Twisted Whitney towers and the higher-order Arf conjecture

Rob Schneiderman – (with Jim Conant & Peter Teichner)

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EMBEDDING CALCULUS AND GROPE COBORDISM OF KNOTS

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Abstract. We show that the invariants ev_n of long knots in a 3-manifold, produced from embedding calculus, are *surjective* for all $n \geq 1$. On one hand, this solves some of the remaining open cases of the connectivity estimates of Goodwillie and Klein, and on the other hand, it confirms one half of the conjecture by Budney, Conant, Scannell and Sinha that for classical knots ev_n are universal additive Vassiliev invariants over the integers.

We actually study long knots in any manifold of dimension at least 3 and develop a geometric understanding of the layers in the embedding calculus tower and their first non-trivial homotopy groups, given as certain groups of decorated trees. Moreover, in dimension 3 we give an explicit interpretation of ev_n using *capped grope cobordisms* and our joint work with Shi and Teichner.

The main theorem of the present paper says that *the first possibly non-vanishing embedding calculus invariant ev_n of a knot which is grope cobordant to the unknot is precisely the equivalence class of the underlying decorated tree of the grope in the homotopy group of the layer.*

As a corollary, we give a sufficient condition for the mentioned conjecture to hold over a coefficient group. By recent results of Boavida de Brito and Horel this is fulfilled for the rationals, and for the p-adic integers in a range, confirming that the *embedding calculus invariants are universal rational additive Vassiliev invariants*, factoring configuration space integrals.

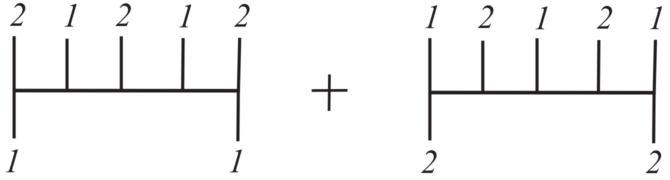
Outline of this talk

- Twisted Whitney towers and their trees
- Intersection invariants for order n twisted Whitney towers
- Classification of order n twisted Whitney towers in B^4
- The Higher-order Arf invariant Conjecture

Preview of end of talk

Key case of the Higher-order Arf invariant Conjecture in the setting of 'finite type' invariants:

The following sum of trees represents a non-trivial finite type concordance invariant of 2-component links (first-non-vanishing, $\mathbb{Z}/2\mathbb{Z}$-coefficients):



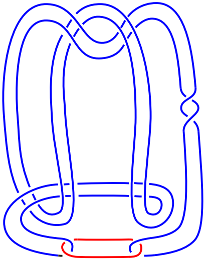
This invariant is finite type *degree* 6.

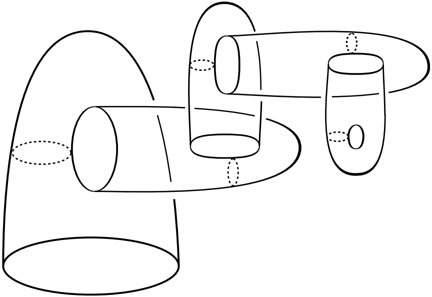
J-B. Meilhan and A. Yasuhara have characterized all finite type concordance invariants of string links in degrees ≤ 5.

Preview of end of talk

Key case of the Higher-order Arf invariant Conjecture
in the setting of 'gropes':

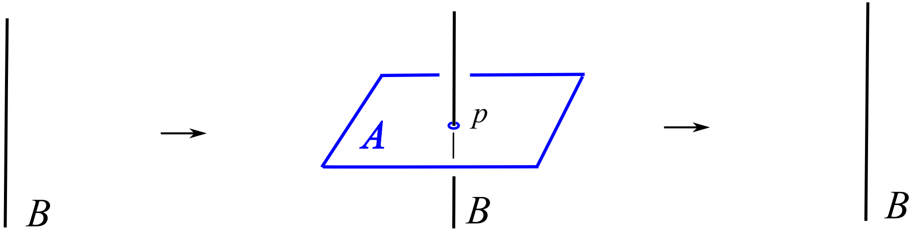
The Bing double of any knot in S^3 having non-trivial Arf invariant does not bound an embedded grope of degree 7 into B^4.





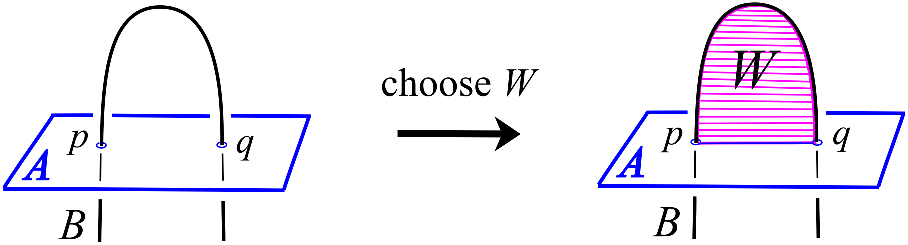
Left: The Bing double of the Figure-8 knot.
Right: One component of a *degree 6 grope*.

