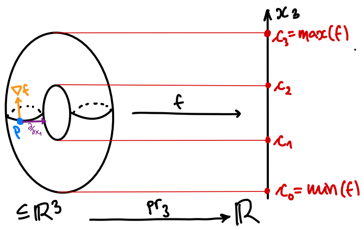
# LECTURE 2

## § Vector Fields — Lee.

**def.** A (smooth) vector field on $N$ is a (smooth) section of $TN \xrightarrow{\pi} N$,
that is, a (smooth) map $X: N \to TN$ with $\pi \circ X = id_N$. Write $X_p := X(p)$.

**Exercise.** In a chart $(U, \varphi)$ around $p$: $X_p = \sum_{i=1}^{n} X^i(p) \frac{\partial}{\partial x_i}\Big|_p$,
where $x_i$ are coordinates in $\varphi(U) \subseteq \mathbb{R}^n$ and $X^i: U \to \mathbb{R}$ smooth functions.

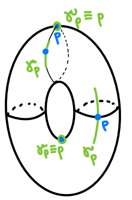
**def.** Let $f: N \to \mathbb{R}$ smooth.
Points $p \in N$ for which $df_p = 0$ are critical points.
Image of a critical point is a critical value.
Values of $f$ that are not critical are regular values.



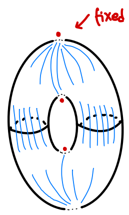

**Thm** [Integration of a vector field]
If $N$ is a smooth $n$-manifold with $\partial N = \emptyset$ and $X: N \to TN$ is a smooth vector field,
then there is a continuous map $\varepsilon: N \to [0, \infty)$, $p \mapsto \varepsilon_p$
and a unique map
$$\gamma: \{(p,t) \in N \times \mathbb{R} : |t| < \varepsilon_p\} \longrightarrow N, \quad (p,t) \mapsto \gamma_p(t)$$
such that $\forall p \in N$ $\gamma_p: [-\varepsilon_p, \varepsilon_p] \longrightarrow N$
satisfies $\gamma_p(0) = p$ and $(d\gamma_p)_t(\partial/\partial t) = X_{\gamma_p(t)}$.

⋆ Idea: solve a system of ordinary differential equations (ODEs)
and glue solutions together using a partition of unity.

Example.

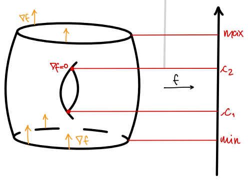



for some $t > 0$
have $\gamma(t): N \to N$




shift by time $t$.

example with boundary:



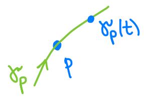

**Exercise.** Boundary of a smooth manifold admits a collar,
i.e. $\partial N \subseteq N$ has a nbhd diffeomorphic to $\partial N \times [0,1)$, with $\partial N$ identified with $\partial N \times \{0\}$.
(Hint: Construct a vector field $X$ on $N$ s.t. $TN|_{\partial N} = T(\partial N) \oplus X$.)

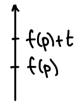
**Thm** [no critical points $\Rightarrow$ cylinder]

We call such $(N, \partial_0 N, \partial_1 N)$ a cobordism.

Assume $N$ compact and $\partial N = \partial_0 N \sqcup \partial_1 N$ for some smooth manifolds $\partial_i N$.
Let $f: N \to \mathbb{R}$ smooth with $f^{-1}(i) = \partial_i N$, for $i = 1, 2$.
If $f$ has no critical points (i.e. $df_p \neq 0$ $\forall p \in N$)
then $N \cong \partial_0 N \times [0,1]$ (and in particular $\partial_0 N \cong \partial_1 N$).

⋆ Idea:




in particular $p \in \partial_0 N$ have $f \circ \gamma_p(t) = t$

Therefore: the map $G: \partial_0 N \times [0,1] \longrightarrow N$, $(p,t) \mapsto \gamma_p(t)$ is well-defined and smooth,
and has a smooth inverse $p \longmapsto (\gamma_p(-f(p)), f(p))$. □

Q: What happens when there are critical points?
A: Morse theory: study how topology changes when passing through a critical point
⇝ a handle is attached, finitely many possibilities.

key **THM** [Smale, Wallace, around 1960] — Handle Decomposition Theorem —
For any cobordism $(W, \partial_0 W, \partial_1 W)$ there exists a sequence of smooth cobordisms
$$\partial_0 W \times [0,1] = W_{-1} \subseteq W_0 \subseteq \dots \subseteq W_m = W$$
such that $W_k$ is obtained from $W_{k-1}$ by attaching a collection of $k$-handles
to its top boundary: $W_k = W_{k-1} \cup h_1^k \cup \dots \cup h_{r_k}^k$

index of a handle

NOTE: Can take $\partial_0 W = \partial_1 W = \emptyset$ so $W$ is a smooth manifold without boundary.
Then theorem says $W$ decomposes into a union of handles.

