LECTURE 6

Step 1.

– Handle Trading Lemma –

Assume $(W, \partial_0 W, \partial_1 W)$ is an h-cobordism with $\dim W = n \geq 6$ and a handle decomposition with no handles of index $\leq k-1$ for some $1 \leq k \leq n-2$.
Then the decomposition can be modified so that precisely one k-handle is removed and precisely one $(k+2)$-handle is added.

proof of Handle Trading Lemma.

Let h^k be the k-handle we wish to remove.
The idea is to use the reverse of the Cancellation Lemma to add a cancelling pair $h^{k+1} \cup h^{k+2}$ so that h^{k+1} cancels our h^k and leaves h^{k+2} behind.
In other words we will have:

($\partial_0 W \times [0,1] \cup$ handles of index $\leq k$ $= W^{\leq k}$)

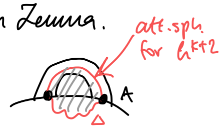
$$W^{\leq k} \cong W^{\leq k} \natural D^n$$ by $\natural D^n$ does nothing (exercise)

$$\cong (W^{\leq k-1} \cup \text{other } k\text{-handles} \cup h^k) \cup_{\varphi_{k+1}} (h^{k+1} \cup_{\varphi_{k+2}} h^{k+2})$$ by reverse of the Cancellation Lemma

$$\cong (W^{\leq k-1} \cup \text{other } k\text{-handles}) \cup_{\varphi_k} (h^k \cup_{\varphi_{k+1}} h^{k+1}) \cup_{\varphi_{k+2}} h^{k+2}$$ by the Reordering Lemma

$$\cong (W^{\leq k-1} \cup \text{other } k\text{-handles}) \cup_{\varphi_{k+2}} h^{k+2}$$ by the Cancellation Lemma.

once we find



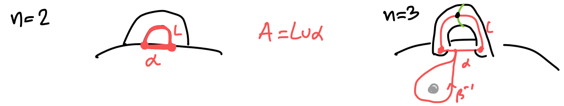
$A := \varphi_{k+1}|_{S^k \times 0} \subseteq \partial_1 W^{\leq k}$ such that:
1° A goes over h^k geom. once (for Canc Z to apply) $\Leftrightarrow$ $A \pitchfork$ belt sphere of h^k = {pt}
2° A is unknotted (for rev. of Canc Z to apply) $\Leftrightarrow$ A isotopic to the unknot.

Let us construct such an A. We need to distinguish the case $k=1$ from $k \geq 2$.

Case k=1.

(works also for $\dim W \geq 5$)

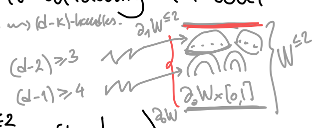
Firstly, let $L \subseteq \partial h^1$ be a push-off of the core of h^1. The endpoints $\partial L \subseteq \partial_0 W$ can be connected by an arc $\alpha \subseteq \partial_0 W$ (by connectedness assumption on $\partial_0 W$) which can be chosen to miss attaching regions of all other 1-handles. Then $A := L \cup \alpha$ is a circle in $\partial_1 W^{\leq 1}$ which can be assumed to be smooth and disjoint from all att. circles of 2-handles, so lives in $\partial_1 W^{\leq 2}$.
By construction, A goes over h^1 geometrically once. 1° ✓



$\delta : D^2 \hookrightarrow \partial_1 W^{\leq 2}$

Lemma. The arc α can be chosen so that $A := L \cup \alpha : S^1 \hookrightarrow \partial_1 W^{\leq 2}$ is nullhomotopic.
Assuming this, we will have that A is unknotted since $\dim(\partial_1 W^{\leq 2}) \geq 4$. 2° ✓ □.

proof of Lemma. Since attaching a k-handle is homotopy equivalent to attaching a k-cell, only 1- and 2-handles can change π_1.
Thus: $\pi_1 W^{\leq 2} \xrightarrow{\cong} \pi_1 W$
and $\pi_1 \partial_1 (W^{\leq 2}) \xleftarrow{\cong} \pi_1 W^{\leq 2}$ (by turning $W^{\leq 2}$ upside down).

