2021-01-20, Building bridges seminar

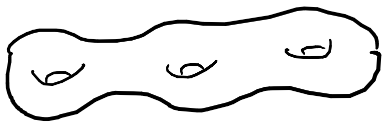
Unknotting 2-spheres in S^4 with Finger- & Whitney moves

with Jason Joseph,
Michael Klug & Hannah Schwartz

everything [the manifolds, embeddings, ...] is smooth here

Knotted 2-spheres: $S^2 \hookrightarrow S^4$

smooth embedding

Knotted (orientable) surfaces:  $= \Sigma_g \hookrightarrow S^4$

up to smooth ambient isotopy

There is a difference between $\dfrac{\text{topologically locally flat embedded surfaces}}{\text{topological isotopy}}$

and $\dfrac{\text{smoothly embedded surfaces}}{\text{smooth isotopy}}$

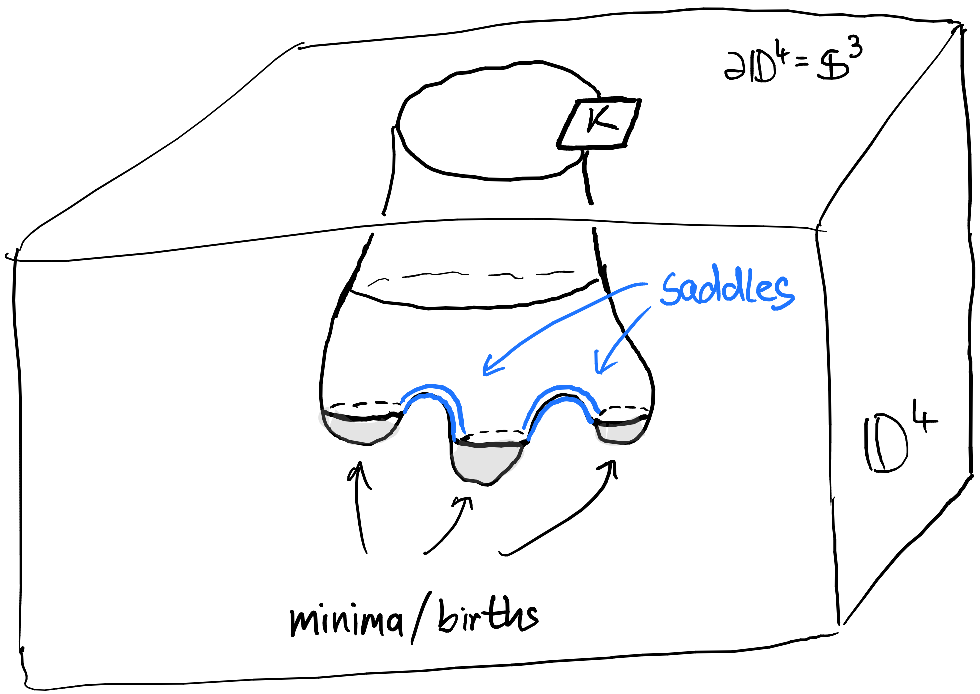
"exotic knotting"

Classical ribbon knots in S^3

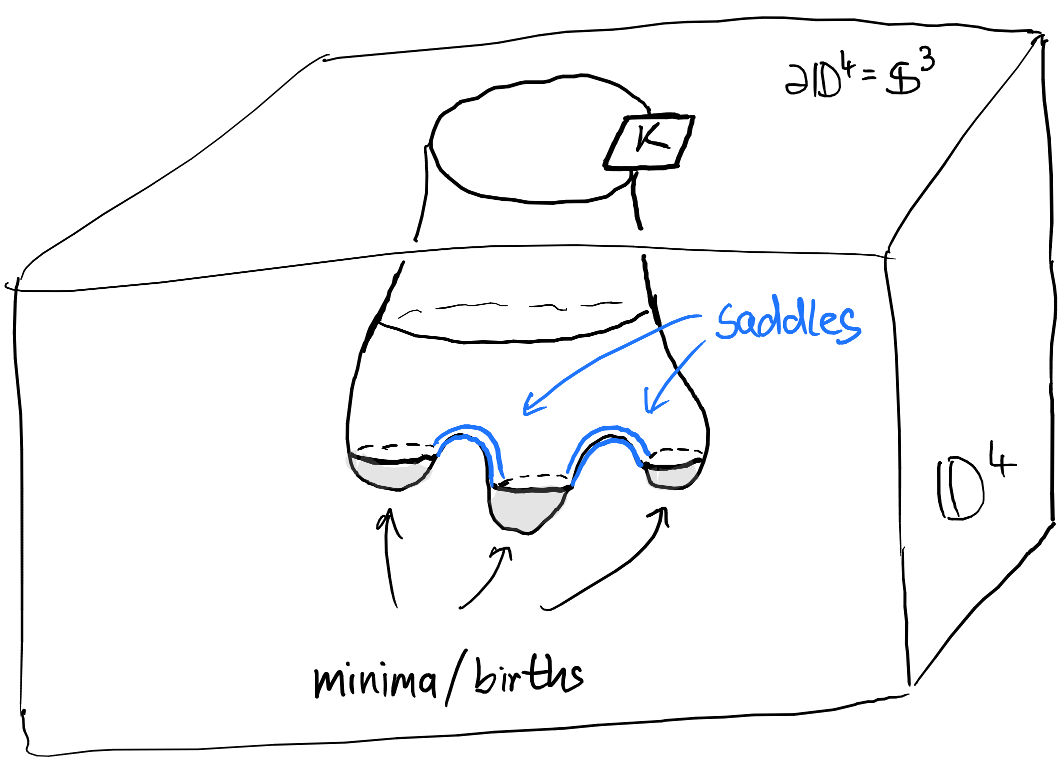
Start with unlink

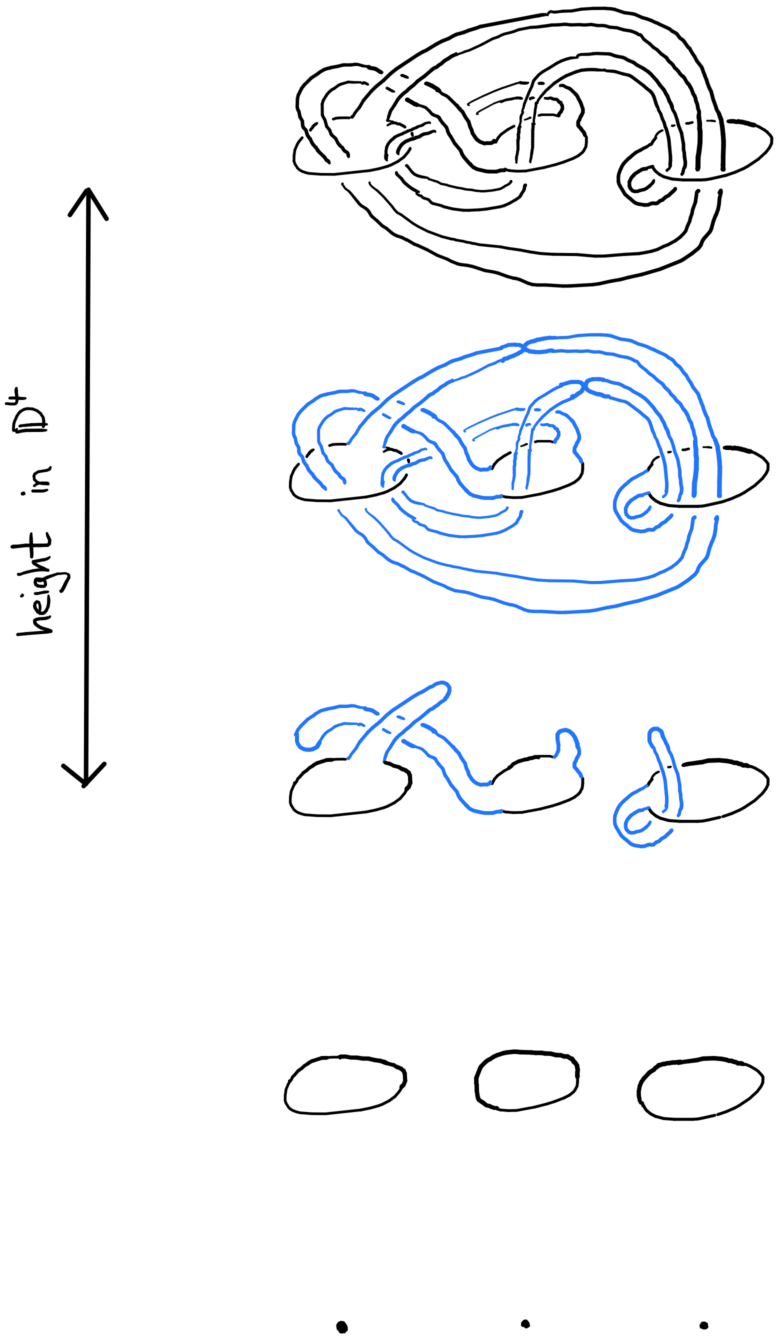
Join components with fusion bands

Ribbon disk:



