LECTURE 3

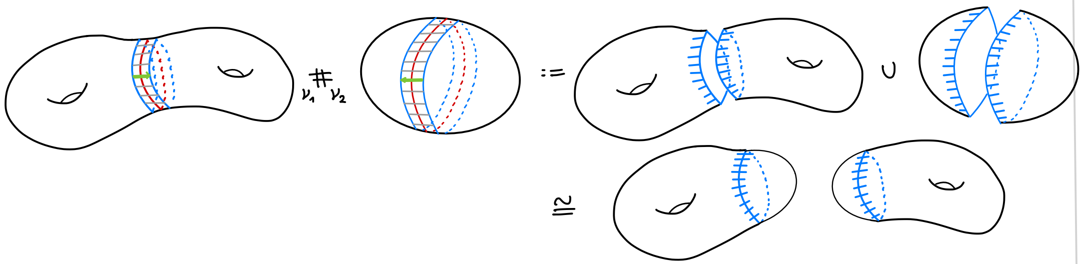
§ How to (de)construct manifolds

Def (GLUE) Given two smooth manifolds with $\dim N_1 = \dim N_2 = n$, a rank $(m-n)$ bundle $E \to Y$ over a smooth m-manifold, $m \le n$, and neat tub. nbhds $\nu_i : E \hookrightarrow N$ of neat submanifolds $\nu_i(Y)$.

Define:

$$N_1 \,{}_{\nu_1}\#_{\nu_2}\, N_2 := N_1 \setminus \nu_1(Y) \sqcup N_2 \setminus \nu_2(Y) \Big/ \nu_1(v) = \nu_2\left(\mathrm{rev}(|v|)\cdot \tfrac{v}{|v|}\right), \ \forall v \in E$$

where $\mathrm{rev}: (0,\infty) \to (0,\infty)$ is an orientation reversing diffeomorphism.



NOTE: For $\nu_i : E \hookrightarrow \partial N_i$ can still define $N_1 \,{}_{\nu_1}\#_{\nu_2}\, N_2$.
We can first glue the boundaries: $\partial N_1 \,{}_{\nu_1}\#_{\nu_2}\, \partial N_2$
But then we have to make sure we can "put N_1 and N_2 back in" and still have a smooth structure. We can do this using "half-tubular" neighbourhoods

THM. This operation yields a well-defined smooth manifold, which up to diffeomorphism does not depend on rev and depends only on ν_i up to isotopy.

Some special cases:

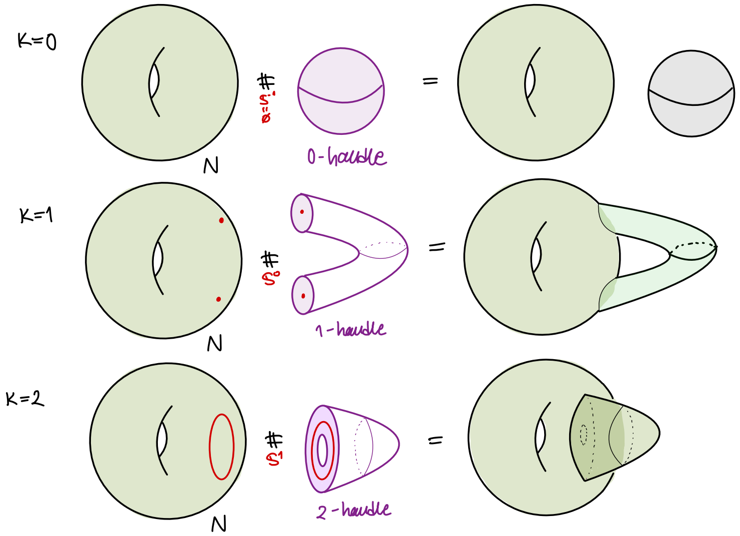
1° connected sum $N_1 \# N_2$ for $Y = pt$, $E = \mathbb{R}^n$, $\nu_i : \mathbb{R}^n \hookrightarrow \mathrm{int} N_i$

2° boundary connected sum $N_1 \natural N_2$ for $Y = pt$, $E = \mathbb{R}^{n-1}$, $\nu_i : \mathbb{R}^{n-1} \hookrightarrow \partial N_i$

THM [Palais]
Any two $D^n \hookrightarrow N$, either both orient. preserving or reversing, are ambiently isotopic.
Hence, connected & boundary connected sum are well-defined and indep. of all choices.