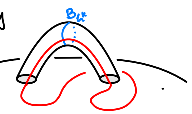


By the h-cobordism assumption $\pi_1 \partial_0 W \xrightarrow{\cong} \pi_1 W$. Therefore,
$$\pi_1 \partial_1 W^{\leq 2} \cong \pi_1 \partial_0 W.$$

- If $\pi_1 \partial_0 W \cong \{1\}$ we immediately have $A \sim *$ in $\partial_1 W^{\leq 2}$.
- More generally: A might be nontrivial $[A] \neq 0 \in \pi_1 \partial_1 W^{\leq 2} \cong \pi_1 W^{\leq 2} \cong \pi_1 \partial_0 W$.
  Let β be a loop in $\partial_0 W$ realizing this class, chosen so that it misses all att. spheres of 1- and 2-handles. Thus, β lives in $\partial_1 W^{\leq 2}$, and replacing α with $\alpha\beta^{-1}$ gives $A := L \cdot \alpha\beta^{-1} \simeq *$ in $\partial_1 W^{\leq 2}$ □

Case k≥2.

IDEA: Start from $A :=$ small unknot and isotope it using handle slides until it goes over h^k geometrically once.



Since $H_*(\tilde{W}, \partial_0 \tilde{W}; \mathbb{Z}) = 0$ (h-cob. assumption) we have that $\cdots \to C^{\tilde{u}}_{k+1} \xrightarrow{\delta^{\tilde{u}}_{k+1}} C^{\tilde{u}}_k \xrightarrow{\delta^{\tilde{u}}_k} C^{\tilde{u}}_{k-1} \to \cdots$ is exact.
Then $C^{\tilde{u}}_{k-1} = 0$ implies that $\delta^{\tilde{u}}_{k+1}$ is surjective.
So: $\exists z_j \in \mathbb{Z}, 1 \leq j \leq r_{k+1}, g_j \in \pi$ with $\tilde{H}^k = \delta^{\tilde{u}}_{k+1}\left(\sum_{j=1}^{r_{k+1}} z_j g_j \tilde{H}^{k+1}_j\right)$.

Use HANDLE SLIDES LEMMA: we can start from a small unknot $S^k \hookrightarrow \partial_1 W^{\leq k-1}$ and slide it over handles h^{k+1}_j with coefficients $z_j g_j$ until we have $A : S^k \hookrightarrow \partial_1 W^{\leq k-1}$ with $[\tilde{A}] = \sum_{j=1}^{r_{k+1}} z_j g_j \tilde{H}^{k+1}_j$. Since handle slides are isotopies, A is unknotted.
On the other hand, $\delta^{\tilde{u}}_{k+1}[\tilde{A}] = \tilde{H}^k$ says that A goes over h^k algebraically once.
Then the Whitney Trick Lemma finishes the proof:
A can be improved to go over h^k geom. once. □

need: $\dim \partial_1 W^{\leq k-1} \geq 5$ so $\dim W \geq 6$

LECTURE 7

key Thm [Smale 1961] – s-cobordism Theorem –
If $(W, \partial_0 W, \partial_1 W)$ is an s-cobordism with $\dim W = n \geq 6$,
then it is smoothly trivial,
i.e. there is a diffeomorphism $(W, \partial_0 W, \partial_1 W) \cong (\partial_0 W \times [0,1], \partial_0 W \times \{0\}, \partial_0 W \times \{1\})$.

Proof.

Pick a handle decomposition of $(W, \partial_0 W, \partial_1 W)$.
Thanks to Remove 0- and n-handles Lemma, we can assume NO 0- and n-handles.