Describing knotted surfaces via movies



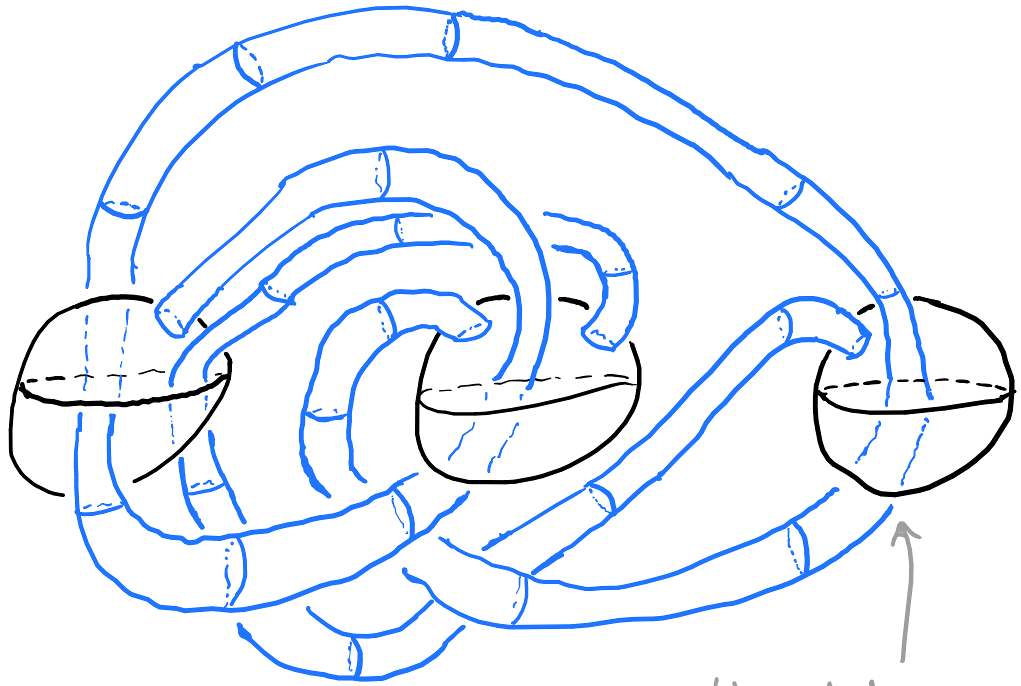


Ribbon 2-knots in S^4

↗ Satoh's tube map

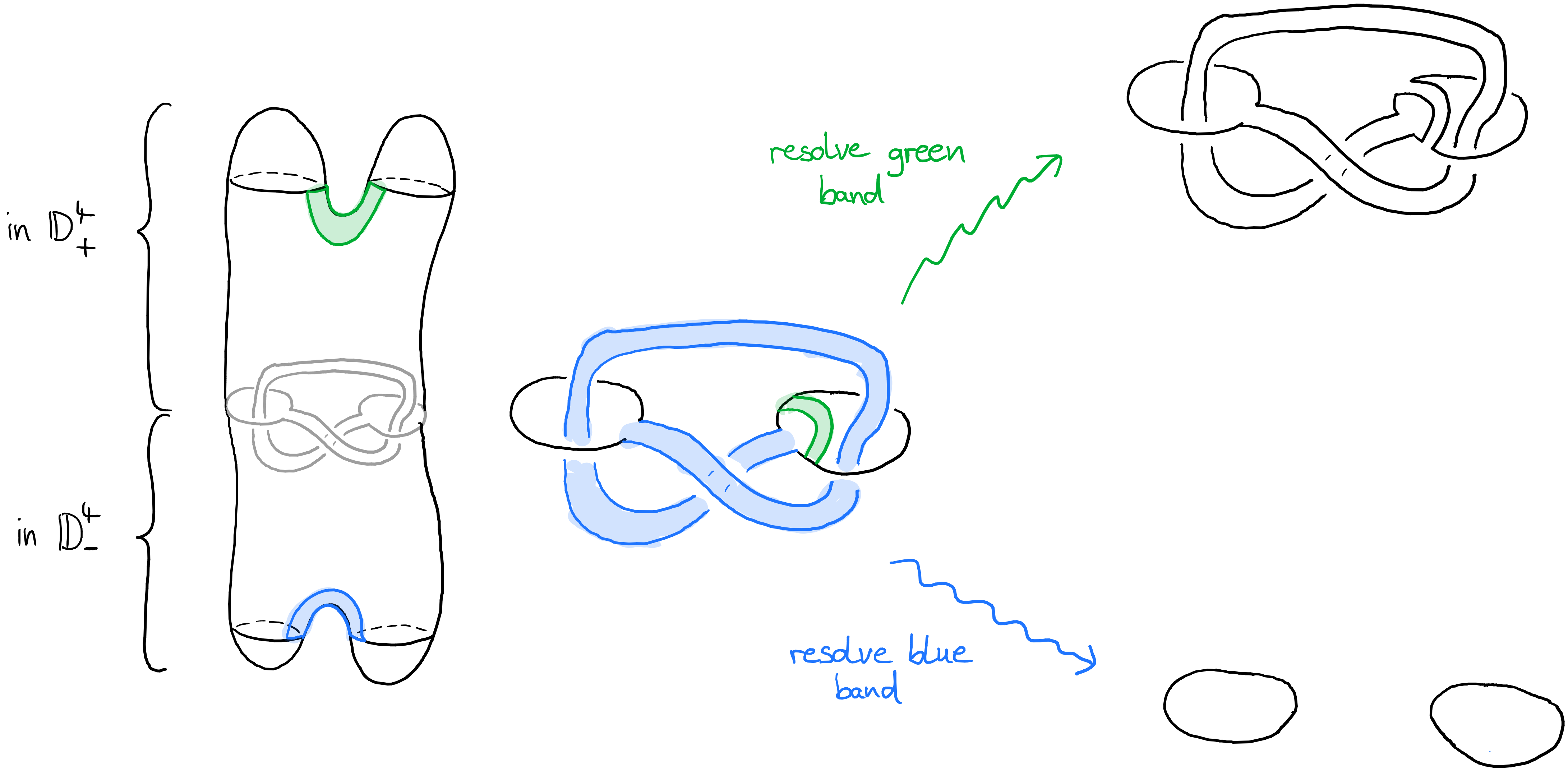
Start with an unlink of 2-spheres

Attach fusion tubes



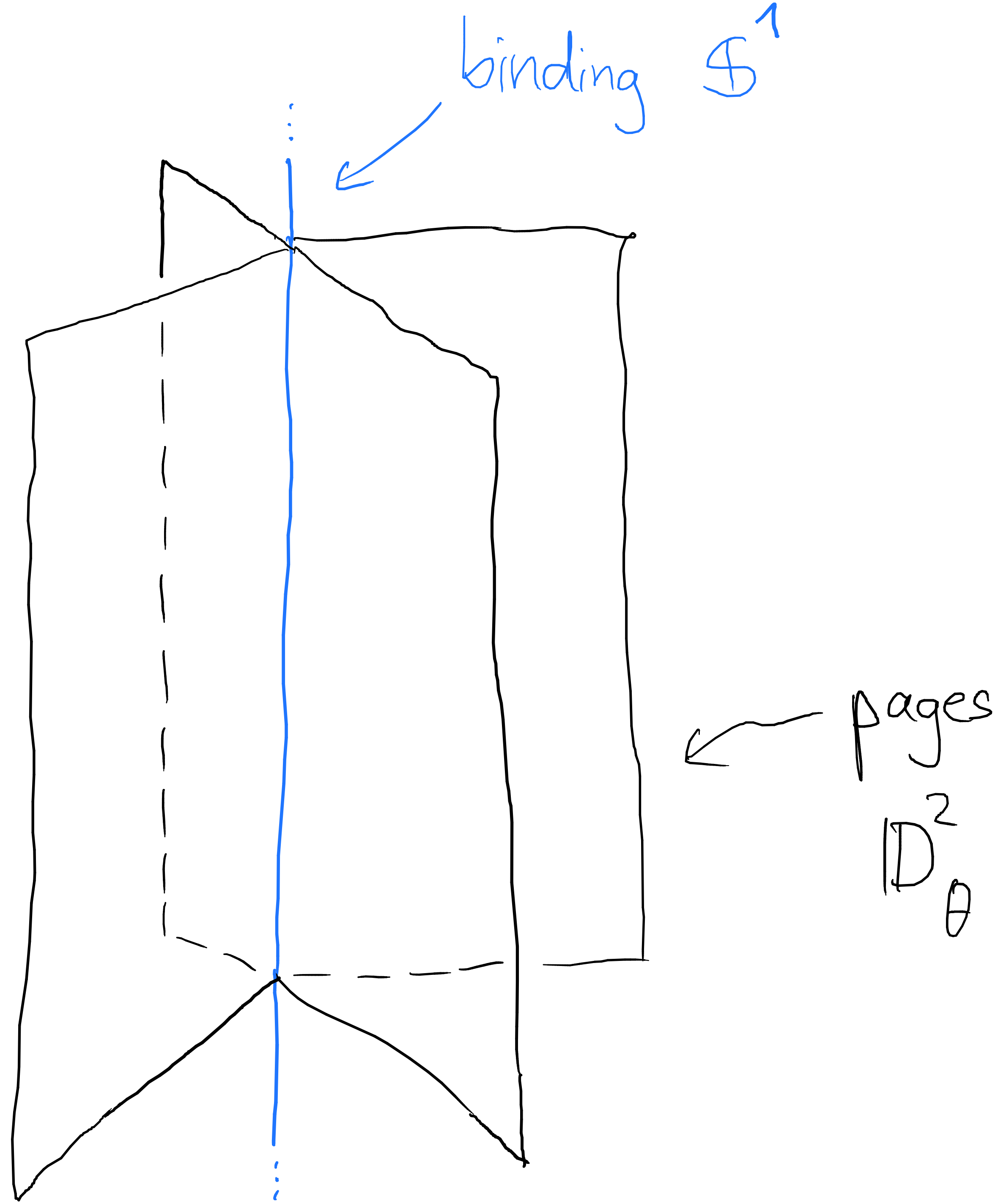
the tubes usually link with the spheres

Banded unlink diagrams



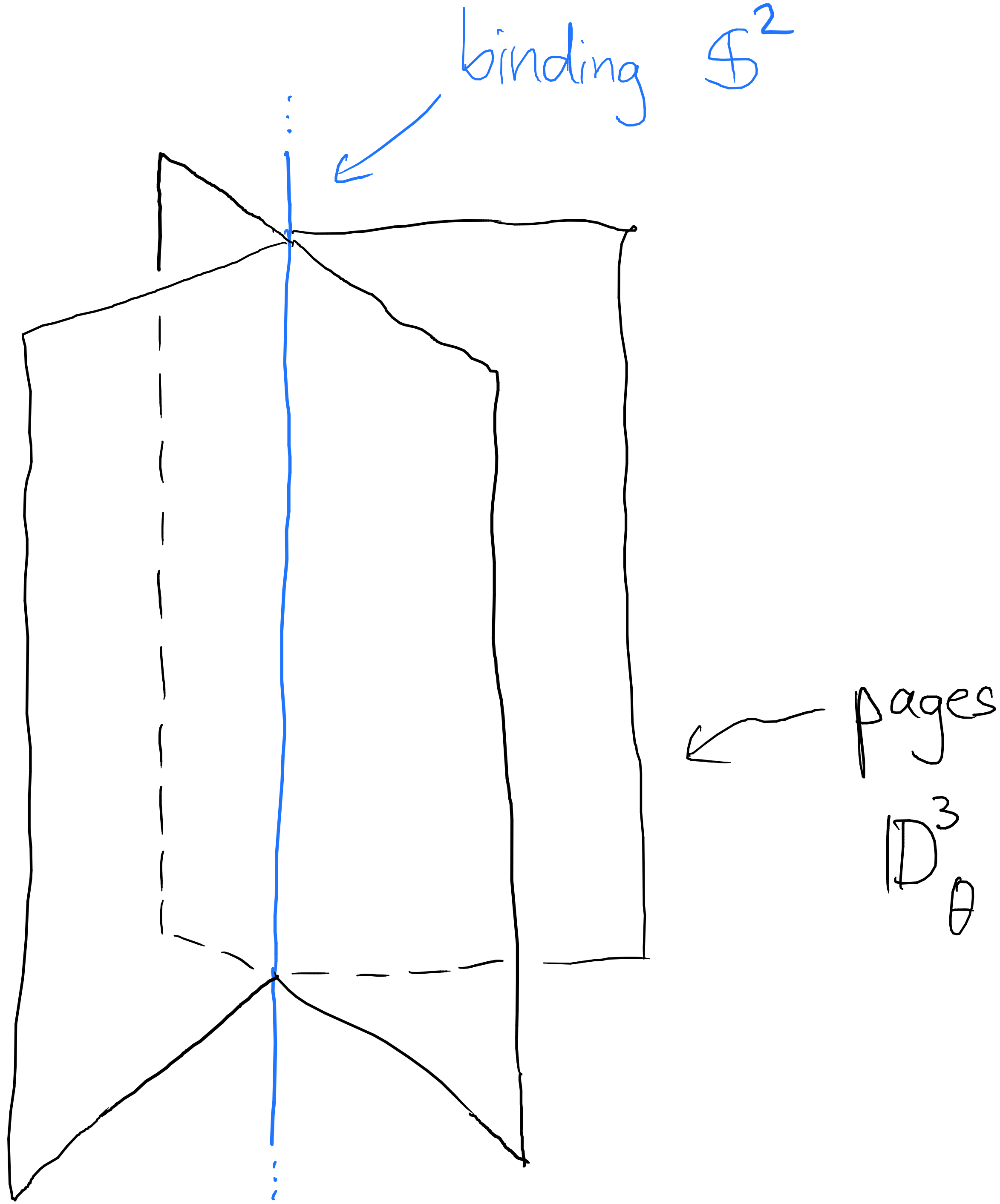
Spinning

open book decomposition of $\mathbb{S}^3$ =



