Lemma G. There exist a collection of framed immersed spheres $\{G_m\}$ that is algebraically dual to the collection $\{W_m\}$.

i.e. $\tilde{I}(G_n \pitchfork W_m) = \delta_{nm}$.

Moreover, G_m are disjoint from all A_j and B_i.

key Thm [Freedman 1982] – Disc Embedding Thm –

If M is a smooth connected 4-manifold with $\partial M = \emptyset$ and $\pi_1 M$ a good group, and $W_m : (\mathbb{D}^2, \partial \mathbb{D}^2) \looparrowright (M, \partial M)$ is a framed immersed collection with emb. boundary which has a framed immersed collection $\{G_m\}$ of algebraic duals, then

there exists a locally flat embedded collection $\{\overline{W_m}\}$ with the same framed boundary as $\{W_m\}$

not needed in the current proof $\Big\{$ and with a framed immersed collection $\{\overline{G_m}\}$ of geometric duals with $\overline{G_m} \underset{htpc}{\simeq} G_m$.

PROOF VERY HARD.

We now apply Disc Emb. Thm:

to W_m and G_m in $M := W_{1/2} \setminus \left(\bigcup \nu B_i \cup \bigcup \nu A_j \right)$.

Note: $\pi_1 M \cong \pi_1 W_{1/2}$ ($\cong \pi_1 W$) since A_j and B_i have duals (so their meridians are nullhomotopic in M).

Therefore, we can perform Whitney moves on A_j along the framed loc. flat discs $\overline{W_m}$ to remove all unwanted intersections with B_i. This is a loc. flat isotopy of A_j, making it into a geometric dual of B_j, so that 2- and 3-handles geom. cancel. There are no other handles in $(W, \partial_0 W)$, so W is homeomorphic to $\partial_0 W \times [0,1]$.

$\square$

LECTURE 13.

Lemma #. Let C stand either for A or B.
There is an unframed immersed collections of spheres $\{C_i^\#\}$ that is geometrically dual to the framed embedded collection $\{C_i\}$
i.e. $C_i^\# \pitchfork C_j = \emptyset$ unless $i=j$ when $=\{pt\}$.

proof of Lemma #.
Since $\pi_1(\partial_0 W) \hookrightarrow \pi_1(W^{\le 2})$ is an isomorphism,
2-handles of W are attached along homotopically trivial circles in $\partial_0 W$.
So in $W_{1/2} := \partial_1 W^{\le 2}$ the att. circle of the i-th 2-handle bounds both an immersed disc and a push-off of the core of the 2-handle. These glue to an immersed sphere $B_i^\#$ that intersects the belt sphere B_i of the 2-handle in a single point.
Since 2-handles are mutually disjoint, the sphere $B_i^\#$ is disjoint from B_j for $j \neq i$.
Thus, the collection $\{B_i^\#\}$ is geom. dual to the embedded collection $\{B_i\}$.
The same argument applied to $W^{\le 2}$ turned upside down produces immersed collection $\{A_i^\#\}$ that is geom. dual to the embedded collection $\{A_i\}$. □

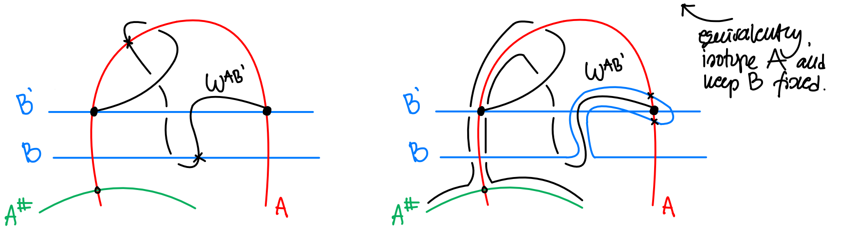
NOTE: We have no control over the framing of #-spheres.
We also have no control over $B^\#/A$ or $A^\#/B$ or $B^\#/A^\#$ or $B^\#/B^\#$ - intersections.

Lemma ^. After an isotopy of $\{A_i\}$,
There is a framed immersed collection of spheres $\{\hat{B}_i\} \cup \{\hat{A}_i\}$ that is geometrically dual to the framed immersed collection $\{B_i\} \cup \{A_i\}$,
i.e. for $C, D \in \{A, B\}$ $\hat{C}_i \pitchfork D_j = \emptyset$ unless $i=j$ and $C=D$ when $=\{pt\}$.

NOTE: Not only this produces framed spheres $\hat{B}_i$ and $\hat{A}_i$,
but also we control all intersections apart from $\hat{B}/\hat{A}$ - intersections.

proof of Lemma ^.
1° Let B_i' be a parallel copy of B_i, and A_i' a parallel copy of A_i.
Since $\{B_i\}$ and $\{A_i\}$ are each framed embedded,
we have that $\{B_i\} \cup \{B_i'\}$ and $\{A_i\} \cup \{A_i'\}$ are each framed embedded.
Moreover, $\tilde{I}(A_j' \pitchfork B_i) = \tilde{I}(A_j \pitchfork B_i') = \delta_{ij}$,
so all but one A'/B- and A/B'-intersections can be paired up by immersed Whitney discs $\{W_e^{A'B}\}$ and $\{W_e^{AB'}\}$ respectively.