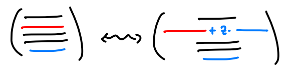
Step 1. – Normal Form Lemma –
For every h-cobordism of dimension $n \geq 6$ and any $2 \leq \ell \leq n-3$
there is a handle decomposition of the form $\partial_0 W \times [0,1] \cup \bigcup_{i=1}^{r_\ell} h_i^\ell \cup \bigcup_{j=1}^{r_{\ell+1}} h_j^{\ell+1}$.

proof of Normal Form Lemma. We first prove we can remove all handles of index $\leq \ell - 1$.
Indeed, using HANDLE TRADING LEMMA we trade
1- for 3-handles, then 2- for 4-handles, etc., $(\ell-1)$- for $(\ell+1)$-handles.
Thus, we have $W \underset{\text{rel } \partial_0}{\cong} \partial_0 W \times [0,1] \cup \ell\text{-handles} \cup (\ell+1)\text{-handles} \cup \dots \cup (n-1)\text{-handles}$
Now, we can turn this handle decomposition upside down and repeat the procedure:
in effect, we will be trading $(n-1)$- for $(n-3)$-handles, ..., $(\ell+2)$- for ℓ-handles.
Thus, we are left with only ℓ- and $(\ell+1)$-handles, as desired. □

Step 2.
We are left with $0 \to C_{k+1}^{\tilde{W}} \xrightarrow{\delta_{k+1}^{\tilde{W}}} C_k^{\tilde{W}} \to 0$ and we wish to remove these as well.
Since $H_*(C_*^{\tilde{W}}, \delta_*^{\tilde{W}}) = 0$, $\delta_{k+1}^{\tilde{W}}$ is an isomorphism $(\mathbb{Z}\pi)^{r_k} \longrightarrow (\mathbb{Z}\pi)^{r_k}$
represented by the equivariant intersection matrix $J^{\tilde{W}} := (\tilde{I}(\tilde{A}_i \pitchfork \tilde{B}_j))_{1 \leq i,j \leq r_k}$

Lemma. $J^{\tilde{W}}$ can be modified to the identity matrix $Id_{(\mathbb{Z}\pi)^{r_k}}$ by the moves listed below, if and only if all the remaining handles can be put into alg. cancelling position.

MOVES:
1° interchange rows:

2° add rows:

$z \in \mathbb{Z}[\pi]$
3° (de) stabilise:

4° multiply a row by $g \in \pi$ (or $-g$):


proof. Show that each move on matrices can be realised by a move on handles. [Exercise]. □

def. The Whitehead group $Wh(\pi)$ is the set of equivalence classes under moves 1°–4° of invertible matrices of arbitrary size with entries in $\mathbb{Z}[\pi]$ with group structure $J + J' = \begin{pmatrix} J & 0 \\ 0 & J' \end{pmatrix}$.

NOTE: Equivalently, $Wh(\pi) := GL(\mathbb{Z}\pi)^{ab} / \langle [g], [-g] : g \in \pi \rangle$

where $GL(R) := \operatorname{colim}_{n \to \infty} GL_n(R)$ for a ring R,
and ab denotes abelianization ($K_1(R) := GL(R)^{ab}$)

Examples. $Wh(\{1\}) = 0$ since $\mathbb{Z}\{1\} = \mathbb{Z}$ has Euclidean algorithm
$Wh(\pi) = 0$ for π = free abelian group [Bass-Heller-Swan '64]
$Wh(\mathbb{Z}/5\mathbb{Z}) = \mathbb{Z}$ generated by the unit $t + t^{-1} - 1 \in GL_1$.
Conjecture. $Wh(\pi) = 0$ if π is torsion-free.

def. Whitehead torsion of $(W, \partial_0 W, \partial_1 W)$ is $\tau_W := [J^{\tilde{W}}] \in Wh(\pi_1 W)$.

Remark. $\tau_W = 0$ iff $\partial_i W \longrightarrow W$ are simple homotopy equivalences.
gives the name to the s-cobordism Thm.

Step 3.

We now want to use Whitney moves to turn an algebraically cancelling pair of handles, into a geometrically cancelling pair.