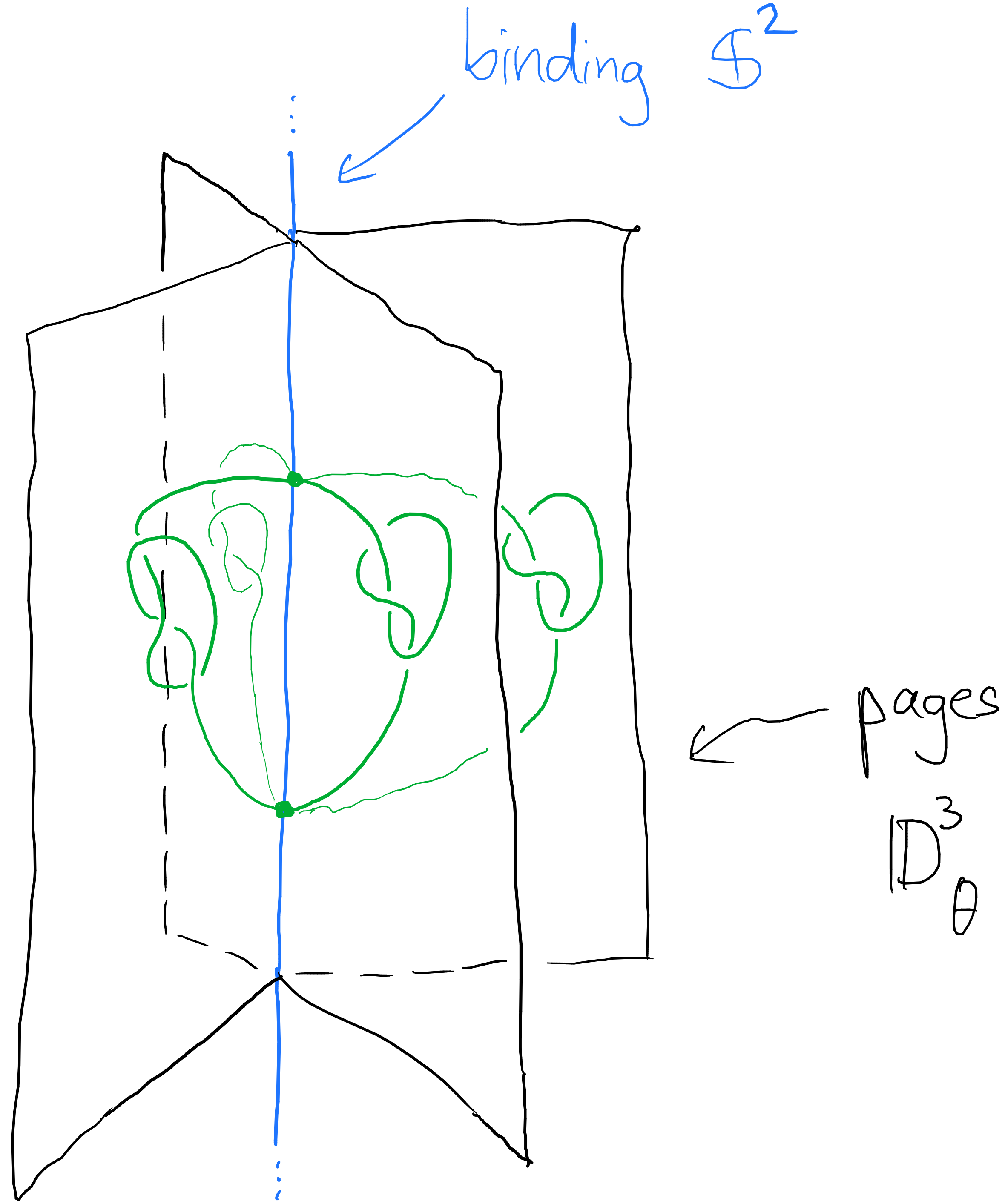
Spinning

open book decomposition of $\mathbb{S}^4$ =



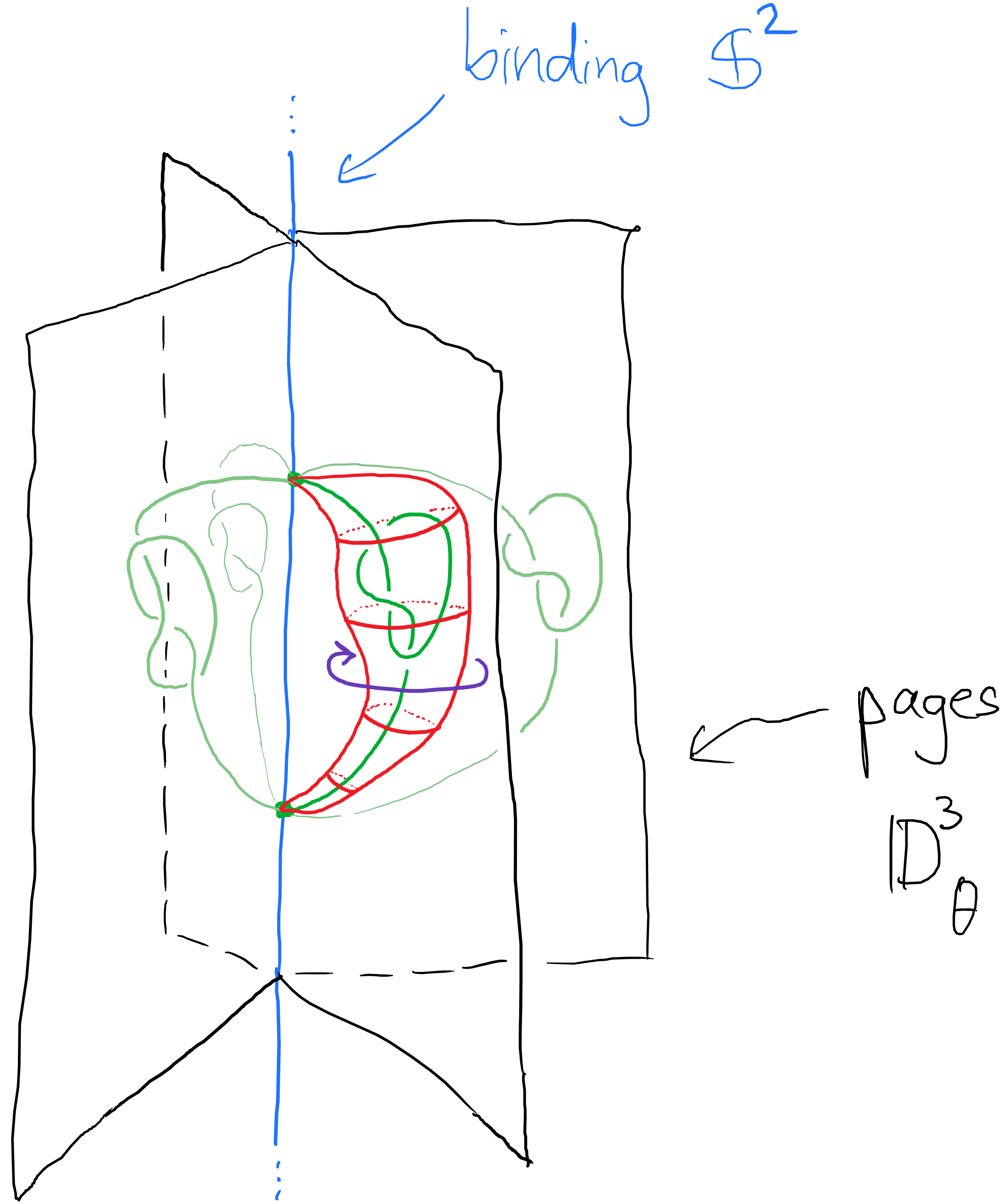
Spinning

open book decomposition of S^4 =



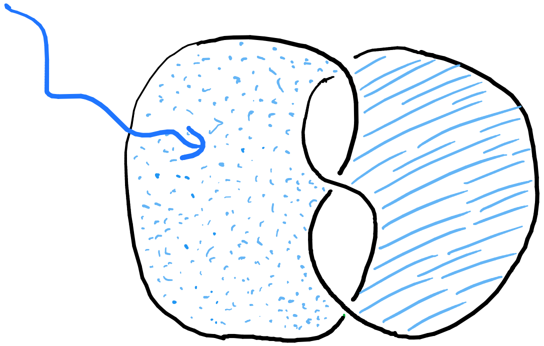
Twist-Spinning

open book decomposition of S^4 =

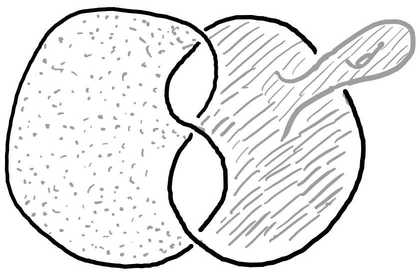


$\boxed{\begin{array}{l}\underline{\text{Idea:}}\ \text{Study codimension} = 2 \text{ knots} \\ \qquad\quad \text{via submanifolds that they bound}\end{array}}$

Just as knots $S^1 \hookrightarrow S^3$ bound <u>Seifert surfaces</u> ...

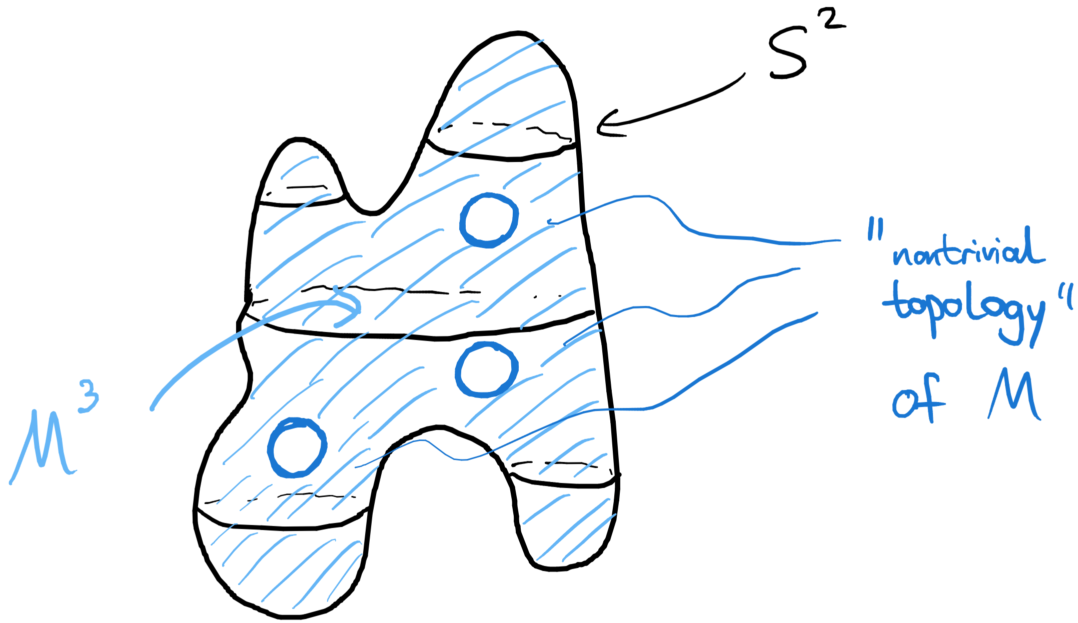


(not unique $\rightsquigarrow$ S-equivalence)

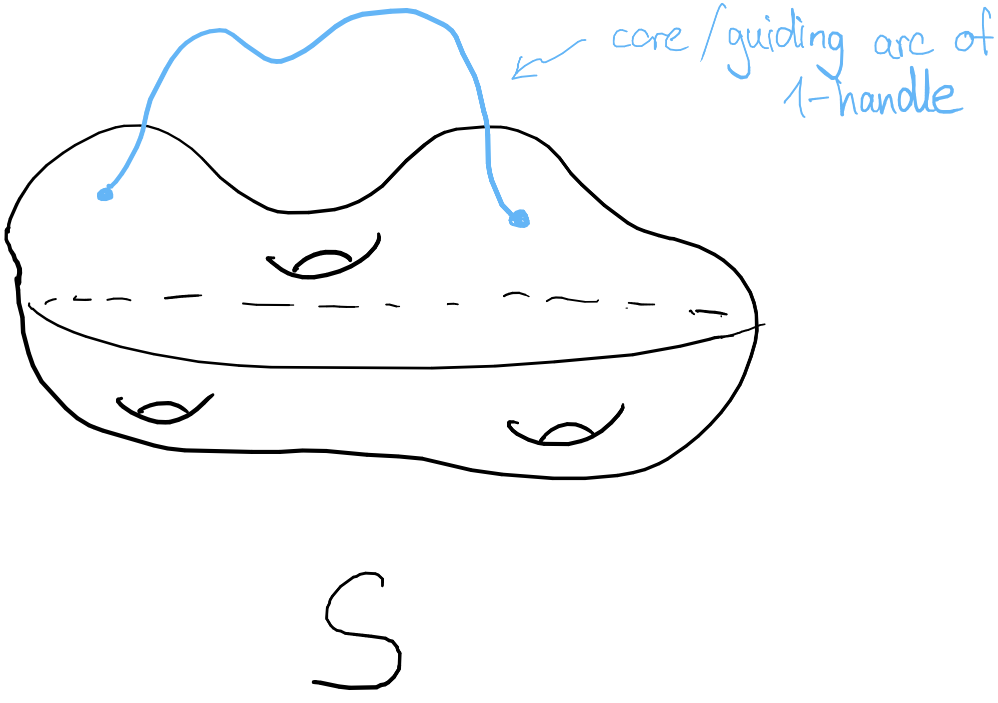


... knotted surfaces $\Sigma_g \overset{S}{\hookrightarrow} S^4$ bound <u>Seifert hypersurfaces</u> / <u>Seifert solids</u>