* *Idea:*
- Pick a function $f: W \to \mathbb{R}$ $\quad f(\partial_i W) = i$ for $i=0,1$
  and denote $W_{\leq s} := f^{-1}((-\infty, s])$ for some $s \in \mathbb{R}$.
- We already saw: $W_{\leq s} \cong W_{\leq t}$ if $f$ has no critical values in $[s,t] \subseteq \mathbb{R}$
- If $p \in f^{-1}([s,t])$ is a unique crit. point for some $[s,t] \subseteq \mathbb{R}$

WANT: $p$ non-degenerate i.e. $\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\big|_p\right) \neq 0$ (Hessian of $f$ at $p$ is nondegenerate)

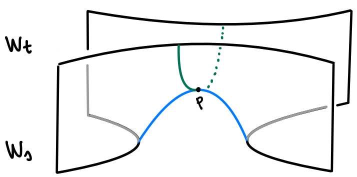
Step 1. There exists a Morse function $h: W \to \mathbb{R}$ with $h^{-1}(j) = W_j$ $\quad j = 0,1$.
meaning, critical values of $h$ are distinct and critical points are non-degenerate, i.e. $\det\left(\frac{\partial^2 h}{\partial x_i \partial x_j}\big|_p\right)_{i,j} \neq 0$

def. $\mathrm{ind}_p(h) :=$ the number of neg. eigenvalues of the Hessian.

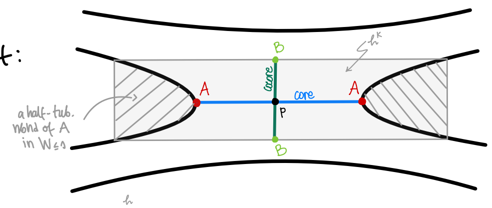
Step 2. – Morse Lemma –
For a critical point $p \in W$ of $h$ of index $k$, there exists a chart $(U, \varphi)$ s.t. $h \circ \varphi^{-1}(x_1, \dots, x_n) = -x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$.

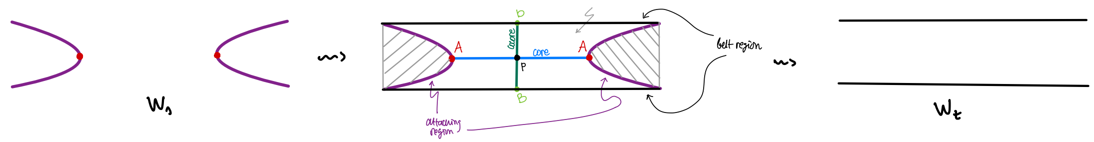
Step 3. – Passing a Critical Point Lemma –
If $W_{[s,t]}$ contains a single crit. point $p \in W$ of $h$, and $\mathrm{ind}_p(h) = k$, then $W_{\leq t}$ is obtained from $W_{\leq s}$ by attaching a handle of index $k$.

* *Idea:*
Can reparametrize $h$ so that $W_{\leq t} \setminus W_{\leq s}$ contained in a chart $(U, \varphi) \ni p$




in a chart:
⇝




. NOTE: $W_t$ is obtained from $W_s$ by surgery on the sphere $A$.

this means take out nbhd of $A$ (given by $\varphi: S^{k-1} \times D^{n-k} \hookrightarrow W_s$)

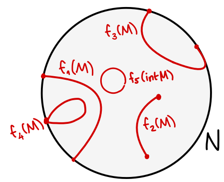
and glue in $D^k \times S^{n-k-1}$ (the belt region).

## § Submanifolds & Transversality

def. A smooth map $f: M \to N$ is an immersion if $Df|_x : TM_x \to TN_{f(x)}$ is injective for every $x \in M$. A smooth embedding is a top. embedding which is an immersion. A smooth embedding is neat if
1° $f(M) \cap \partial N = f(\partial M)$
2° $\forall p \in \partial N \ \exists (U, \varphi: U \hookrightarrow \mathbb{R}^n_+)$ s.t. $U \cap M = \varphi^{-1}(0 \times \dots \times 0 \times \mathbb{R}^{m-1} \times \mathbb{R}_+)$.

def. A (neat) submanifold is a closed subset $M \subseteq N$ s.t. the inclusion map is a (neat) smooth embedding. We define $\mathrm{codim}(M,N) := \dim N - \dim M$.

Examples.




$M = D^1, N = D^2$
$f_1, f_2, f_3$ are smooth embeddings
$f_1$ is neat

Note: For $f: M \to \mathbb{R}$ $y$ regular if $\forall x \in f^{-1}(y)$ have: $df_x$ surjective.