2-disks A and B in $B^4 = B^3 \times I$ with $p = A \pitchfork B$ and $A \subset B^3 \times *$

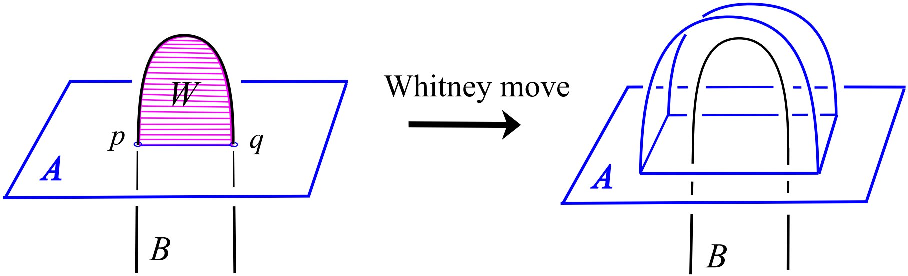


Visualize: Hopf link $= \partial A \cup \partial B \subset S^3 = \partial(B^3 \times I)$

Intersections $p, q \in A \pitchfork B$ and a *Whitney disk* W pairing them:

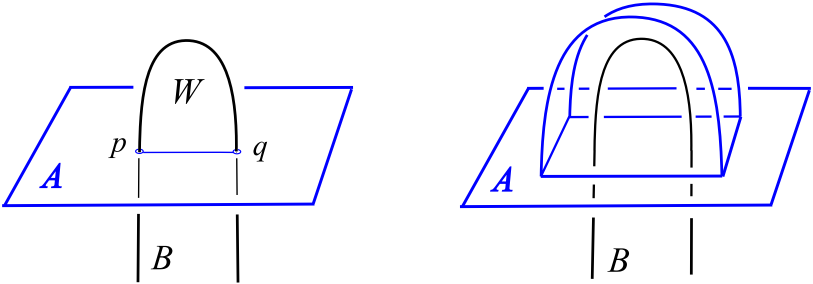


Before and after a *Whitney move*:



Successful Whitney move: W is 'clean' and 'framed'

Eliminates $p, q \in A \pitchfork B$ without creating new intersections in A or B:

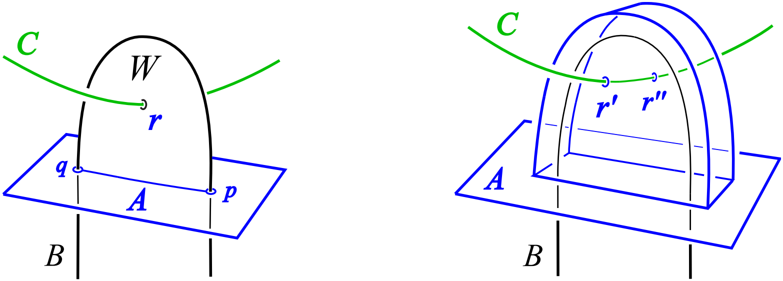


W is *clean* = embedded & interior disjoint from all surfaces.
W is *framed* = W has appropriate parallels.

Want to 'measure' obstructions to successful Whitney moves...

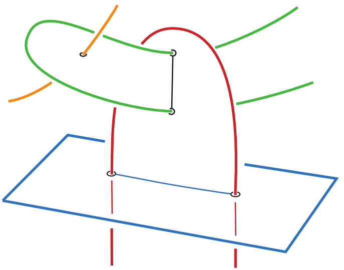
W not clean $\rightsquigarrow$ Whitney move creates new intersections:

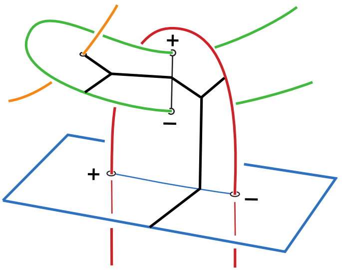
$r \in W \pitchfork C \quad \rightsquigarrow \quad r', r'' \in A \pitchfork C$ after W-move on A:



Visualize: The Borromean Rings $\partial A \cup \partial B \cup \partial C \subset \partial B^4$

‘higher-order Whitney disks’ ⇝‘higher-order intersections’ ⇝ trees...





Visualize: The Bing-double of the Hopf link in ∂B^4.

Definition:

A *Whitney tower* on $A^2 \looparrowright X^4$ is defined by:

1. A itself is a Whitney tower.
2. If $\mathcal{W}$ is a Whitney tower and W is a Whitney disk pairing intersections in $\mathcal{W}$, then the union $\mathcal{W} \cup W$ is a Whitney tower.

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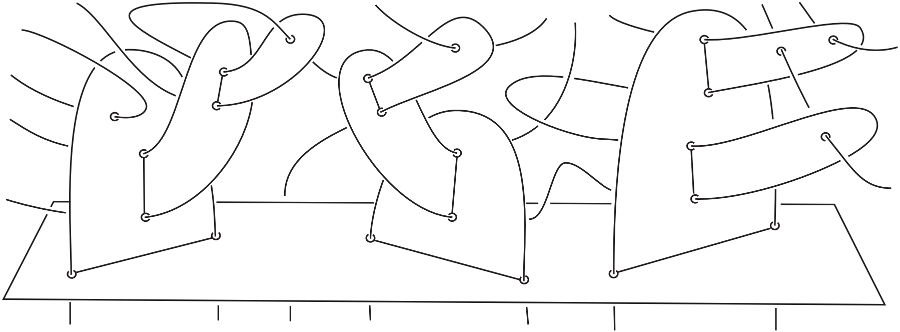
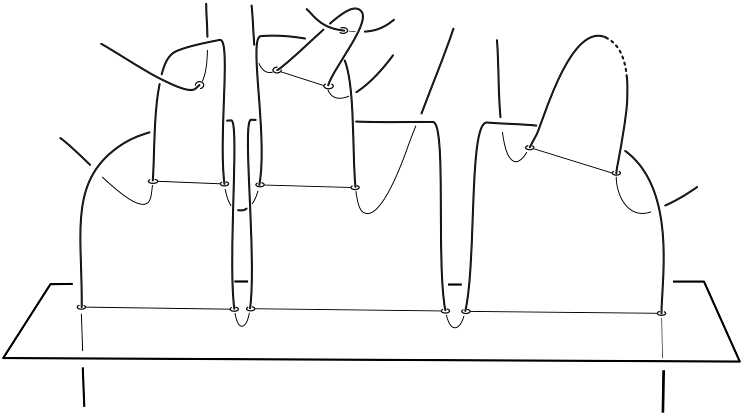


Part of a Whitney tower!