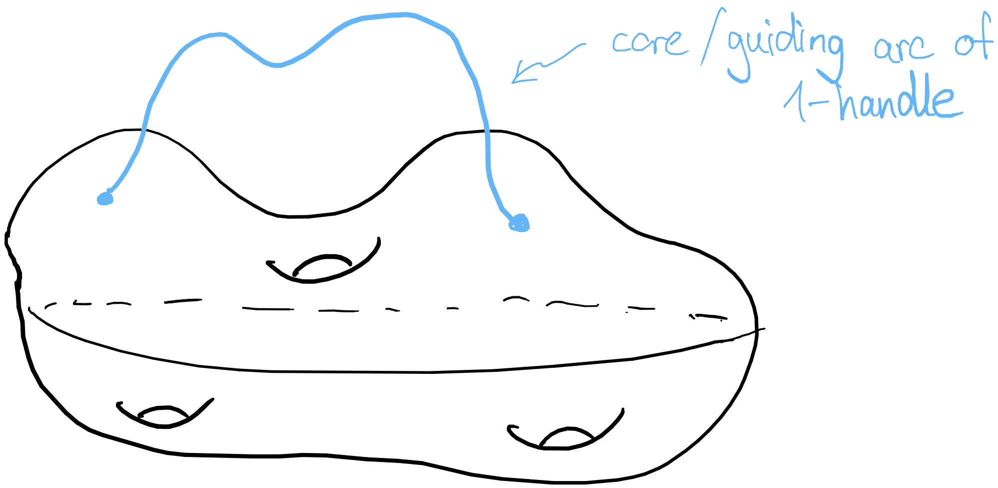
$$\boxed{\begin{array}{c}\text{oriented, smooth compact 3-mflds} \\ M^3 \hookrightarrow S^4 \text{ with } \partial M = S.\end{array}}$$



1-handle stabilization of a surface



1-handle stabilization of a surface

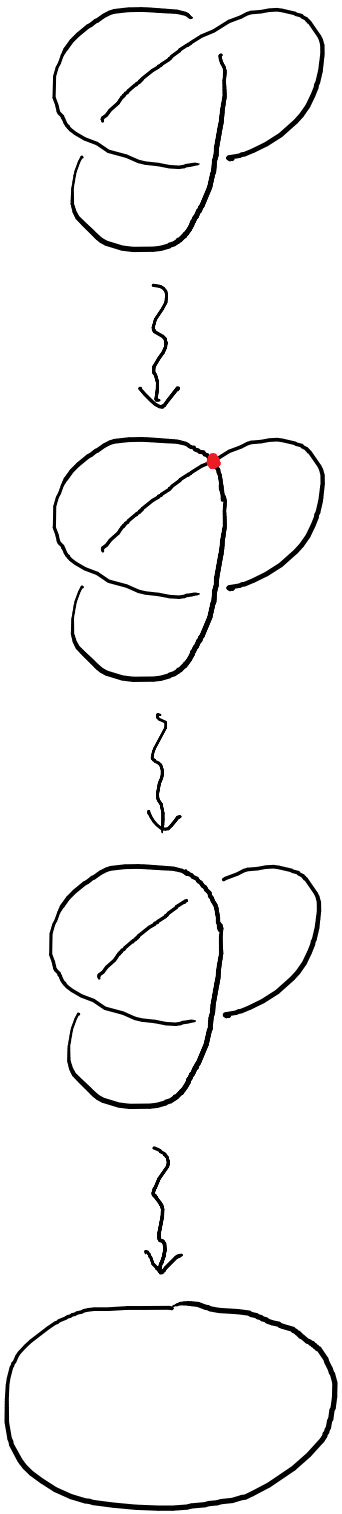


$S \rightsquigarrow S + h^1$

Fact: Any surface $S \subset S^4$ can be unknotted with enough 1-handle stabilizations.

A surface $S : \Sigma_g \hookrightarrow S^4$ is unknotted if it bounds a handle body

Idea: Study knots via regular homotopies to the unknot



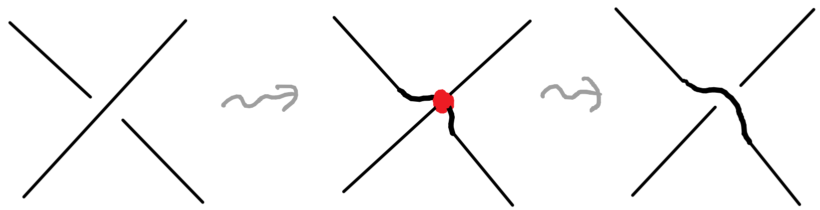
knot K in S^3

<u>homotopic</u> to unknot $\bigcirc$

$\pi_1(S^3) = \{1\}$

(of course if K non-trivial, not isotopic to unknot)

sequence of isotopies and crossing changes:



Unknotting by Finger- & Whitney moves:

Similarly, any 2-knot $S^2 \overset{S}{\hookrightarrow} S^4$ is (regularly) homotopic to unknot

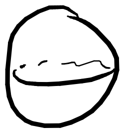
$\pi_2(S^4) = \{0\}$

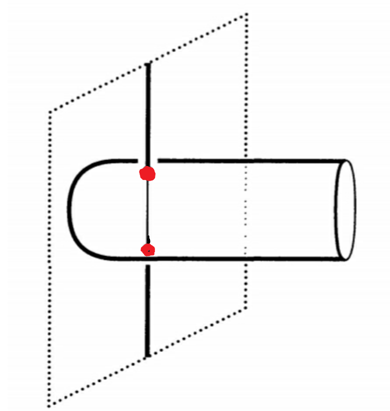
2-knot S

Finger moves

immersed middle stage

Whitney moves

unknot  $\subset S^4$



[from Scorpan: The wild world of 4-manifolds]