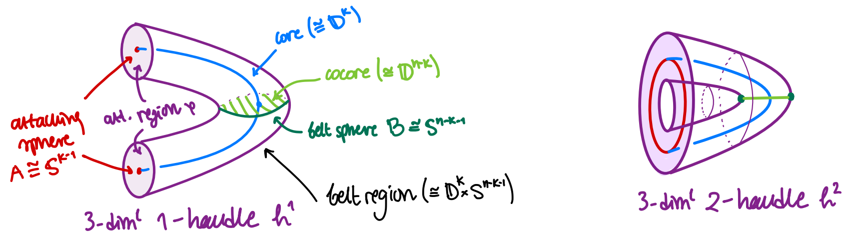
3° handle attachment: $N_1 = N$, $N_2 = D^n$, $Y = S^{k-1}$
$\nu_1 : S^{k-1} \times \mathbb{R}^{n-k} \hookrightarrow \partial N$, $\nu_2 : S^{k-1} \times \mathbb{R}^{n-k} \hookrightarrow S^{k-1} \times D^{n-k} \subseteq D^k \times D^{n-k} \approx D^n$.

Examples.



K=3 this is boundary connected sum with D^3 (so not possible for $N = S^1 \times D^2$)

NOTE: We simplify by thinking of handle attachment as just $N \cup_\varphi h^k \equiv N \,{}_{\nu_1}\#_{\nu_2}\, D^n$
where $h^k := D^k \times D^{n-k}$ is the handle of index k (k-handle)
$\varphi := \overline{\nu}_1 : S^{k-1} \times D^{n-k} \hookrightarrow \partial N$ is the attaching region



NOTE: The usual def^n of $N \cup_\varphi h^k$ as the gluing of top. spaces is apriori not a smooth manifold, but our def^n $N \,{}_{\nu_1}\#_{\nu_2}\, D^n$ is!

4° surgery along a sphere: $N_1 = N$, $N_2 = S^n$, $Y = S^{k-1}$, $\nu_1 : S^{k-1} \times \mathbb{R}^{n-k} \hookrightarrow \partial N$, $\nu_2 : S^{k-1} \times \mathbb{R}^{n-k} \underset{std}{\hookrightarrow} S^n$

Exercise. Use the Handlebody Decomposition Thm to prove the classification of compact surfaces.

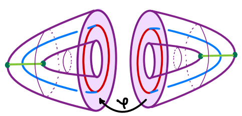
Exercise. Relate surgery on a $(k-1)$-sphere and handle attachment of a k-handle.

§ HANDLE CALCULUS

Cor of Thm. – **Isotopy Lemma** – If $\varphi_i : S^{k-1}\times D^{n-k} \hookrightarrow \partial N$ are isotopic $i=1,2$, then $N \cup_{\varphi_i} h^k$ are diffeomorphic $i=1,2$.

– **Unknot Lemma** –
If $N := D^n \cup_\varphi h^k$ and $A := \varphi|_{S^{k-1}\times 0} : S^{k-1} \hookrightarrow \partial D^n$ bounds an embedding $\Delta : D^k \hookrightarrow \partial D^n$ then N is a D^{n-k}-bundle over a smooth manifold homeomorphic to S^k.

proof. Push the interior of Δ into $\mathring{D}^n$, so $\Delta' : D^k \hookrightarrow D^n$, $\partial\Delta' = A$. Then D^n can be viewed as a tub nbhd $\nu_\Delta : D^k \times D^{n-k} \cong D^n$. Then: $N = D^n \cup_\varphi h^k \cong (D^k \times D^{n-k}) \underset{S^{k-1}\times D^{n-k} \supset \varphi}{\cup} (D^k \times D^{n-k})$
Now, the projections $D^k \times D^{n-k} \to D^k$ glue together along $(\partial D^k)\times D^{n-k}$ to a well-defined map $N \longrightarrow \underbrace{(D^k \cup_A D^k)}_{\text{see Lemma below}}$ which is a fibre bundle with fibre D^{n-k}. □



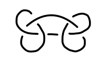
def. For a diffeomorphism $A : S^{k-1} \circlearrowleft \cong$ define the smooth manifold $S(A) := D^k \cup_A D^k$.

Lemma. $S(A)$ is always homeomorphic to S^k.

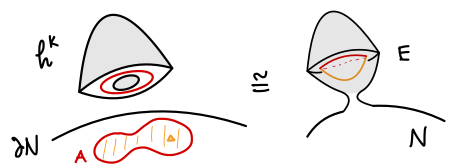
proof. Define a homeomorphism $D^k \cup_A D^k \xrightarrow[\approx]{\bar{A} \cup \mathrm{id}} D^k \cup_{\mathrm{id}_{S^{k-1}}} D^k = S^k$
where $\bar{A} : D^k \to D^k$, $\bar{A}(r,v) = (r, A(v))$ is a homeomorphism extending A radially. □
$r \in [0,1]$, $v \in S^{k-1}$

NOTE: $S(A)$ is not diffeomorphic to S^k in general. It is called a "twisted sphere".
We will see:
Smale's h-cobordism Thm $\Rightarrow$ Every exotic sphere of dim ≥ 5 is a twisted sphere.