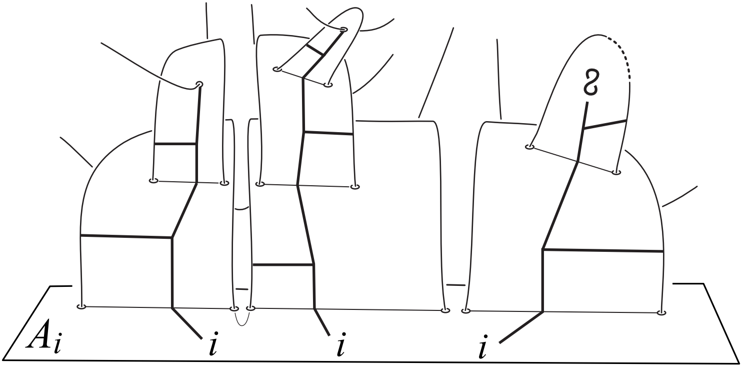
Goal: Study $\mathcal{W}$ to get info about A...

So a Whitney tower $\mathcal{W} \subset X^4$ on a properly immersed surface $A^2 \looparrowright X^4$ is the union of $A = \cup_i A_i$ and 'layers' of Whitney disks.



The *intersection forest* multiset $t(\mathcal{W})$ of a Whitney tower $\mathcal{W}$

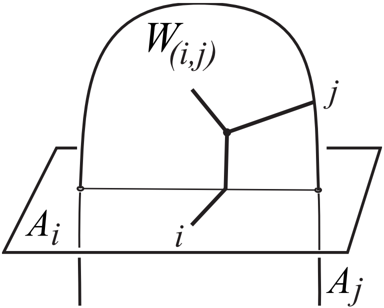
$$\mathcal{W} \mapsto t(\mathcal{W}) = \sum \epsilon_p \cdot t_p + \sum \omega(W_J) \cdot J^{\infty}$$



'framed tree' $t_p \leftarrow\!\!\dashv p$ unpaired intersection with sign $\epsilon_p = \pm 1$,
'twisted tree' $J^{\infty} := J \text{---} \infty \leftarrow\!\!\dashv W_J$ with twisting $\omega(W_J) \neq 0 \in \mathbb{Z}$.

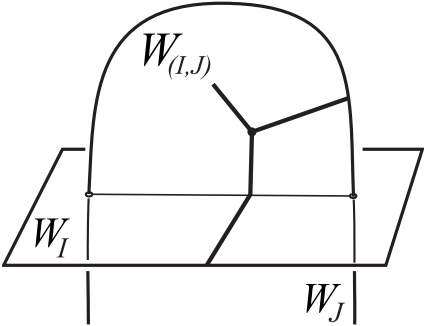
Paired intersections $\longrightarrow$ rooted trees

$W_{(i,j)}$ pairing $A_i \pitchfork A_j \quad \longmapsto \quad$ rooted tree $\prec^j_i = (i,j)$



Paired intersections → rooted trees

Recursively: $W_{(I,J)}$ pairing $W_I \pitchfork W_J \quad \longmapsto \quad -\!\!<{}^{J}_{I} = (I, J)$

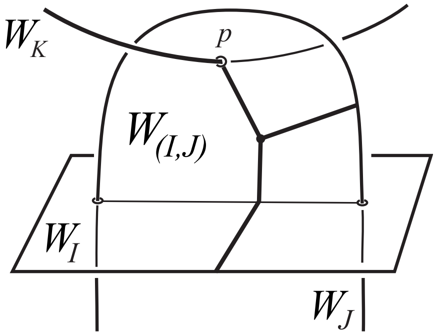


Rooted trees I, J = non-associative bracketings from $\{1, 2, 3, \ldots, m\}$
Notation convention: Singleton subscript W_i denotes component A_i.

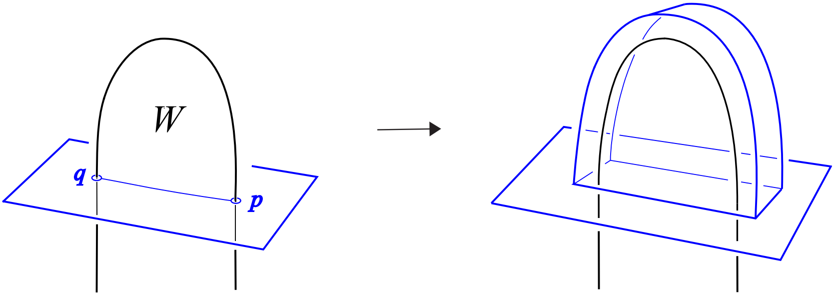
Un-paired intersections → un-rooted trees

