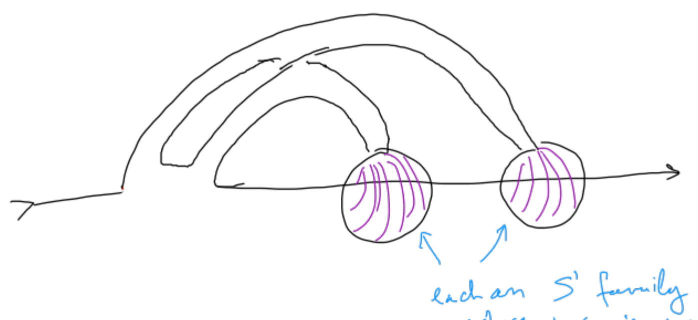
Homotopy

Conjecture All (Whitehead product indecomposable) classes in $\pi_0 \mathrm{Emb}(\mathbb{I}, \mathbb{I}^d)$ are represented by families of clasper surgeries.

EX $\pi_2(\mathrm{Emb}(\mathbb{I}, \mathbb{I}^4))$.   (after Kosanović)



Further evidence
- Kosanović's thesis for $\pi_0 P_n \mathrm{Emb}(\mathbb{I}, \mathbb{I}^3)$.
- Haefliger trefoils (∂o π_0)
- Graph models of Fresse - Turchin - Willwacher (& Tsopméné - Turchin).
- Watanabe's families for disproof of 4-D Smale Conjecture.

Application (Haefliger; Budney): "spinning" π_n Emb $\rightsquigarrow$ π_0 Emb.

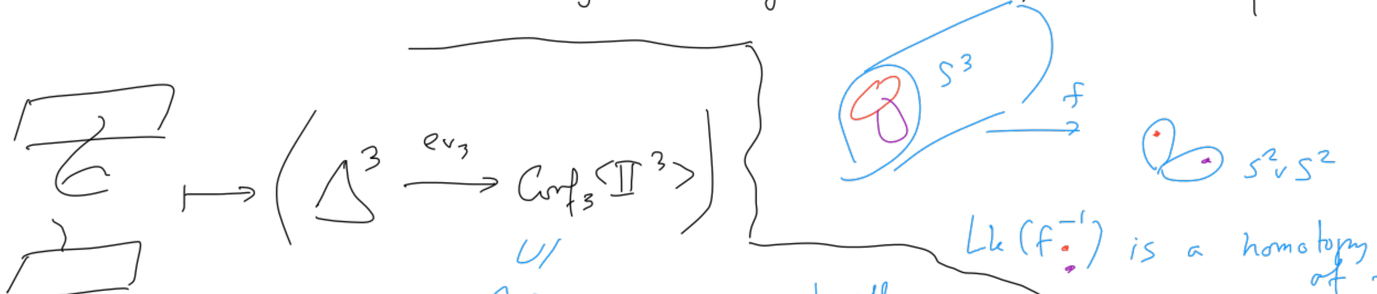
What about invariants?

How do we understand a knot through htpy class of evaluation?

Consider $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^3) \to AM_3 = \mathrm{Maps}(\Delta^3, \mathrm{Conf}_3\langle \mathbb{I}^3\rangle)$ w/ ∂ conditions.

A /R, integrals
But...

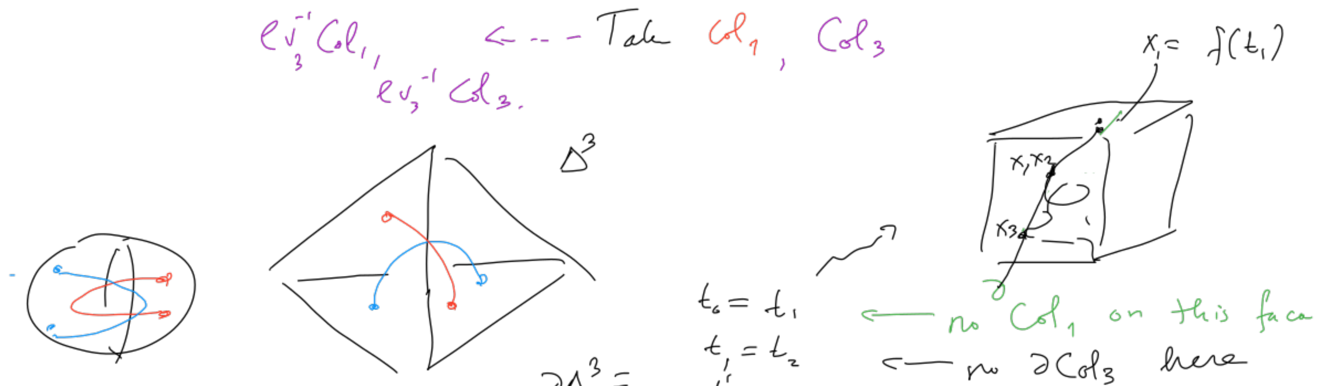
$\pi_* SS$: $\pi_0(AM_3) \cong \mathbb{Z}$. Associated graded is $\pi_3(\mathrm{fib}\; S^2 \vee S^2 \to S^2 \times S^2)$, which is generated by the (universal) Whitehead product.