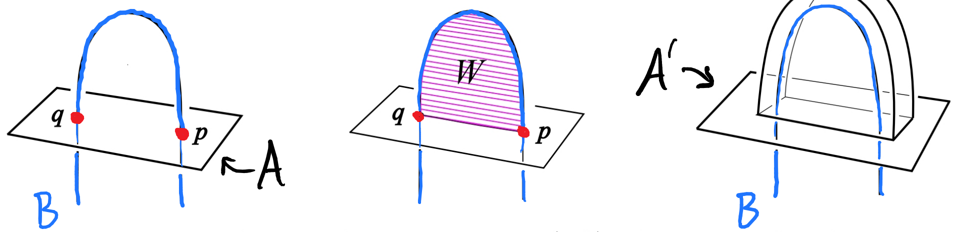
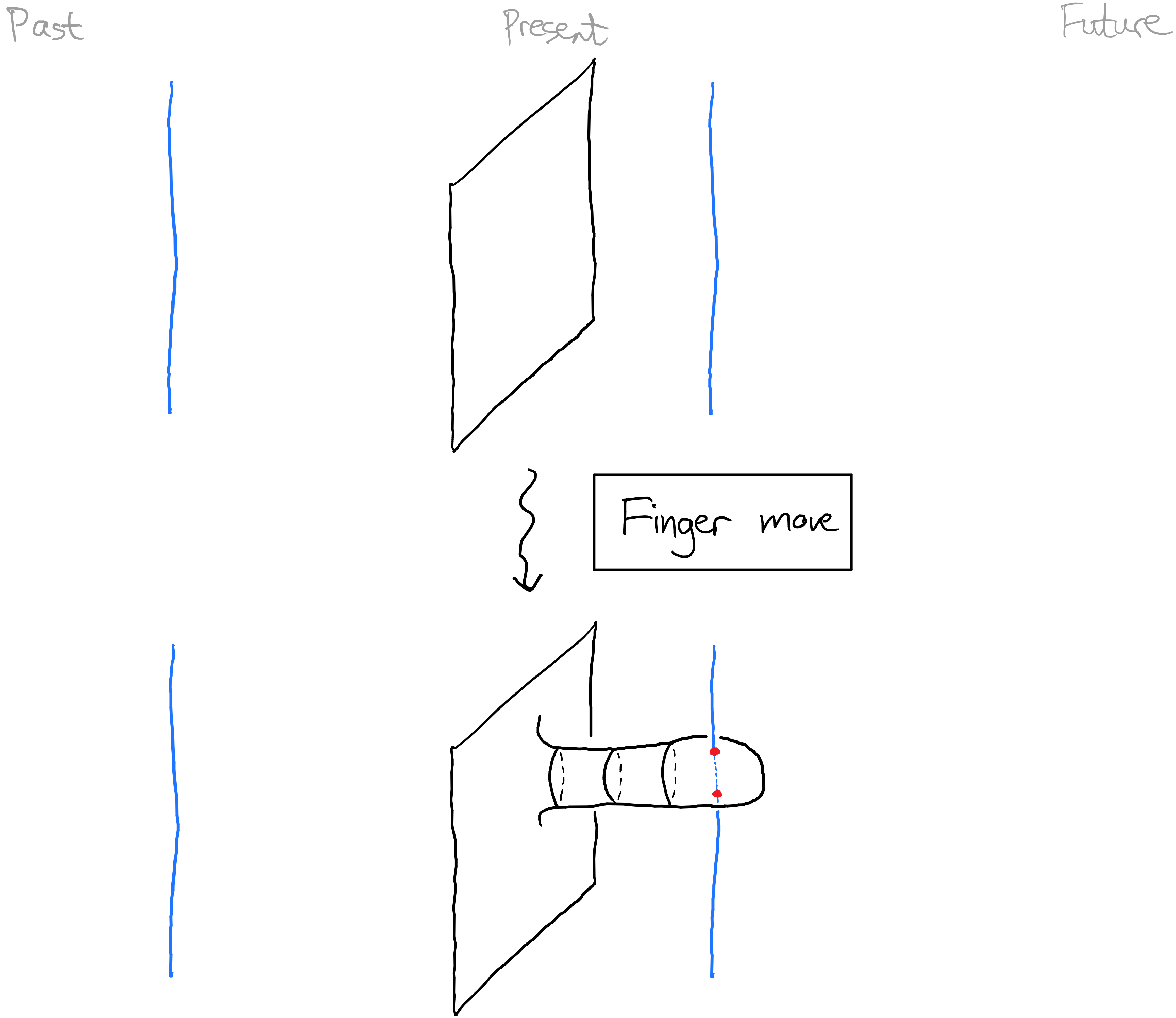
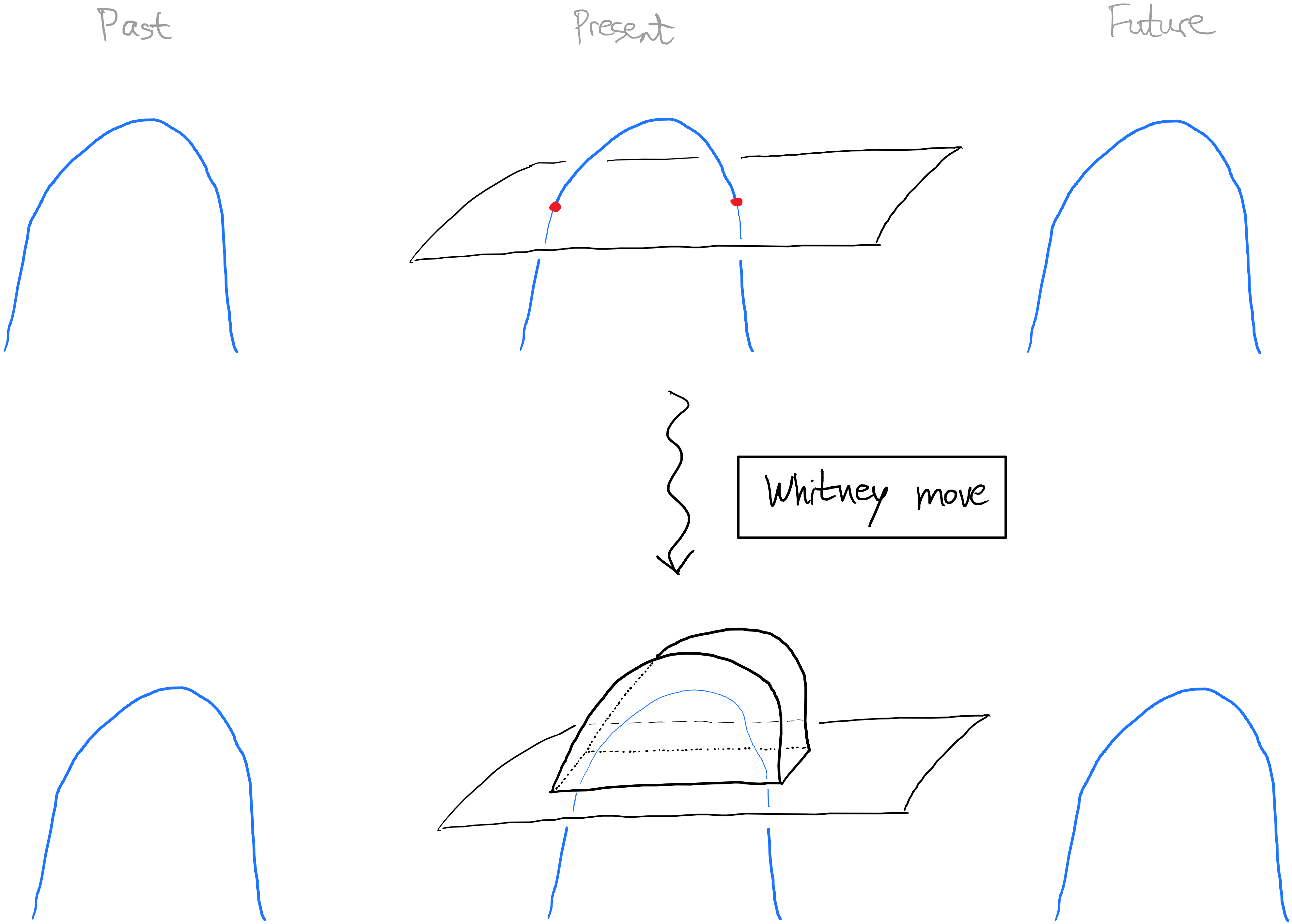


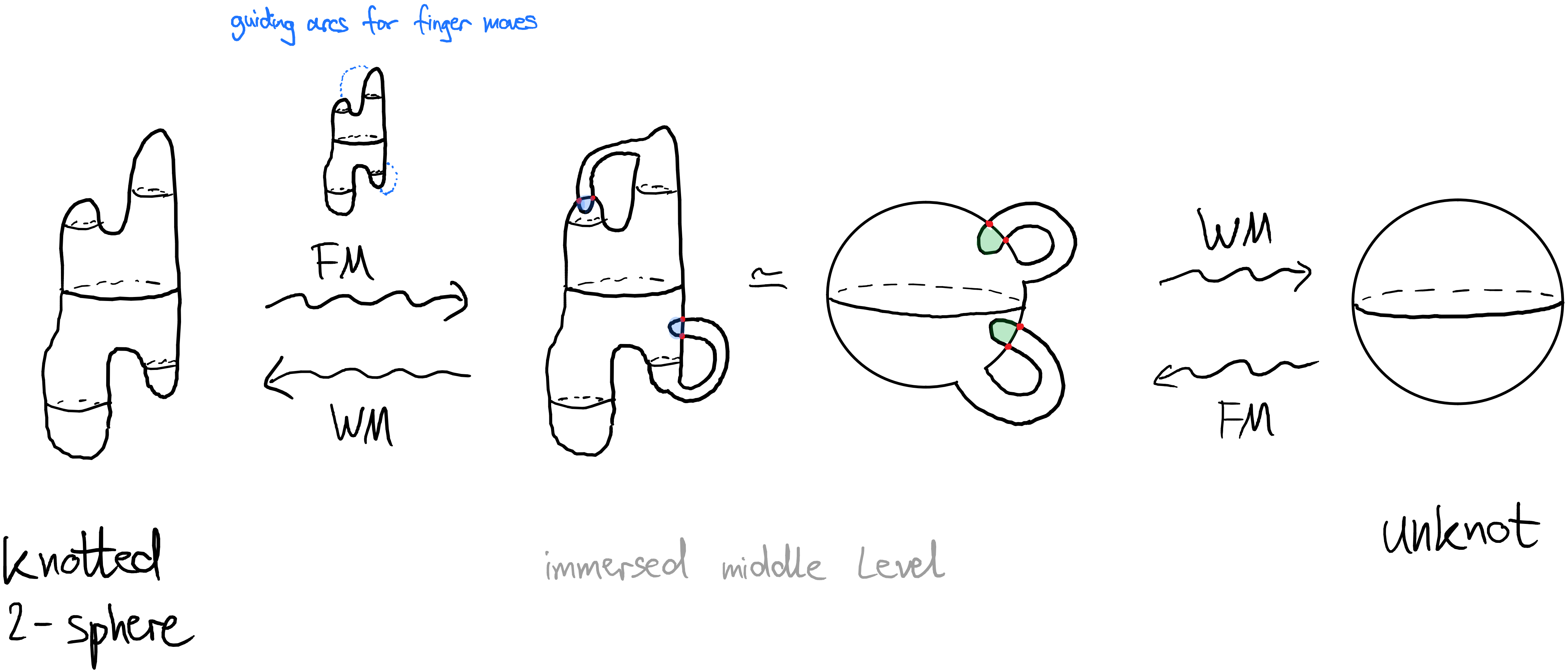
FIGURE 2.3. The pair of intersections p, q (left) admits a purple Whitney disk W (center) which guides a Whitney move eliminating p, q by adding a *Whitney bubble* to the horizontal sheet (right).

