LECTURE 9 & 10

The idea as the Eilenberg swindle in algebra:

Lemma. Any projective R-module A is stably free.

proof. A projective $\iff \exists B$ s.t. $A \oplus B = F$ free

Then $(A\oplus B)\oplus(A\oplus B)\oplus \dots = F^\infty$

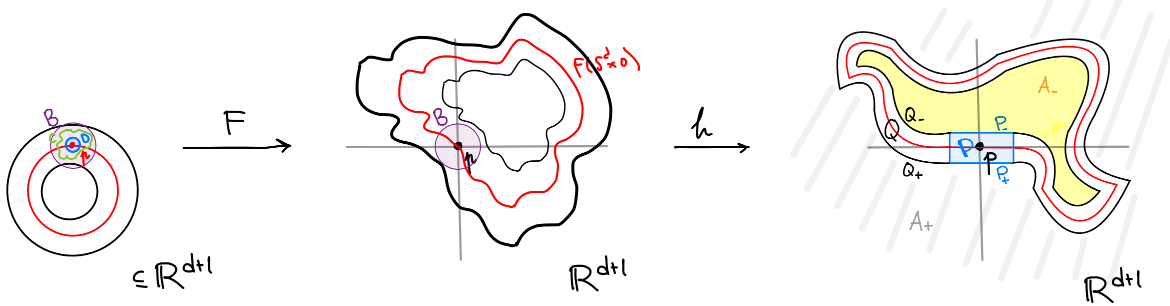
$= A \oplus (B\oplus A)\oplus(B\oplus \dots) = A \oplus F^\infty$. □

Next, we remove the condition that there is a standard spot.

Thm [Morse 1960]
For any $d \geq 1$ and a bicollared embedding $F: S^d \times [-1,1] \xhookrightarrow{top} \mathbb{R}^{d+1}$ there exists a self-homeomorphism $h: \mathbb{R}^{d+1} \circlearrowleft \approx$ and an $\varepsilon > 0$ so that $h \circ F|_{S^d \times [-\varepsilon,\varepsilon]}$ has a standard spot.

proof. We fix $p \in S^d$ (say, north pole) and include $S^d \times [-1,1] \subseteq \mathbb{R}^{d+1}$ so that $p = 0 \in \mathbb{R}^{d+1}$.
Let $D \subseteq S^d \times (-1,1)$ be a small round ball around p.
We can assume $F(p,0) = 0 \in \mathbb{R}^{d+1}$ (otherwise, translate).

- Let $B \subseteq \mathbb{R}^{d+1}$ be a standard round ball in $\mathbb{R}^{d+1}$ centered at p, such that: $D \subseteq B$
  $B \subseteq F(S^d \times (-1,1))$
- Let $C := F^{-1}(B) \subseteq S^d \times (-1,1)$.



Then define

$$h: \mathbb{R}^{d+1} \xrightarrow[\approx]{g} B \xrightarrow[\approx]{(F|_C)^{-1}} C$$

where g is a homeomorphism s.t. $g|_D = \mathrm{Id}_D$.

Note that h is a homeomorphism from $\mathbb{R}^{d+1}$ to an open subset of $\mathbb{R}^{d+1}$ and that:

$$\begin{array}{ccccccc}
h\circ F: S^d \times [-1,1] & \hookrightarrow & \mathbb{R}^{d+1} & \longrightarrow & B & \longrightarrow & C \subseteq \mathbb{R}^{d+1} \\
\cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid \\
F^{-1}(D) & \xmapsto{F} & D & \xmapsto{\mathrm{Id}} & D & \xmapsto{F^{-1}} & F^{-1}(D)
\end{array}$$

Thus, $h \circ F|_D = \mathrm{Id}_D$

If we now restrict F to $S^d \times [-\varepsilon, \varepsilon]$ such that: $D^d \times [-\varepsilon,\varepsilon] \subseteq F^{-1}(D)$ then we will have a standard spot. □.