2° Claim. The collection $\{W_e^{AB'}\}$ can be made framed and disjoint from $\{B_i\} \cup \{A_i\}$.
proof. First make $W_e^{AB'}$ disjoint from A by tubing into $A^\#$.
Then make $W_e^{AB'}$ disjoint from B by finger-moving B into A.



Then make $W_e^{AB'}$ framed by boundary-twisting around B' (so more $W^{AB'}/B'$-inter.) □

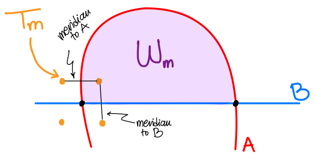
3° Claim. The collection $\{W_e^{A'B}\}$ can be made framed and disjoint from $\{B_i\} \cup \{A_i\}$.
proof. analogous.

4° Define $\{\hat{A}_i\}$ as the collection obtained from $\{B_i'\}$ by performing Wh. moves along $\{W_e^{AB'}\}$.
Define $\{\hat{B}_i\}$ as the collection obtained from $\{A_i'\}$ by performing Wh. moves along $\{W_e^{A'B}\}$.

Note: $\{\hat{A}_i\}$ is a framed immersed collection geom. dual to $\{A_i\}$ (since B_i' was alg. dual to it and $W_e^{AB'}$ disjoint from A_i) and disjoint from $\{B_i\}$ (since B_i was and $W_e^{AB'}$ disjoint from B_i)
and similarly for $\{\hat{B}_i\}$. □

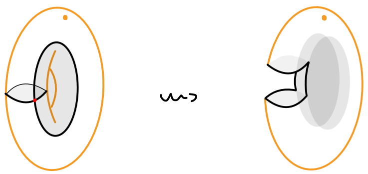
Lemma G. There exists a framed immersed collection of spheres $\{G_m\}$ that is algebraically dual to the collection $\{W_m\}$
i.e. $\tilde{I}(G_n \pitchfork W_m) = \delta_{nm}$.
Moreover, G_m are disjoint from all A_j and B_i.

proof of Lemma G.
We use Clifford tori:



$^+$in this half-dim' picture the torus is $T = S^0 \times S^0$ where one S^0 = meridian to A, other S^0 = meridian to B $^+$

Each W_m has associated a torus $T_m = S^1 \times S^1$ at one of the intersection point in $A \cap B$.
Think locally around the point: $S^1 \times S^1 \subseteq \mathbb{R}^2 \times \mathbb{R}^2$.
We have $T_m \cap W_m = \{pt\}$
and $S^1 \times \{pt\}, \{pt\} \times S^1 \subseteq T_m$ bound discs in $W_{1/2}$ that are meridian to A, B resp.
So these discs are embedded and intersect only A, B respectively.
We can tube those intersections into $\hat{A}$ and $\hat{B}$ respectively. Call these caps.
Then we do the symmetric surgery to T_m along those caps:



This results in a framed immersed sphere that still intersects W_m algebraically once: we use each cap twice, with opposite sign, so any intersection of a cap with W_m will appear again with the opposite sign. □

§ COROLLARIES

Thm [Freedman 1982] – Diff-to-Top Poincaré Conj in Dim 4 –
Every closed smooth 4-manifold homotopy equivalent to S^4 is homeomorphic to it.

Proof. Claim. Such a 4-manifold M bounds a smooth contractible 5-manifold.
Remove a small ball from it to obtain a 5-dim' cobordism W from S^4 to M. By Top s-cob. thm this is homeo to $S^4 \times [0,1]$, so M is homeo to S^4. □

uses surgery theory of Wall's Thm, see Sec 20.2 in DET-book.

Remark. We saw how Smale's h-cob. Thm shows Diff Poincaré Conj in dim ≥ 6.
For dim = 5 one needs a similar argument to the above, showing that M^5 bounds a contractible 6-manifold, then remove a ball, and use Smale's h-cob. Thm in dim = 6.

Note: Diff 5D s-cob Thm is known to be false (so there are smooth 5d s-cob that are nontriv.)
Top 4D s-cob Thm is known to be false (so there are top. 4d s-cob that are nontriv.)

Open problems.
4D Diff Poincaré Conj. Every closed smooth 4-manifold homotopy equivalent to S^4 is homeomorphic to it.
Equivalently: Every smooth 4-manifold homeomorphic to S^4 is diffeomorphic to it.
Equivalently: S^4 has no exotic smooth structures.
Diff 4D s-cob. Thm.