homotopy equivalent to S^n) diffeomorphic to S^n?

key Thm [Cor. of Smale 1962] – Top Generalized Poincaré Conjecture –
Any smooth manifold homotopy equivalent to S^n is homeomorphic to it.

Thm [Milnor 1957, Kervaire-Milnor 1962, Hill-Hopkins-Ravenel 2009]
For MANY $n \geq 1$ there exists a smooth n-manifold homotopy equivalent to S^n but that is not diffeomorphic to it. For example, all odd $n > 61$.

Cor. [of these two thms] $\exists$ non-diffeomorphic smooth structures on S^n.
(Those different from the standard one are called exotic.)

Milnor's Conjecture. For $n\geq 5$ smooth structure on S^n unique iff $n = 5, 6, 12, 56, 61$.
note: $\Leftarrow$ known, and $\Rightarrow$ known for n odd.

4D Smooth Poincaré Conjecture: S^4 has a unique smooth structure.

note: this should be compared to the following: (see Gompf-Stipsicz, Chapter 9)

Thm [Stallings 1961, Kirby-Siebenmann 1970, Casson 1973, Gompf 1985, Taubes 1987...]
$\mathbb{R}^n$ has a unique smooth structure for every $n\neq 4$.
$\mathbb{R}^4$ has uncountably many exotic structures.

note: we will prove Diff Schönflies and Top Poincaré
using: key Thm [Smale 1962] – h-cobordism Thm –
Then we prove Top Schönflies using Mazur's swindle and Morse's push-pull
Finally, we will discuss 4-manifolds.

def. A cobordism $(W, \partial_0 W, \partial_1 W)$ is an h-cobordism
if the inclusions $\partial_i W \hookrightarrow W$ are homotopy equivalences.
It is an s-cobordism if they are simple homotopy equivalences.

key Thm [Smale 1961] – ~~h~~ s-cobordism Theorem – [Barden, Mazur, Stallings 1963]
Assume $(W, \partial_0 W, \partial_1 W)$ is a ~~simply connected~~ s- h-cobordism with $\dim W \geq 6$.
Then it is smoothly trivial, i.e. there is a diffeomorphism
$$(W, \partial_0 W, \partial_1 W) \cong (\partial_0 W \times [0,1], \partial_0 W \times \{0\}, \partial_0 W \times \{1\})$$

For the proof we will need:
submanifolds, transversality, handle decompositions, handle calculus
intersection numbers, Whitney trick.

§ ORIENTATIONS

See links

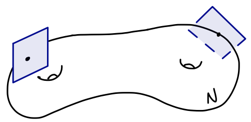
def. A top. n-manifold N is orientable iff it can be covered by a collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ of orientation-compatible charts, i.e. $\varphi_\beta \circ \varphi_\alpha^{-1}$ is orientation preserving.
A choice of maximal such is an orientation.
(If N is a smooth n-manifold, then we ask for a subcollection of its smooth str.)

Exercise. A compact n-manifold N is orientable iff $H_n(N, \partial N; \mathbb{Z}) \cong \mathbb{Z}$
and an orientation corresponds to the choice of a generator
$[N, \partial N] \in H_n(N, \partial N; \mathbb{Z})$, called a fundamental class.

Exercise. The boundary of an oriented top. n-manifold is orientable and has a canonical orientation (s.t. $\partial(\mathbb{R}^n_+)$ is the positively oriented $\mathbb{R}^{n-1}$).

§ DIFF: TANGENT BUNDLE

See Lee



IDEA: tangent space $TN_p \cong \mathbb{R}^n$ at a point $p \in N$:
$X_p \in TN_p$ is an equiv. class of "germs of curves at p"
i.e. $X_p = \frac{d\gamma}{dt}\big|_0$ for some $\gamma: \mathbb{R} \to U \subseteq N$ s.t. $\gamma(0) = p$.
tangent bundle $TN = \bigsqcup_{p\in N} TN_p \longrightarrow N$ is a smooth vector bundle
$\Rightarrow$ TN is a smooth 2n-manifold.

NOTE: for $F: M \xrightarrow{sm} N$ have $dF: TM \to TN$ a smooth map of v. bundles
key cases: 1° $f: N \to \mathbb{R}$ then $df: TN \to T\mathbb{R}$ has shape $df(X_p) = \frac{d(f\circ\gamma)}{dt}\big|_0 \cdot \partial/\partial t$
where $T\mathbb{R}_{f(p)} \cong \mathbb{R}$ gen. by $\partial/\partial t$.
2° $\gamma: \mathbb{R} \to N$ then $d\gamma: T\mathbb{R} \to TN$ has shape $d\gamma_{t_0}(\partial/\partial t) = \frac{d\gamma}{dt}\big|_{t_0}$ in a chart.
– the velocity vector of γ at $t_0 \in \mathbb{R}$.