Cor. Top Schönflies Conjecture is true in all dimensions.

proof. By Connelly every loc. flat sphere has a collar.
By Morse we can find a collar $F: S^d \times [-\varepsilon,\varepsilon] \to \mathbb{R}^{d+1}$ so that $h \circ F$ has a standard spot.
By Mazur, the closures of the complement of $h \circ F|_{S^d \times 0}$ are homeomorphic to D^{d+1}.
Precomposing those homeomorphisms with h we get desired result.
□.

In order to get a feeling for topological manifold we prove the following result. The proof relies on a standard tool called the push-pull argument.

THM. Let X and Y be compact top. manifold.
If $X \times \mathbb{R}$ is homeomorphic to $Y \times \mathbb{R}$, then $X \times S^1$ is homeomorphic to $Y \times S^1$.

proof Fix $h: X \times \mathbb{R} \xrightarrow[\approx]{\text{top}} Y \times \mathbb{R}$.
Denote:

$$X_t := X \times \{t\} \quad \text{and} \quad X_{[t,u]} := X \times [t,u], \qquad t < u \in \mathbb{R}$$
$$Y_a := Y \times \{a\} \quad \text{and} \quad Y_{[a,b]} := Y \times [a,b] \qquad a < b \in \mathbb{R}.$$

Step 1. There exist $r < s$ and $a < b < c$ such that

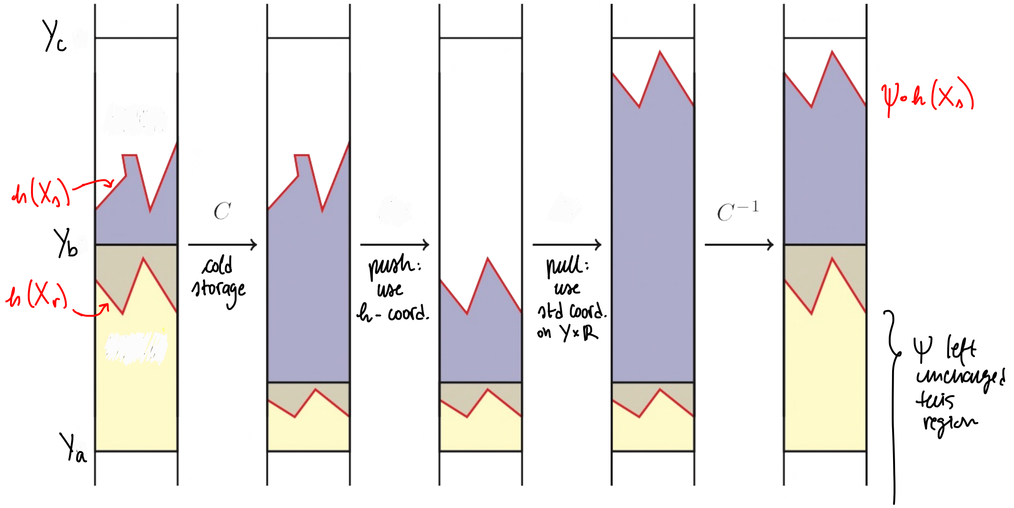
$$h(X_r) \subseteq Y_{[a,b]}$$
$$h(X_s) \subseteq Y_{[b,c]}$$
$$Y_b \subseteq h(X_{[r,s]})$$

Namely: Fix r arbitrary,
then by compactness of X, Y find $a \in \mathbb{R}$ s.t. $h(X_r) \cap Y_{(-\infty,a]} = \emptyset$
then —‖— $b > a$ s.t. $h(X_r) \cap Y_{[b,\infty)} = \emptyset$
$s > r$ s.t. $Y_b \subseteq h(X_{[r,s]})$
$c > b$ s.t. $h(X_s) \cap Y_{[c,\infty)} = \emptyset$

Step 2. Construct a homeomorphism $\psi: Y \times \mathbb{R} \to Y \times \mathbb{R}$
s.t.

$$\psi|_{Y \times (\mathbb{R} \setminus [a,c])} = \mathrm{Id} \quad \text{and} \quad \psi(h(X_s)) = \text{translate of } h(X_r)$$

in the usual $Y \times \mathbb{R}$ coordinate.