[picture borrowed from Schneiderman-Teichner]

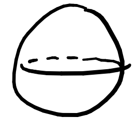

Past
Present
Future
Finger move


Past
Present
Future
Whitney move

Schematic of a regular homotopy



$\pi_2(S^4) = \{0\} \rightsquigarrow$ any knotted 2-sphere $K: S^2 \hookrightarrow S^4$

is (regularly) homotopic to the unknot 

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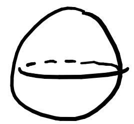
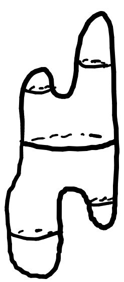
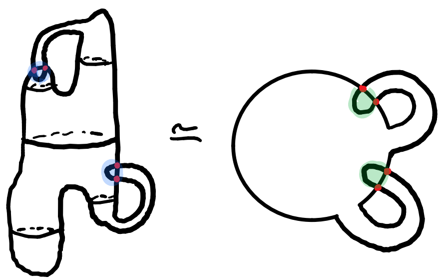
is (regularly) homotopic to the unknot 

$\boxed{\text{We define the } \underline{\text{Cassan-Whitney number}} \quad \underline{u_{CW}(K)} \quad \text{as the minimal number of Finger moves in a regular homotopy from } K \text{ to the unknot}}$



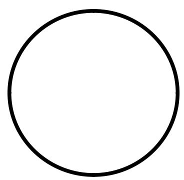
knotted 2-sphere

$\xrightarrow{\text{two finger moves}}$



immersed middle level

$\xrightarrow{\text{two Whitney moves}}$



unknot

If this is the shortest seq.

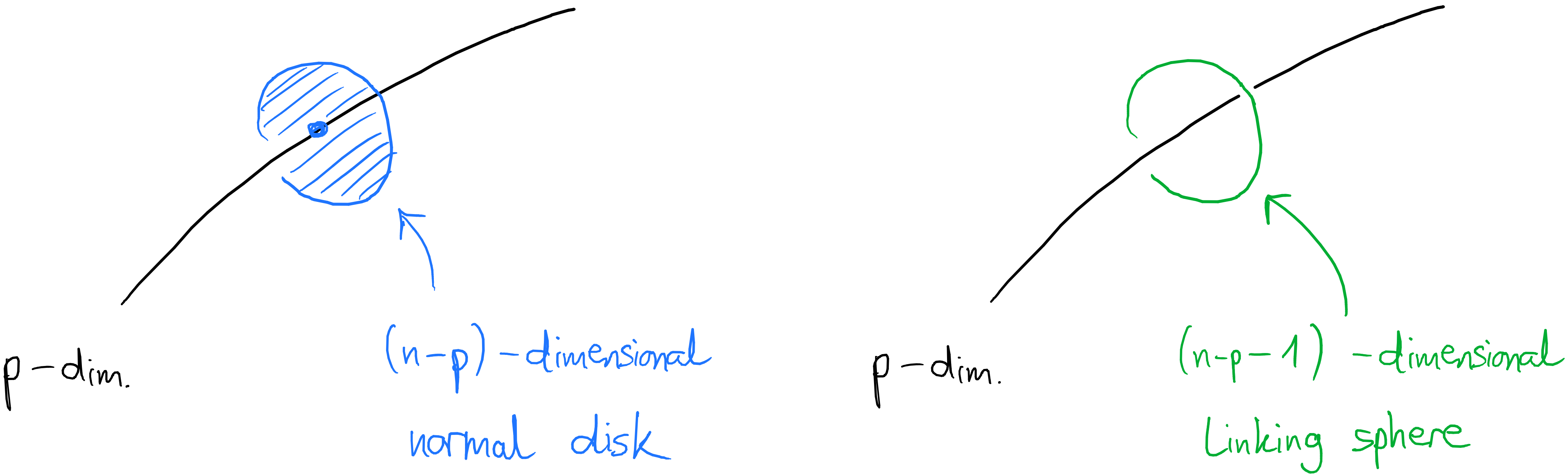
then <u>here</u>: $u_{CW}(K) = 2$

Idea: Study knotted surfaces $S \subset S^4$

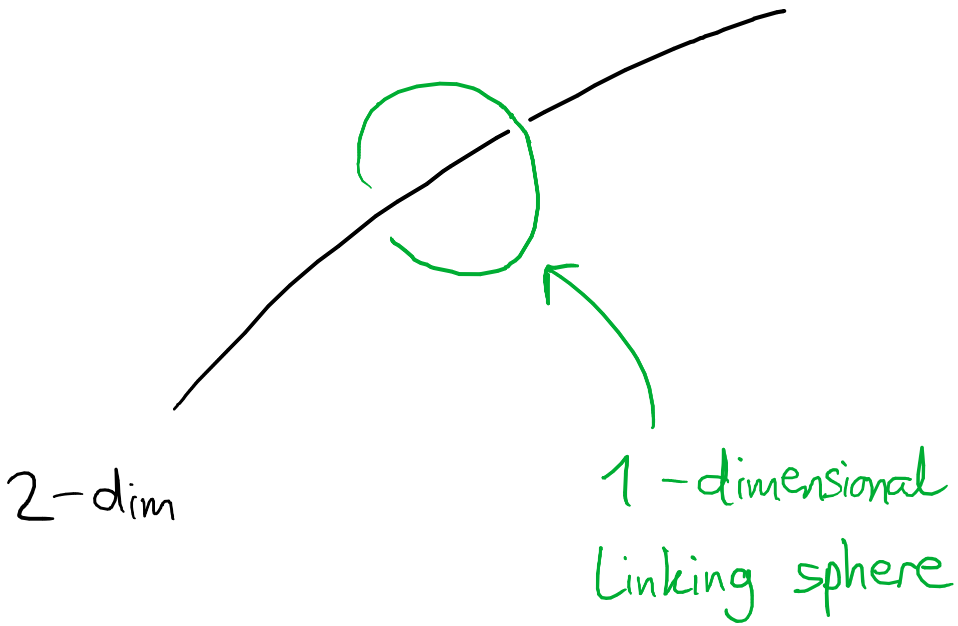
via the fundamental group of their complement

$$\pi_1(S^4 - S, *)$$

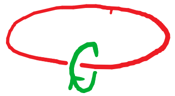
ambient space $\mathbb{R}^n$



If ambient dimension is 4-dimensional:



unknot

$$\pi_1\Big(\, S^3 \setminus$$  $$\Big) \cong \mathbb{Z}$$

generated by a meridian

Corollary of Dehn's lemma:

$\pi_1(S^3 \setminus K) \cong \mathbb{Z}$

$\Rightarrow K$ is unknotted

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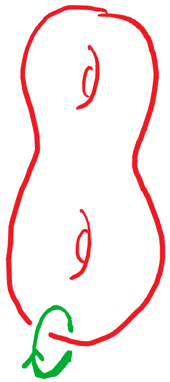
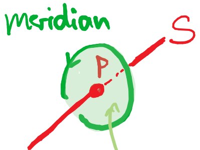
unknotted surface S

$$\pi_1\Big(\, S^4 \setminus$$  $$\Big) \cong \mathbb{Z}$$

meridian: boundary of a normal 2-disk of S at point p



fiber of the normal disk bundle

BIG open question:

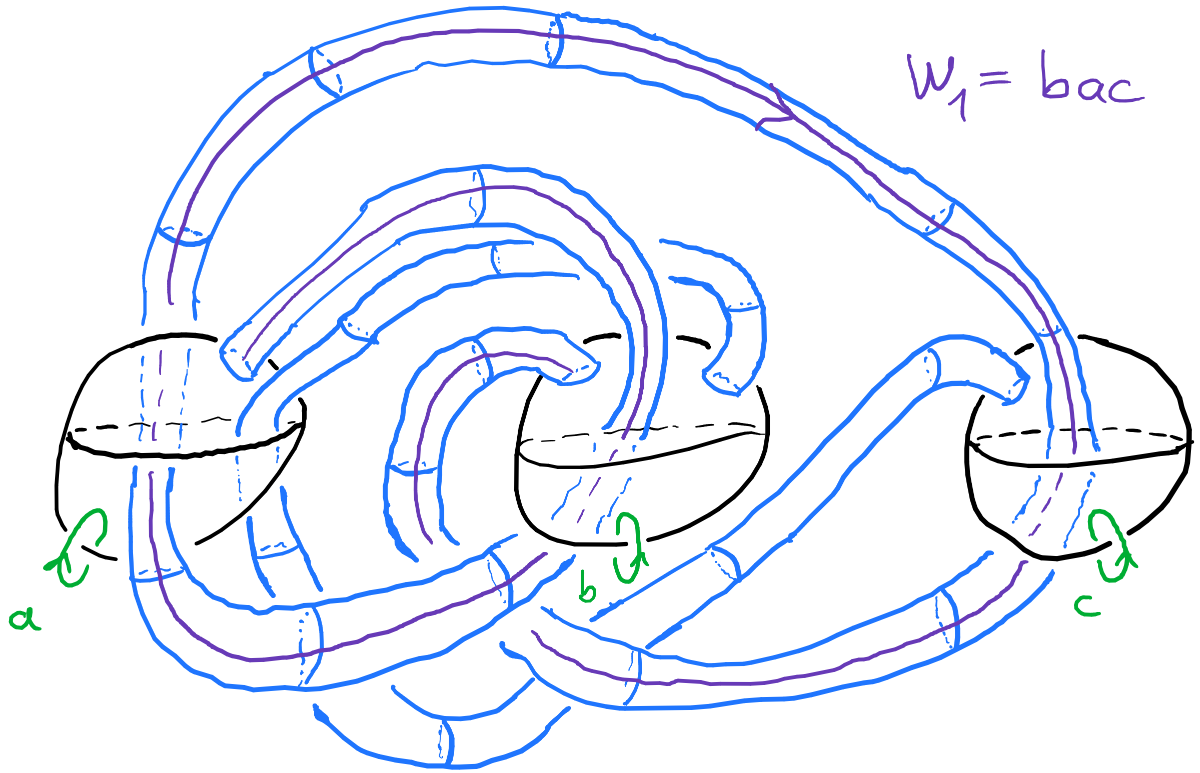
Does π_1 characterize smoothly unknotted surfaces in 4-dim. space?