– Whitney Trick Lemma –

based spheres: $\tilde{A} = A \cup w_A$, $\tilde{B} = B \cup w_B$

If $\dim N \geq 5$ and $\tilde{A}: S^{n_1} \hookrightarrow N$, $\tilde{B}: S^{n_2} \hookrightarrow N$ have $\tilde{I}(\tilde{A} \pitchfork \tilde{B}) = +1$, then there is an isotopy of $\tilde{A}$ such that $\tilde{A}' \pitchfork \tilde{B} = \{pt\}$. $(n_1 + n_2 = n = \dim N \geq 5)$

proof. Having $\tilde{I}(\tilde{A} \pitchfork \tilde{B}) = \sum_{p \in A \pitchfork B} \varepsilon_p g_p = +1 \quad = (\overbrace{\varepsilon_p g_p + \varepsilon_q g_q}^{0}) + \dots + \underbrace{\varepsilon_r}_{+1} \underbrace{g_r}_{1}$

implies that we can find pairs $p, q \in A \pitchfork B$ such that $\varepsilon_q g_q = -\varepsilon_p g_p$

$\Rightarrow$ $\exists$ Whitney circle $\gamma_1 \cdot \gamma_2^{-1}$ through p and q, which is nullhomotopic in N

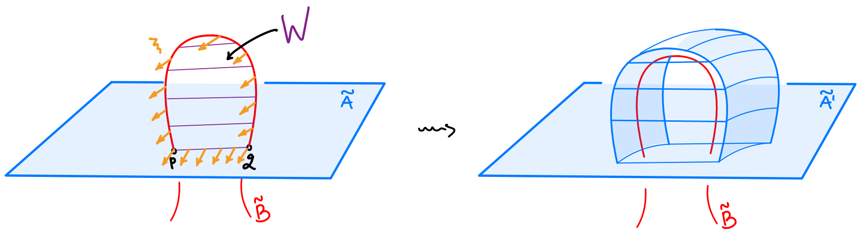
Since $n_i \leq n-3$ $i = 1,2$ $\Rightarrow$ $\pi_1(N \setminus (A \cup B)) \cong \pi_1 N$ $\Rightarrow$ $\gamma_1 \gamma_2^{-1} \simeq *$ in $N \setminus A \cup B$.

$\Rightarrow$ $\gamma_1 \gamma_2^{-1}$ bounds an immersed disc in $N \setminus A \cup B$.

Since $n \geq 5$ $\Rightarrow$ $\gamma_1 \gamma_2^{-1}$ bounds an embedded disc $W: D^2 \hookrightarrow N$ with $\operatorname{int} W \pitchfork A \cup B = \emptyset$

Since $\varepsilon_q = -\varepsilon_p$ and $n > 4$ $\Rightarrow$ W can be framed.

$\Rightarrow$ We can perform the Whitney move to remove p, q. Continue with other pairs, until precisely one intersection p with $\varepsilon_p g_p = +1$ left. □



Corollaries.

Thm – Top Poincaré Conjecture in dim ≥ 6 –

If N is a smooth homotopy n-sphere and $n \geq 6$, then N is homeomorphic to S^n (i.e. N is an exotic n-sphere).

proof. Remove two small discs from N. The resulting manifold is a simply connected h-cobordism from S^{n-1} to itself, so by the h-cobordism theorem:

$$(N \setminus D^n_1 \cup D^n_2, \partial D^n_1, \partial D^n_2) \cong (\partial D^n_1 \times [0,1], \partial D^n_1 \times \{0\}, \partial D^n_1 \times \{1\})$$

We can glue back D^n_1 by $id_{\partial D^n_1}$, but D^n_2 has to be glued back by a homeomorphism extending the diffeomorphism $\partial D^n_2 \to \partial D^n_1 \times \{1\}$ (use the radial extension, see Lecture 3) □.

Thm [Diff Schoenflies Conjecture in dim ≥ 6]

If $K: S^{n-1} \hookrightarrow S^n$ is a smooth embedding and $n \geq 6$, then the closure of each component of $S^n \setminus K(S^{n-1})$ is diffeomorphic to D^n.

proof. Since K has a tubular neighbourhood, we see that the closure of each component of $S^n \setminus K(S^{n-1})$ is a smooth manifold with boundary S^{n-1}. It is simply connected by Seifert-van-Kampen Theorem. Thus, if we remove from it a small disc we get a simply connected h-cobordism. By the h-cobordism Theorem this is diffeomorphic to $S^{n-1} \times [0,1]$, and we can put back the disc by the identity to get a diffeomorphism to D^n. □