**def.** Let $f: M \to N$ smooth. We call $y \in N$ a regular value if
$df_x : TM_x \to TN_{f(x)}$ is surjective for all $x \in f^{-1}(y)$ and
$df_x| : T(\partial M)_x \to TN_{f(x)}$ is surjective for all $x \in f^{-1}(y) \cap \partial M$.

**Thm.** [Cor. of Implicit Function Theorem]
If $y \in N \setminus \partial N$ is a regular value of a smooth map $f: M \to N$, and of $f|_{\partial M}$
then $f^{-1}(y)$ is a neat smooth submanifold of $M$.
Moreover, $\operatorname{codim}(f^{-1}(y), M) = \dim N$

**def.** Two smooth maps $f: M_1 \to N$ and $g: M_2 \to N$ are transverse, $f \pitchfork g$.
if $(\forall x_1 \in M_1, x_2 \in M_2)\ f(x_1) = g(x_2) =: y \Rightarrow df(TM_1)_{x_1} + dg(TM_2)_{x_2} = TN_y$

In particular:
1° $\dim M_1 + \dim M_2 < \dim N$ then $f \pitchfork g$ iff $f(M_1) \cap g(M_2) = \emptyset$
2° $\dim M_1 + \dim M_2 = \dim N$ then $f \pitchfork g$ iff $f(x_1) = g(x_2) \Rightarrow df(TM_{x_1}) \oplus dg(TM_{x_2}) \cong TN_y$.
3° $\dim N = 2 \dim M$ then $f \pitchfork f$ iff $f(x_1) = f(x_2) \Rightarrow df(TM_{x_1}) \oplus df(TM_{x_2}) = TN_y$ $\infty$
4° $g: \{y\} \hookrightarrow N$ then $f \pitchfork \{y\}$ iff $y$ is a regular value of $f$.

**def.** An ambient isotopy of $N$ is a smooth map
$F: N \times [0,1] \to N$
s.t. $\forall t \in [0,1]$ $F_t : N \to N$ is a diffeomorphism.
Given an isotopy $f_t : M \to N$ we say that
$F_t: N \to N$ is an ambient extension of $f_t$ if: $\forall t \in [0,1]$ $F_t \circ f_0 = f_t$.

**Thm** [Cerf 1961, Palais 1960] – Ambient Isotopy Extension –
If $M$ is compact, then any $f_t: M \to N$ admits an ambient extension.
NOTE: This is useful when we want to "move" not only
a submanifold but also its tubular nbhd.

## § NORMAL BUNDLES & TUB. NEIGHBOURHOODS

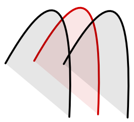
**Thm.** For a smooth manifold $N$ there is a neighbourhood $U$ of $N \subseteq TN$ (zero-section)
and a smooth map $\gamma: U \times [0,1] \to N$ s.t. for $v \in U$ and $s,t \in [0,1]$:
1) $\gamma_v(0) = \pi(v)$, $\dot{\gamma}_v(0) = v$ $\quad$ $\gamma_v : [0,1] \to N$
2) $\gamma_{sv}(t) = \gamma_v(st)$ $\quad$ $d\gamma_v(\partial/\partial t)_0 = v$
Moreover, if $N$ is compact one can take $U = TN$.
We then define $\exp: U \to N$ by $\exp(v) = \gamma_v(1)$.

**def.** The normal bundle $\nu_{N \subseteq M}$ of a smooth submanifold $M \subseteq N$
is the quotient bundle $TN|_M / TM$.
If we fix a Riemannian metric on $TN$ then $\nu_{N \subseteq M} = (TM)^\perp \subseteq TN$.

**Thm.** There is a neighbourhood of $M \subseteq \nu_{M \subseteq N}$ on which $\exp: TN \to N$
is an embedding. Thus, $\exists\, U_M \subseteq N$, $M \subseteq U_M$ and $U_M \to M$
has a structure of a vector bundle, isomorphic to $\nu_{M \subseteq N}$,
and $U_M \cap \partial N \to \partial M$ has a structure of a v. bundle, isomorphic to $\nu_{\partial M \subseteq \partial N}$.

**def.** Such $U_M$ is called a neat tubular neighbourhood.

NOTE: A normal vector field is a section of the bundle $\nu_{N \subseteq M} \longrightarrow N$
and allows us to create "parallel push-offs" of submanifolds:



**Exercise.** We saw for $f: M \to N$, $y \in N$ regular value: $M^y := f^{-1}(y) \subseteq M$ submanifold.
Show that $\nu_{M^y \subseteq M}$ is a trivial bundle and has a canonical trivialisation.

NOTE: - For $M \subseteq N$ if two out of $TM, \nu_{M \subseteq N}, TN$ are oriented, then is also the third, via:
$TM \oplus \nu_{M \subseteq N} = TN|_M$.