$\mathrm{Col}_2 := x_1, x_2, x_3$ collinear, x_2 in middle.

Lk(f^{-1}(·)) is a homotopy invt of f

$ev_3^{-1}\mathrm{Col}_1$, $ev_3^{-1}\mathrm{Col}_3$. ← -- Take Col_1, Col_3



$\partial\Delta^3 =$
$t_0 = t_1$ ← no Col_1 on this face
$t_1 = t_2$ ← no $\partial \mathrm{Col}_3$ here
$t_2 = t_3$ ← no $\partial \mathrm{Col}_3$ here
$t_3 = t_4$ ← no $\partial \mathrm{Col}_3$ here.

Inspired by Hopf invariant take $Lk(ev_3^{-1}\mathrm{Col}_1, ev_3^{-1}\mathrm{Col}_3)$

project t_2: x_1, x_2, x_3 collin, 1 in middle
x_1, x_2', x_3 collinear, 3 in middle.

Definition 2.1.
- *A quadrisecant Q on a knot f is a collection of four points on the knot $\{f(t_i)\}_{i=1}^4$ which are collinear.*
- *Define a permutation σ_Q by orienting the line such so that $f(t_2) - f(t_1)$ is positively oriented and setting $\sigma_Q(i) = j$ if $f(t_i)$ is the jth point on the line. The permutations achieved are precisely the permutations such that $\sigma(2) > \sigma(1)$. An alternating quadrisecant has permutation 2431.*
- *Define a sign ε_Q associated to Q as the sign of the determinant which vanishes when a quadrisecant is not generic.*

Theorem 2.2. [BCSS05] *The signed count of alternating quadrisecants, namely $\sum_{Q:\sigma_Q=2413} \varepsilon_Q$ is equal to the coefficient of z^2 in the Conway-normalized Alexander polynomial of f.*

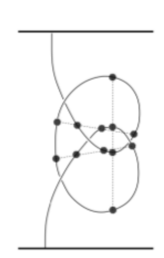
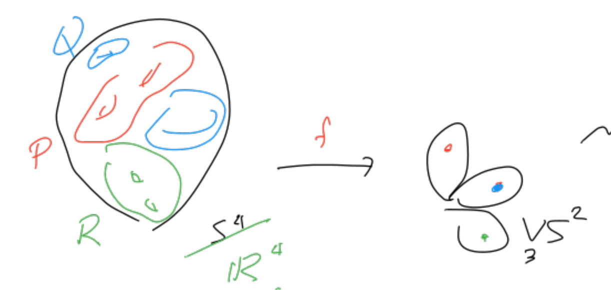


Figure 2.1: Alternating quadrisecants on the figure-eight knot.

Next case requires invariants of $\pi_4(S^2 \vee S^2 \vee S^2)$ $d^{-1}p$ is a choice...



invts.
$d^{-1}P \cap d^{-1}Q \cap R$
$d^{-1}(d^{-1}P \cap Q) \cap R$
$[P \times Q \times R] \to \mathrm{Conf}_3(\mathbb{R}^4)$
(evaluate H^* to get #)

Summary
- generalized linking - w/ correction
- configuration spaces
- Quillen Lie coalgebraic bar construction ← after Sullivan, Hain, Chen.

arxiv: math: 0809.5084 $\mathrm{Hom}(\pi_n(X), \mathbb{Q})$.   {idea: a|b|c, $a, b, c \in C(X)$ ↦ ...

"All rational cohomotopy is given by linking invariants of cochains."
Simply ctd... but starting to "see" free π_1 through cochains similarly (with applications to mapping class groups). ← Johnson filtrations.