Inner product 'fuses' rooted edges into single edge:

$$p \in W_{(I,J)} \pitchfork W_k \quad \longmapsto \quad t_p = \langle (I,J), K \rangle = {}^{I}_{J}\!> \!\!- K$$

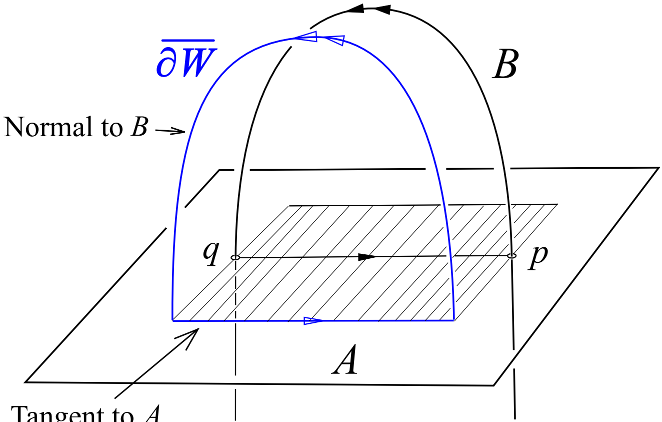


Recall: Whitney move uses two parallel copies of W:



∞-trees for twisted Whitney disks

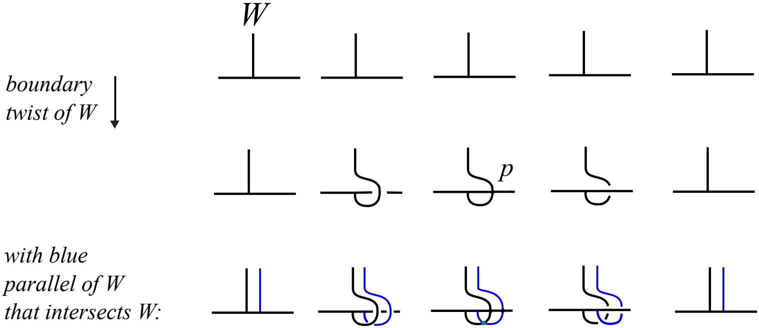
The *twisting* $\omega(W) \in \mathbb{Z}$ of W is the relative Euler number of a normal section $\overline{\partial W}$ over ∂W determined by the sheets:



$$W_J \quad \mapsto \quad J^\infty := J —\infty \quad \text{if } \omega(W_J) \neq 0.$$

Boundary twist **on W changes $\omega(W)$ by ± 1, creates intersection p between W and a sheet paired by W**

'Side view' near a point in ∂W:

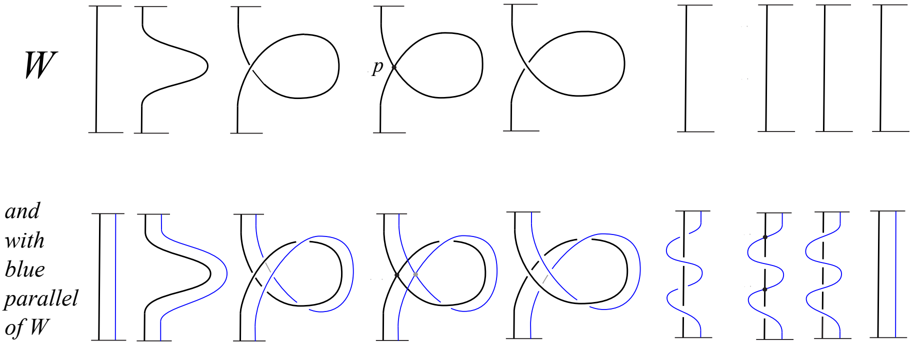


Can create any clean $W_{(I,J)}$ by finger moves,
then boundary twist into J-sheet changes $t(\mathcal{W})$ by:

$$I \prec^{J}_{J} \quad \pm \quad I \prec^{J}_{\infty}$$

$\pm$-*interior twist* on W changes $\omega(W)$ by ∓ 2 and creates $p \in W \pitchfork W$

After the interior twist,
near an arc in W that runs between the two sheets:



Can create any clean W_J by finger moves,
then $\pm$-interior twist changes $t(\mathcal{W})$ by:

$$\pm\langle J, J\rangle \quad \mp \quad 2 \cdot J^{\infty}$$

Obstruction theory for links bounding twisted Whitney towers

- $\mathcal{W}$ is an *order n twisted Whitney tower* if $t(\mathcal{W})$ contains only framed trees of order $\geq n$ and twisted trees of order $\geq n/2$, where <u>*order*</u> := number of trivalent vertices.

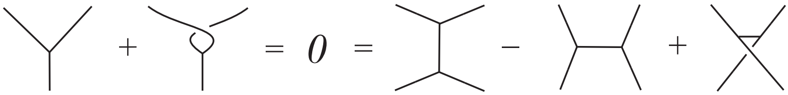
- Will define abelian groups $\mathcal{T}^{\infty}_n$
  and intersection invariants $\tau^{\infty}_n(\mathcal{W}) := [t(\mathcal{W})] \in \mathcal{T}^{\infty}_n$
  such that:

  L bounds an order n twisted $\mathcal{W}$ with $\tau^{\infty}_n(L) := \tau^{\infty}_n(\mathcal{W}) = 0$
  if and only if L bounds an order $n+1$ twisted Whitney tower.