Step 3. $H := \psi \circ h|_{X_{[r,s]}} : X \times [r,s] \xrightarrow{\text{top}} Y \times \mathbb{R}$
has $H(X \times \{r\}) = H(X \times \{s\})$
so descends to a continuous map

$$X \times S^1 \longrightarrow Y \times S^1.$$

This is a bijection (we omit the proof).
Its inverse is continuous by
"Continuous map from from compact to Hausdorff is closed.

□

§ OTHER APPLICATIONS OF PUSH-PULL

- Push-pull technique was used by Brown in his proof of Collar and Bicollar Theorems. [1962]
- Armstrong '70 used it to show that (bi) collars are unique up to isotopy.

- Kister used it to prove the Stretching Lemma — this is the key step for his theorem:

KISTER'S THEOREM. Every topological n-manifold admits an $\mathbb{R}^n$-fibre bundle $\mathbb{R}^n \to \tau_M \xrightarrow{p} M$ with structure group $\mathrm{Homeo}_0 \mathbb{R}^n$ and an embedding $e: \tau_M \hookrightarrow M \times M$ onto a neighbourhood of $\Delta_M \subseteq M \times M$ s.t. the following commutes for $s =$ 0-section

$$\begin{tikzcd} M \arrow[r, hook, "s"] \arrow[d, equal] & \tau_M \arrow[r, "p"] \arrow[d, "e"] & M \arrow[d, equal] \\ M \arrow[r, hook, "\Delta"] & M \times M \arrow[r, "pr_1"] & M \end{tikzcd}$$

(τ_M, s, p) fitting into this diagram is called a tangent microbundle.

Moreover, such τ_M is unique up to isomorphism.

def. Such a fibre bundle $\tau_M \longrightarrow M$ is called the topological tangent bundle of M.

Proposition. The tangent vector bundle of a smooth manifold is a top. tangent bundle in this sense.

§ 4-MANIFOLDS

We saw: Smale's h-cobordism theorem + Barden-Mazur-Stallings s-cobordism theorem apply to cobordisms W with $\dim W \geq 6$.
For $\dim W = 5$ we could prove the Normal Form Lemma, but could not proceed further since the Whitney trick fails.

key **Thm** [Freedman 1982] – s-cobordism Theorem in dim 5 –
If $(W, \partial_0 W, \partial_1 W)$ is an h-cobordism with $\dim W \geq 6$ and trivial Whitehead torsion $Wh(W, \partial_0 W) \in Wh(\pi_1 W)$ and $\pi_1 W$ is a good group, then W is topologically trivial,
i.e. there is a homeomorphism $(W, \partial_0 W, \partial_1 W) \cong (\partial_0 W \times [0,1], \partial_0 W \times \{0\}, \partial_0 W \times \{1\})$.

e.g. all finite groups
all abelian groups

proof. As before (see Lecture 6): Step 0 Remove 0- and 5-handles Lemma.
Step 1 Normal Form Lemma (using Handle Trading Lemma).
Step 2 Algebraically cancelling pairs:

$$0 \to C_3^{\tilde{u}} \xrightarrow{\partial_3^{\tilde{u}}} C_2^{\tilde{u}} \to 0$$

with $\partial_3^{\tilde{u}}$ represented by the identity matrix (using $Wh(W, \partial_0 W) = 0$ and Handle Slides).

need to revisit the proof to see if all works.

$\Longrightarrow$ In the middle level $W_{1/2} := \partial_1 (W^{\leq 2})$ where $W^{\leq 2} = \partial_0 W \times [0,1] \cup$ 2-handles
we have the belt spheres $B_1, \dots, B_r : S^2 \hookrightarrow W_{1/2}$ of 2-handles $(\{0\} \times S^2 \subseteq D^2 \times D^3)$
and the attaching spheres $A_1, \dots, A_r : S^2 \hookrightarrow W_{1/2}$ of 3-handles $(S^2 \times \{0\} \subseteq D^3 \times D^2)$
so that:

- each $\{B_i\}$ and $\{A_j\}$ is a collection of pairwise disjoint, framed, embedded spheres
- $\tilde{I}(A_j \pitchfork B_i) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \in \mathbb{Z}[\pi_1 W_{1/2}]$

WANT: Isotope A_j so that there intersection numbers are realised geometrically, so that we can cancel each pair of handles, $i = 1, \dots, r$.