?? $\mathrm{Hom}(\pi_1(X), \mathbb{R}/\mathbb{Z})$? Dream

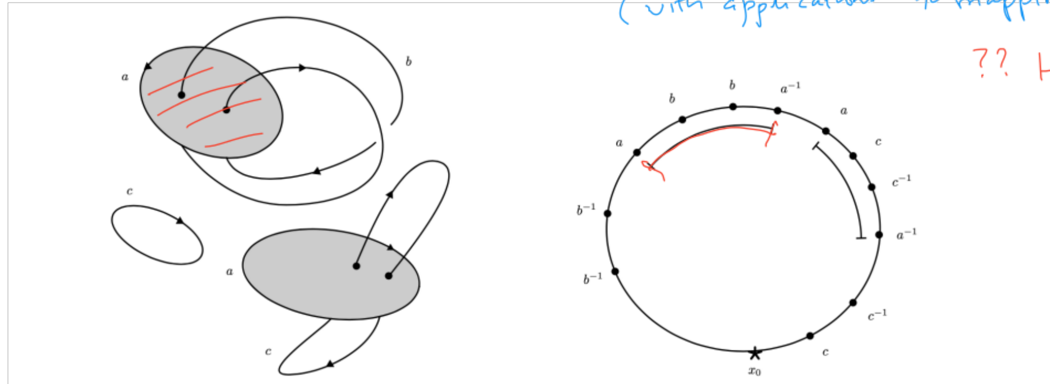


FIGURE 1. Classical linking is on the left; linking of letters on the right.

Thm (BCKS arxiv: 1411.1832)

$\pi_0\,\mathrm{Emb}(\mathbb{I}, \mathbb{I}^3) \to \pi_0\, AM_n$ is a map of abelian monoids. target abelian gp.

Resulting invt. is type $n-1$.

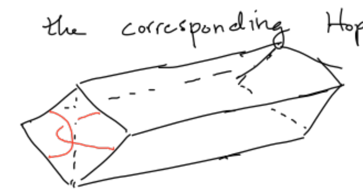
Conjecture Universal!   (now more cases...)

Spectral sequence in degrees 0,1 is SS of abelian gps.

Calculations ⇒ (multi-relative...) Hopf invariants can detect $\mathrm{Hom}(\pi_0 AM_n, \mathbb{Z})$

Conjecture: Hopf invts ~~One approach~~ result in Goussarov-Polyak-Viro arrow diagram counts.

Back to quadrisecant result, the corresponding Hopf invt is denoted $\mathrm{Col}_1 \mid \mathrm{Col}_3$

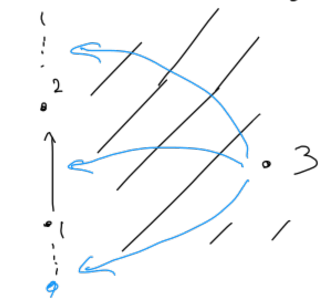


Switch to long tangles, standard at infinity

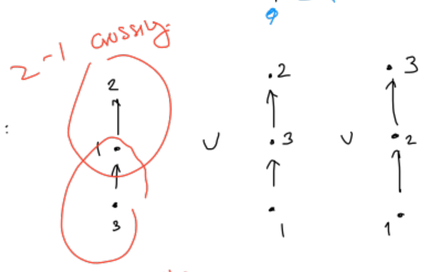
Evaluation map $[-\infty, \infty]^3 \to \mathrm{Conf}_3(\mathbb{R}^3)$. $\mathbb{R}^3 =$ z, y

Consider subspace w/ ∂ of (x_1, x_2, x_3) | • $\frac{x_2 - x_1}{\|\;\|} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ • $x_3 \in$ half-plane to their right.
W

View from below



Boundary:



parameter space $[0,1]^3$

So we consider "linking" of ev_3^{-1} and ev^{-1}
Hopf invt.
1-over-3 crossing   2-over-1 crossing

But these can intersect...
Same for 1 → 3 and 3 → 2
And 1 → 2 and 2 → 3

Thm (Polyak; Gadish - S-)

μ_{123} = 1-2 crossing after 2-1 + 2-3 after 1-3 + 3-2 after 2+1
+ 2-1 crossings 4 stuff 3 to right.

also ∃ quadrisecant formula.

Conjecture - all Milnor invts. (Vassiliev) for tangles through ev_n.

⇝ main conjecture about knots
universal invt. + geometry (Hopf invts).