Algebraic effect of stabilization:

$$\boxed{\pi_1\left(S^4 - (S + h^1)\right) \cong \pi_1(S^4 - S) \big/ \langle\langle w^{-1} a w = b \rangle\rangle}$$

So a stabilization can make two meridians <u>equal</u>

Example: $\pi_1(S^4 - \text{ribbon 2-knot})$



$$\langle a, b, c \mid b = w_1^{-1} a w_1 , \quad c = w_2^{-1} b w_2 \rangle$$

$$\Leftrightarrow b = (bac)^{-1} a (bac) \qquad \Leftrightarrow c = a^{-1} b a$$

Algebraic effect of finger move:

$$\pi_1(S^4 - S^{\text{fing.}}) \cong \pi_1(S^4 - S) \Big/ \langle\langle [w^{-1} a w, b] \rangle\rangle$$

Immersion after finger move on S

A finger move can make a pair of meridians <u>commute</u>

Algebraic versions of the unknotting #s:

Finger move: $\pi_1(S^4 - S^{\text{fing}}) \cong \pi_1(S^4 - S) / \langle\langle [w^{-1}aw, a] \rangle\rangle$

Stabilization: $\pi_1(S^4 - S^{\text{stab}}) \cong \pi_1(S^4 - S) / \langle\langle w^{-1}aw = a \rangle\rangle$

$a_{CW}(K)$:= min. # of Finger move relations $[w_i^{-1}\underbrace{a}_{\text{meridians}}w_i, \underbrace{a_i}_{\text{meridians}}]$

such that $\pi_1(S^4 - K) / \langle\langle [w_1^{-1}a_1w_1, a_1], [w_2^{-1}a_2w_2, a_2], \dots, [w_k^{-1}a_kw_k, a_k] \rangle\rangle$

is abelian $\;(\Leftrightarrow \cong \mathbb{Z})$

$a_{stab}(K)$:= min. # of 1-handle relations $a_i = w_i^{-1} \cdot a_i \cdot w_i$

such that $\pi_1(S^4 - K) / \langle\langle a_1 = w_1^{-1}a_1w_1, a_2 = w_2^{-1}a_2w_2, \dots, a_k = w_k^{-1}a_kw_k \rangle\rangle$

is abelian

Some bounds:

this is the best lower bound for the Casson-Whitney number we know of

$$
\begin{array}{ccc}
a_{cw}(K) & \leq & u_{cw}(K) \\
\rotatebox{90}{$\leq$} & & \\
a_{stab}(K) & \leq & u_{stab}(K) \\
\rotatebox{90}{$\leq$} & & \\
\boxed{\text{minimal size of generating set of Alexander module of } K \text{ (Nakanishi index)}} & &
\end{array}
$$

Some bounds:

$$
\begin{array}{ccc}
a_{cw}(K) & \leq & u_{cw}(K) \\
\geq & & ?? \\
a_{stab}(K) & \leq & u_{stab}(K)
\end{array}
$$

Oliver Singh's paper was very inspirational

DISTANCES BETWEEN SURFACES IN 4-MANIFOLDS

OLIVER SINGH

Abstract. If Σ and Σ' are homotopic embedded surfaces in a 4-manifold then they may be related by a regular homotopy (at the expense of introducing double points) or by a sequence of stabilisations and destabilisations (at the expense of adding genus). This naturally gives rise to two integer-valued notions of distance between the embeddings: the singularity distance $d_{sing}(\Sigma, \Sigma')$ and the stabilisation distance $d_{st}(\Sigma, \Sigma')$. Using techniques similar to those used by Gabai in his proof of the 4-dimensional light-bulb theorem, we prove that $d_{st}(\Sigma, \Sigma') \leq d_{sing}(\Sigma, \Sigma') + 1$.

1. Introduction

Let X be a smooth, compact, orientable 4-manifold, possibly with boundary. Let Σ, Σ' be connected, oriented, compact, smooth, properly embedded surfaces in X. We say that Σ' is a *stabilisation* of Σ if there is an embedded solid tube $D^1 \times D^2 \subset X$ such that $\Sigma \cap (D^1 \times D^2) = \{0, 1\} \times D^2$, and Σ' is obtained from Σ by removing these two discs and replacing them with $D^1 \times S^1$, as in Figure 1, and then smoothing corners. In this situation we say that Σ is a *destabilisation* of Σ'.

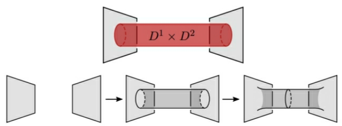


Figure 1. A stabilisation. Given $D^1 \times D^2 \subset X$ which intersects Σ on $S^0 \times D^2$, we remove the two discs $S^0 \times D^2$, add the tube $D^1 \times S^1$, then smooth corners.

Definition 1.1. Given Σ, Σ' as above, both of genus g, define the *stabilisation distance* between Σ and Σ' to be

$$d_{st}(\Sigma, \Sigma') = \min_{\mathbb{S}} |\max\{g(P_1), \ldots, g(P_k)\} - g|,$$

where $\mathbb{S}$ is the set of sequences $P_1, \ldots, P_k$ of connected, oriented, embedded surfaces where $\Sigma = P_1$, $\Sigma' = P_k$ and P_{i+1} differs from P_i by one of, i) stabilisation, ii) destabilisation, or iii) ambient isotopy. If no such sequence exists we declare $d_{st}(\Sigma, \Sigma') = \infty$.

1

By carefully manipulating the regular homotopies to the unknot, we can show

$$\boxed{u_{stab}(K) \leq u_{cw}(K) + 1}$$

⌜the smooth unknotting conjecture would imply that the +1 is not necessary⌟

and

$$\boxed{u_{cw}(K) = 1 \Rightarrow u_{stab}(K) = 1}$$

Have examples with $\underbrace{u_{stab}(K)}_{=\,1} \neq \underbrace{u_{cw}(K)}_{=\,2}$

Used $a_{cw}(K)$ to find the lower bound by showing that one finger move relation is not enough to abelianize the group:

positive generator of the evaluation of the Alexander ideal at $t=-1$

<u>Thm.:</u> For K_1, K_2 2-knots with determinants $\Delta(K_i)\big|_{-1} \neq 1$ have

$$u_{cw}(K_1 \# K_2) \geq 2$$

$$\boxed{\underline{\text{Prop.}}: \quad u_{cw}(\tau^n k) \leq u(k)}$$

n-twist spin of $k: S^1 \hookrightarrow S^3$

classical unknotting number of the 1-knot k

<u>Corollary:</u> $a_{cw}(\tau^n k)$ is a lower bound for the classical unknotting number.

Pf. sketch that $u_{cw}(K_1 \# K_2) \geq 2$: K_1, K_2 2-knots with determinants $\Delta(K_i)|_{-1} \neq 1$

Will show that a relation of the form $[\text{mer.}, w^{-1} \text{mer.} \, w]$ does not abelianize $\pi(K_1 \# K_2)$

•) Determinant condition $\rightsquigarrow$ $\pi K_i \twoheadrightarrow \mathrm{Dih}_{p_i} \cong \mathbb{Z}/p_i \rtimes \mathbb{Z}/2$

•) Group of connected sum admits surjection $\pi(K_1 \# K_2) \xrightarrow{\phi} \underbrace{(\mathbb{Z}/p_1 * \mathbb{Z}/p_2) \rtimes \mathbb{Z}/2}_{=:G}$

•) Enough: Induced image $G / \langle\langle \phi([\text{mer.}, w^{-1} \text{mer.} \, w]) \rangle\rangle$ not abelian

$z = \phi(\text{mer.})$ $v = \phi(w)$

•) Look at commutator subgroup: Want to show $\mathbb{Z}/p_1 * \mathbb{Z}/p_2 / \langle\langle [z, v^{-1} z v] \rangle\rangle$

is <u>not</u> trivial

•) Rewrite $[z, v^{-1} z v] = [z, v]^2$, show this normally gen.

$\rightsquigarrow$ then use a Freiheitssatz of [Fine, Howie, Rosenberger (1988)]

to conclude that $\mathbb{Z}/p_1 * \mathbb{Z}/p_2 / \langle\langle g^2 \rangle\rangle$ is nontrivial for any $g \in \mathbb{Z}/p_1 * \mathbb{Z}/p_2$

$\square$

Last slide