Exercise. A twisted sphere $S(A) = D^k \cup_A D^k$ is diffeomorphic to S^k if and only if $A : S^{k-1} \to S^{k-1}$ extends to a diffeomorphism $D^k \to D^k$.

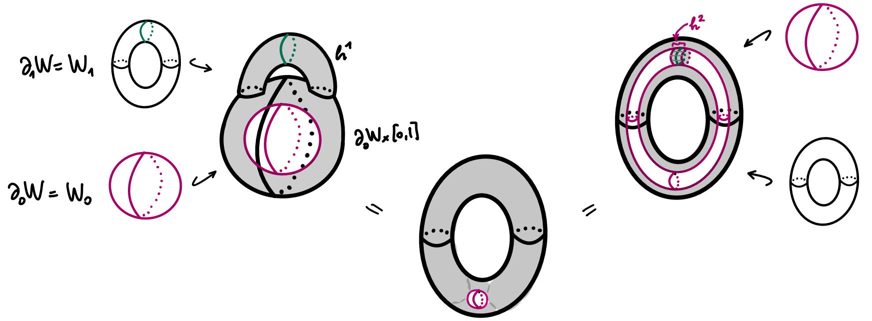
NOTE: The Unknot Lemma is not true if $\Delta \subseteq D^n$ instead of $\Delta \subseteq \partial D^n$.
The condition $\Delta \subseteq \partial D^n$ is equivalent to A being "unknotted", $A \cong U$
whereas $\Delta \subseteq D^n$ is equivalent to A being "slice".
For example: $\exists$ many $A : S^1 \hookrightarrow S^3$ s.t. $A \not\cong U$ but A is slice e.g. 

Cor. If $A : S^{k-1} \hookrightarrow \partial N$ bounds a disc $\Delta : D^k \hookrightarrow \partial N$, then
$$N \cup_\varphi h^k \cong N \natural E$$
where $E \to S(A)$ is a D^{n-k}-bundle.



– **Upside Down Lemma** – For every handle decomposition of $(W, \partial_0 W, \partial_1 W)$ there is an "upside-down" decomposition of $(W, \partial_1 W, \partial_0 W)$ with handles of index $n-k$ attached along the belt spheres of k-handles of the original decomposition.

proof. FACT: Every handle decomposition corresponds to a Morse function, call it h.
Then $-h$ yields a decomposition of the upside-down cobordism.
We just observe that turning a k-handle upside-down turns its belt region into the attaching region.



– Reordering Lemma – If $k \le \ell$ then $(N \cup_{\varphi_1} h_1^\ell) \cup_{\varphi_2} h_2^k \cong (N \cup_{\varphi_2''} h_2^k) \cup_{\varphi_1'} h_1^\ell$ for some isotopic attaching map $\varphi_2'' \cong \varphi_2$, with $\mathrm{im}\,\varphi_2 \subseteq \partial N$, and φ_1' has the same image as φ_1.

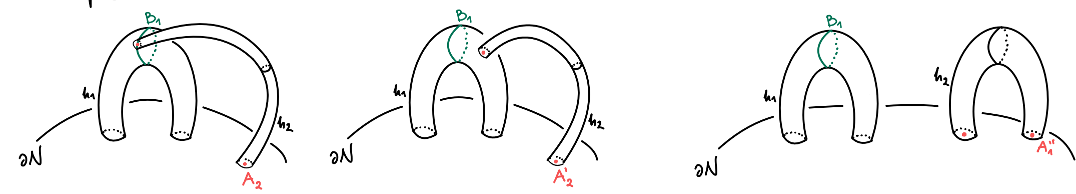
proof. Denote $A_2 :=$ the attaching sphere of h_2, $B_1 :=$ the belt sphere of h_1.

Thm [Thom] If $A : M \to N$ a smooth map and $B \subseteq N$ a compact submanifold then there is an ambient isotopy of N, taking A to A' such that $A' \pitchfork B$. Moreover, the isotopy can be assumed to be the identity outside of any open nbhd of B.