- $\tau^{\infty}_n(L) \longleftrightarrow$ Milnor invariants and higher-order Arf invariants

Towards intersection invariants $\tau_n^\infty(\mathcal{W}) = [t(\mathcal{W})] \in \mathcal{T}_n^\infty$ for order n twisted Whitney towers $\mathcal{W} \subset B^4$ bounded by $L \subset S^3$

$\mathcal{T}_n :=$ free abelian group on order n framed trees modulo local *antisymmetry* (AS) and *Jacobi* (IHX) relations:



AS relations $\Rightarrow$ signs of the framed trees in $t(\mathcal{W})$ only depend on the orientation of $L = \cup_i \partial D^2 \subset \cup_i D^2 \overset{A_i}{\looparrowright} B^4$ after mapping to $\mathcal{T}_n$.

IHX trees can be created locally by controlled manipulations of Whitney disks.

The odd order target groups $\mathcal{T}^{\infty}_{2j-1}$

Obstructions to raising twisted order from $2j-1$ to $2j$:

Definition:
$\mathcal{T}^{\infty}_{2j-1}$ is the quotient of $\mathcal{T}_{2j-1}$ by *boundary-twist relations:*

$$i \,{-\!\!<}^{J}_{J} = 0$$

where J ranges over all order $j-1$ subtrees.

Since via boundary-twisting:

$$i \,{-\!\!<}^{J}_{J} \;\mapsto\; i \,{-\!\!<}^{J}_{\infty} \; + \text{trees of order } \geq 2j$$

and the trees on the right are allowed in order $2j$ twisted $\mathcal{W}$.

The even order target groups $\mathcal{T}_{2j}^{\infty}$

Obstructions to raising twisted order from $2j$ to $2j+1$:

Definition:
$\mathcal{T}_{2j}^{\infty}$ is the quotient of the free abelian group on framed trees of order $2j$ and ∞-trees of order j by the following relations:

1. AS and IHX relations on order $2j$ framed trees
2. *symmetry* relations: $(-J)^{\infty} = J^{\infty}$
3. *twisted IHX* relations: $I^{\infty} = H^{\infty} + X^{\infty} - \langle H, X \rangle$
4. *interior-twist* relations: $2 \cdot J^{\infty} = \langle J, J \rangle$

Remark: $\infty \!-\!\!<^{J}_{J}$ generate the torsion subgroup of $\mathcal{T}^{\infty} := \oplus \mathcal{T}_{n}^{\infty}$.

Intersection/obstruction theory for order n twisted Whitney towers

Definition:

For an order n twisted Whitney tower $\mathcal{W}$ define

$$\tau_n^{\infty}(\mathcal{W}) := [t(\mathcal{W})] \in \mathcal{T}_n^{\infty}$$

Theorem:

If $L \subset S^3$ bounds an order n twisted $\mathcal{W} \subset B^4$ with $\tau_n^{\infty}(\mathcal{W}) = 0 \in \mathcal{T}_n^{\infty}$, then L bounds an order $n+1$ twisted Whitney tower.

Idea of proof: Realize relations by geometric constructions to turn 'algebraic cancellation' in $\mathcal{T}_n^{\infty}$ into 'geometric cancellation' by new layer of Whitney disks.

Quick review of Milnor invariants

For $L = L_1 \cup L_2 \cup \cdots \cup L_m \subset S^3$ and $G = \pi_1(S^3 \setminus L)$:

$$[L_i] \in G_{n+1} \ \textit{(n+1)th lower central subroup} \implies \frac{G_{n+1}}{G_{n+2}} \cong \mathcal{L}_{n+1}$$

$\mathcal{L} = \oplus_n \mathcal{L}_n$ the free $\mathbb{Z}$-Lie algebra on $\{X_1, X_2, \ldots, X_m\}$.

Define the *order n Milnor invariant* $\mu_n(L)$:

$$\mu_n(L) := \sum_{i=1}^{m} X_i \otimes \ell_i \in \mathcal{L}_1 \otimes \mathcal{L}_{n+1}$$

where ℓ_i is the image in $\mathcal{L}_{n+1}$ of the i-th longitude $[L_i] \in \frac{G_{n+1}}{G_{n+2}}$.

Turns out: $\mu_n(L) \in \mathcal{D}_n := \ker\{\mathcal{L}_1 \otimes \mathcal{L}_{n+1} \xrightarrow{\text{bracket}} \mathcal{L}_{n+2}\}$.

Summation maps η_n 'connect' $\tau_n^\infty(\mathcal{W})$ and $\mu_n(L)$

Definition:

The map $\eta_n : \mathcal{T}_n^\infty \to \mathcal{L}_1 \otimes \mathcal{L}_{n+1}$ is defined on generators by

$$\eta_n(t) := \sum_{v \in t} X_{\mathrm{label}(v)} \otimes \mathrm{Bracket}_v(t) \qquad \eta_n(J^\infty) := \frac{1}{2}\,\eta_n(\langle J, J\rangle)$$

Here J is a rooted tree of order j for $n = 2j$.

Examples of η_n for $n = 1, 2$

$$
\begin{aligned}
\eta_1\left(1 \!-\!\!< {}^{3}_{2}\right) &= X_1 \otimes \,-\!\!< {}^{3}_{2} \quad + \quad X_2 \otimes 1\!-\!\!<^{3} \quad + \quad X_3 \otimes 1\!-\!\!<_{2} \\
&= X_1 \otimes [X_2, X_3] + X_2 \otimes [X_3, X_1] + X_3 \otimes [X_1, X_2].
\end{aligned}
$$

$$
\begin{aligned}
\eta_2\left(\infty \!-\!\!< {}^{2}_{1}\right) &= \tfrac{1}{2}\,\eta_2\left({}^{1}_{2} >\!\!-\!\!< {}^{2}_{1}\right) \\
&= X_1 \otimes {}_{2}\!>\!\!-\!\!< {}^{2}_{1} + X_2 \otimes {}^{1}\!>\!\!-\!\!< {}^{2}_{1} \\
&= X_1 \otimes [X_2, [X_1, X_2]] + X_2 \otimes [[X_1, X_2], X_1].
\end{aligned}
$$

The summation maps η_n 'connect' $\tau_n^\infty(\mathcal{W})$ and $\mu_n(L)$

The image of η_n is equal to the bracket kernel $\mathcal{D}_n < \mathcal{L}_1 \otimes \mathcal{L}_{n+1}$.

Theorem:
If L bounds a twisted Whitney tower $\mathcal{W}$ of order n, then the order q Milnor invariants $\mu_q(L)$ vanish for $q < n$, and

$$\mu_n(L) = \eta_n \circ \tau_n^\infty(\mathcal{W}) \in \mathcal{D}_n$$

Proof idea: *Gropes* in $B^4 \setminus \mathcal{W}$ display longitudes of L as iterated commutators exactly according to $\eta_n \circ \tau_n^\infty(\mathcal{W})$...

The order n twisted Whitney tower filtration on links

$$\mathsf{W}_n^\infty := \frac{\{\text{links in } S^3 \text{ bounding order } n \text{ twisted Whitney towers in } B^4\}}{\text{order } n{+}1 \text{ twisted Whitney tower concordance}}$$

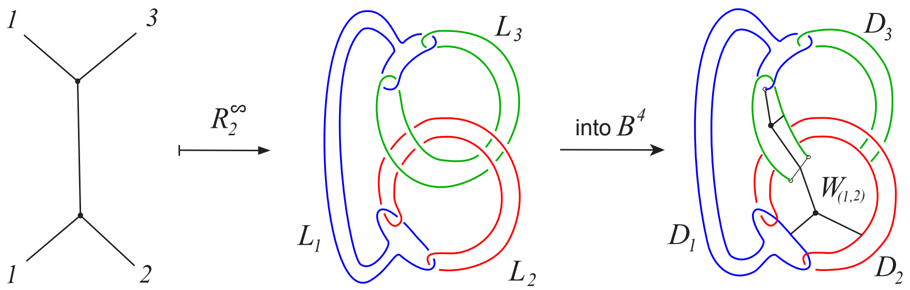
Obstruction theory $\Longrightarrow$ W_n^∞ is a finitely generated abelian group

Via Cochran's Bing-doubling techniques get epimorphisms

$$R_n^\infty : \mathcal{T}_n^\infty \twoheadrightarrow \mathsf{W}_n^\infty$$

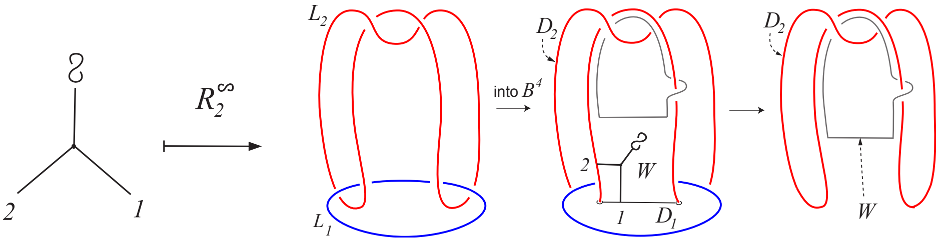
which send $g \in \mathcal{T}_n^\infty$ to the equivalence class of links bounding an order n twisted Whitney tower $\mathcal{W}$ with $\tau_n^\infty(\mathcal{W}) = g$.

Example of $R_n^\infty : \mathcal{T}_n^\infty \twoheadrightarrow \mathsf{W}_n^\infty$ for $n = 2$



L bounds $\mathcal{W}$ with $\tau_2^\infty(\mathcal{W}) = \frac{1}{2} \succ\!\!-\!\!\prec \frac{1}{3}$

Example of $R_n^\infty : \mathcal{T}_n^\infty \twoheadrightarrow \mathsf{W}_n^\infty$ for $n = 2$



L bounds $\mathcal{W}$ with $\tau_2^\infty(\mathcal{W}) = {}^{2}_{1}\!\!>\!\!\text{—}\,\infty$

Computation of W_n^∞ for $n \equiv 0, 1, 3 \mod 4$

Have commutative triangle diagram of epimorphisms:

$$\begin{array}{ccc} \mathcal{T}_n^\infty & \xrightarrow{R_n^\infty} & \mathrm{W}_n^\infty \\ & {\scriptstyle \eta_n} \searrow & \downarrow {\scriptstyle \mu_n} \\ & & \mathcal{D}_n \end{array}$$

Theorem:
The maps $\eta_n : \mathcal{T}_n^\infty \to \mathcal{D}_n$ are isomorphisms for $n \equiv 0, 1, 3 \mod 4$.

Corollary:
For $n \equiv 0, 1, 3 \mod 4$:

- $\mu_n : \mathrm{W}_n^\infty \to \mathcal{D}_n$ *and* $R_n^\infty : \mathcal{T}_n^\infty \to \mathrm{W}_n^\infty$ *are isomorphisms.*
- $\tau_n^\infty(\mathcal{W}) \in \mathcal{T}_n^\infty$ *only depends on* $L = \partial\mathcal{W}$.