Assuming this, we have $A_2' \pitchfork B$ i.e. $dA_2'(TS^{k-1})_a + dB(TS^{n-\ell-1})_b = T\partial(N \cup h_1)_x$ for every a, b s.t. $A_2'(a) = B(b)$. However, since
$\dim B_1 + \dim A_2 = n-\ell-1 + k-1 = (n-1) + (k-\ell) - 1 < n-1 = \dim \partial(N \cup h_1)$
we must have $A_2' \cap B = \emptyset$. We can isotope further, so that $A_2'' \subseteq \partial N$ (i.e. the belt region) away from
By the Ambient Isotopy Extension Thm we have $\varphi_2''(S^{k-1} \times D^{n-k}) \subseteq \partial N$.
Thus, the two handles can be attached in any order (or simultaneously). □

sketch proof of Thom's Thm:
Firstly, find a tubular nbhd U_B of B contained in the given open set $U \supseteq B$.
Then apply to $E = U_B \to B$ the following:
Lem. If $f : M \to E$ is smooth and $E \xrightarrow{\pi} N$ a smooth vector bundle, then there exist a section $s : N \to E$ such that $f \pitchfork s$.
Thus, there is an obvious isotopy from B to $s(B) \subseteq U_B \subseteq U$ and we can extend it by Id on $N \setminus U$.
To prove the Lemma, use Morse-Sard Thm to get the result for trivial bundles, and extend to all bundles using that all vector bundles have stable inverses. □.

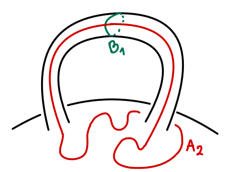
Example. $n=3$, $k=\ell=1$



– Cancellation Lemma – If $\underbrace{A_2 \pitchfork B_1 = \{p\}}$ then $(N \cup_{\varphi_1} h_1^k) \cup_{\varphi_2} h_2^{k+1} \cong N$.

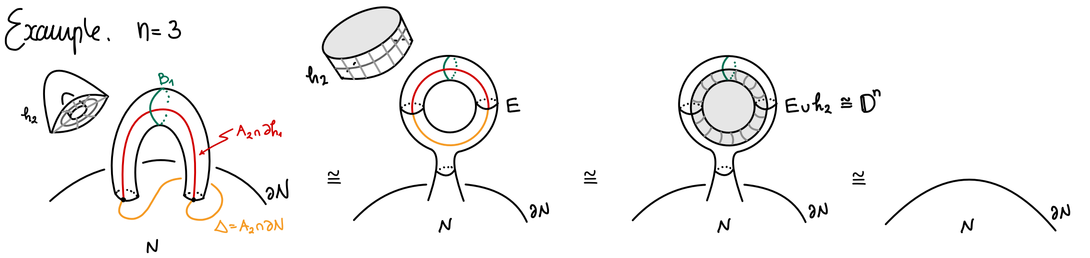
We say that h_1 and h_2 are in a geometrically cancelling position.
Or h_2 goes over h_1 geometrically once.

proof. Since A_2^{k-1} and B_1^{n-k-1} intersect transversely, and $\nu_{B_1 \subseteq N \cup_{\varphi_1} h_1}$ can be identified with the belt region $D^k \times \partial D^{n-k} \subseteq \partial h_1^k$, we can assume $A_2 \cap \partial h_1^k \cong D^k \times \{p\}$ (the fibre of $\nu_{B_1 \subseteq N \cup_{\varphi_1} h_1}$ at $p \in B_1$).



Then by Cor. of Unknot Lemma for
$N' := N \cup_{\varphi_1} h_1^k$ and $A := A_1$ and $\Delta := A_2 \cap \partial N$
we have diffeos:
$$(N \cup_{\varphi_1} h_1^k) \cup_{\varphi_2} h_2^{k+1} \cong (N \natural E) \cup_{\varphi_2} h_2 \cong N \natural (E \cup_{\varphi_2} h_2) \cong N \natural D^n \cong N.$$
since $\mathrm{im}\,\varphi_2 \subseteq \partial E$ — since $D^{n-k} \to E \to S(A_1)$ and h_2 glued onto $A_2 \subseteq \partial E$ which is a section of E. □

Example. $n=3$



Note: We can reverse the argument to show that a cancelling pair can be added.

Example. $\mathbb{D}^n = \mathbb{D}^k \times \mathbb{D}^{n-k}$

$A : S^{k-1} \hookrightarrow \partial \mathbb{D}^n$ as $\{ (x_k, x_{n-k}) : |x_k| = 1,\ x_{n-k} = 0 \}$

then the half-tub. nbhd of A is $\{ (x_k, x_{n-k}) : |x_k| > \varepsilon \}$

can take $\Delta = \{ (x_k, x_{n-k}) : |(x_k, x_{n-k})| = 1,\ |x_{n-k}| \geqslant 0 \}$

can take $\Delta' = \{ (x_k, x_{n-k}) : x_{n-k} = 0 \}$

Then

$\mathbb{D}^n = \mathbb{D}^k \times D^{n-k}$ is a tub. nbhd of $\Delta' = \mathbb{D}^k \times \{0\}$