Towards computation of W_n^∞ for remaining cases $n \equiv 2 \mod 4$

$\mathcal{D}_n$ is a free abelian group of known rank for all n, so have a complete computation of $\mathsf{W}_n^\infty \cong \mathcal{D}_n \cong \mathcal{T}_n^\infty$ in three quarters of the cases.

Towards understanding the remaining cases $n \equiv 2 \mod 4$:

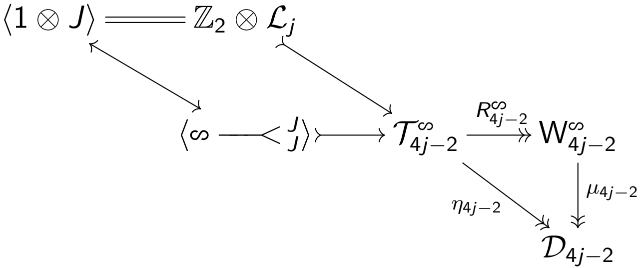
Proposition:

The map $1 \otimes J \mapsto \infty \longrightarrow\!\!<^{J}_{J} \in \mathcal{T}_{4j-2}^\infty$ *induces an isomorphism:*

$$\mathbb{Z}_2 \otimes \mathcal{L}_j \cong \mathrm{Ker}(\eta_{4j-2} : \mathcal{T}_{4j-2}^\infty \to \mathcal{D}_{4j-2})$$

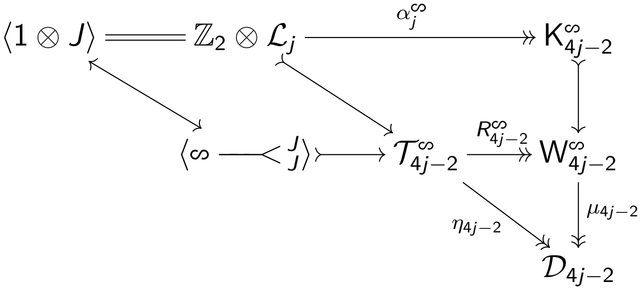
Towards computation of W_n^{∞} for remaining cases $n \equiv 2 \mod 4$

Extending the algebraic side of the triangle:



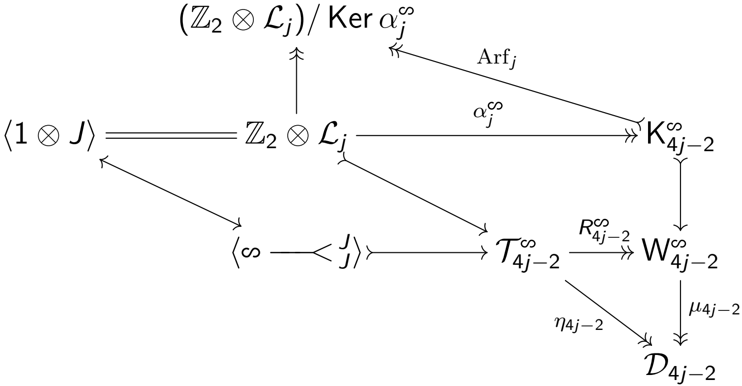
Towards defining higher-order Arf invariants

R^{∞}_{4j-2} induces $\alpha^{\infty}_j : \mathbb{Z}_2 \otimes \mathcal{L}_j \twoheadrightarrow \mathsf{K}^{\infty}_{4j-2} := \ker\{\mu_{4j-2} : \mathsf{W}^{\infty}_{4j-2} \twoheadrightarrow \mathcal{D}_{4j-2}\}$



Higher-order Arf invariant diagram

Also extending the topological side of the triangle:



$$\mathrm{Arf}_j := \mathsf{K}^\infty_{4j-2} \to (\mathbb{Z}_2 \otimes \mathsf{L}_j)/\operatorname{Ker} \alpha_j^\infty$$

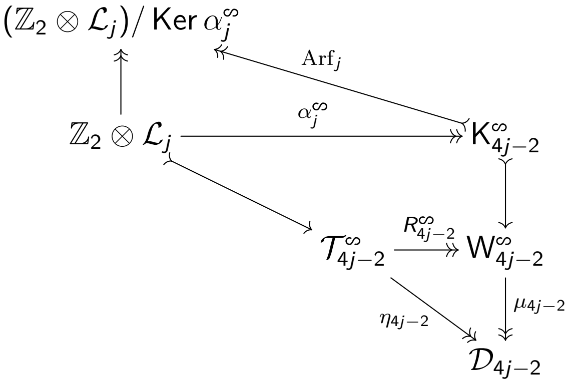
Higher-order Arf invariants and computation of W_n^∞ for all n

Corollary:

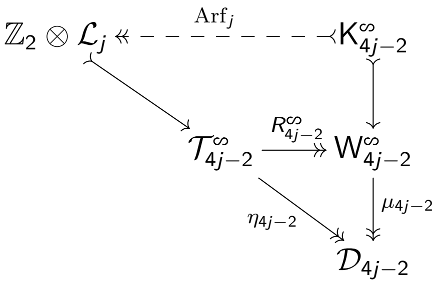
The groups W_n^∞ are classified by Milnor invariants μ_n and, in addition, higher-order Arf invariants Arf_j for $n = 4j - 2$.

In particular, a link bounds an order $n+1$ twisted $\mathcal{W}$ if and only if its Milnor invariants and higher-order Arf invariants vanish up to order n.

Higher-order Arf invariant diagram



Conjectured higher-order Arf invariant diagram



Conjecture: (Higher-order Arf invariant conjecture)

$\mathrm{Arf}_j : \mathsf{K}^{\infty}_{4j-2} \to \mathbb{Z}_2 \otimes \mathsf{L}_j$ *are isomorphisms for all* j.

This conjecture would imply $\mathsf{W}^{\infty}_n \xrightarrow{\tau^{\infty}_n} \mathcal{T}^{\infty}_n$ is an isomorphism for all n.

Determining the image of $2 \leq \mathrm{Arf}_j \leq \mathbb{Z}_2 \otimes \mathcal{L}_j$?

- Arf_1 corresponds to classical Arf invariants of the link components. Are the Arf_j for $j > 1$ also determined by finite type isotopy invariants?

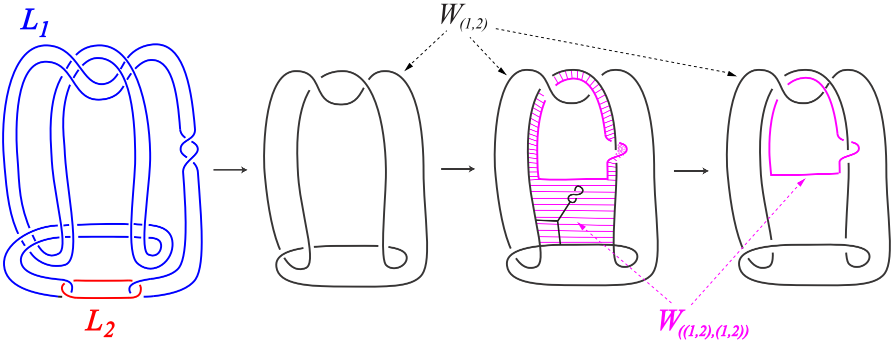
- The links $R^{\infty}_{4j-2}(\infty \!-\!\!\!< {}^{J}_{J}\,)$ realizing the image of Arf_j are known not to be *slice* by work of J.C. Cha.

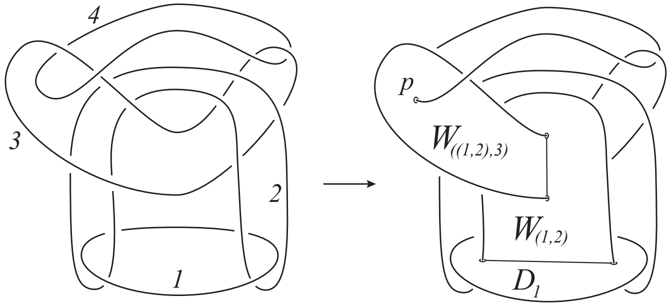
- Fundamental first open test case: Does the Bing double of the Figure-8 knot $R^{\infty}_{6}(\infty \!-\!\!\!< {}^{(1,2)}_{(1,2)}\,) \in \mathsf{W}^{\infty}_{6}$ bound an order 7 twisted Whitney tower?

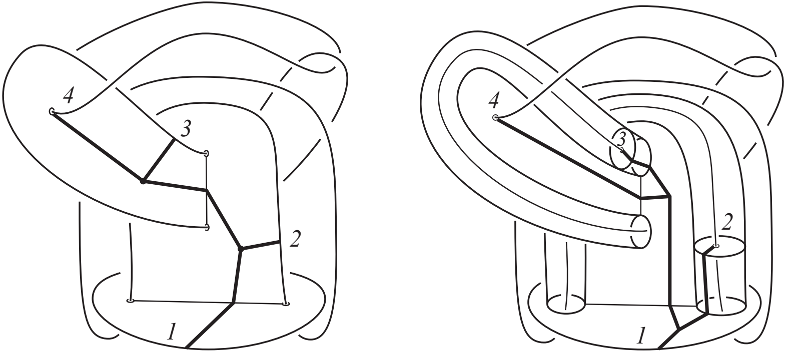
- *If* the Bing double of the Figure-8 knot *does* bound an order 7 twisted Whitney tower, then Arf_j are trivial for all $j \geq 2$.

Bing(Fig8) bounds $\mathcal{W}$ with $t(\mathcal{W}) = ((1,2),(1,2))^\infty$

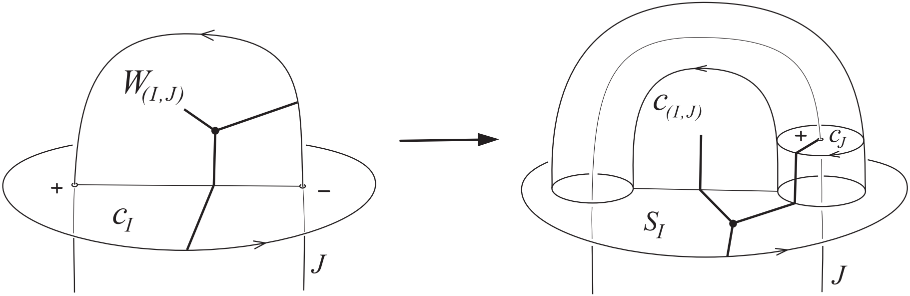
$$\mathcal{W} = D_1 \cup D_2 \cup W_{(1,2)} \cup W_{(1,2),(1,2))}$$




4
3
2
1
p
$W_{((1,2),3)}$
$W_{(1,2)}$
D_1



4
3
2
1
4
3
2
1



$W_{(I,J)}$
+
−
c_I
J
$c_{(I,J)}$
+
c_J
S_I
J