Exotic knotting in dimension 4

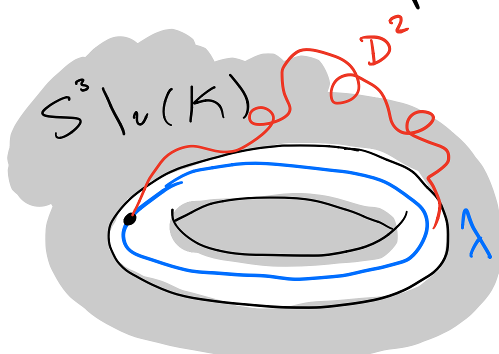
i.e. how different are
"topologically isotopic"
vs
"smoothly isotopic"
for smooth surfaces in 4-mfds?

Short answer:
pretty different

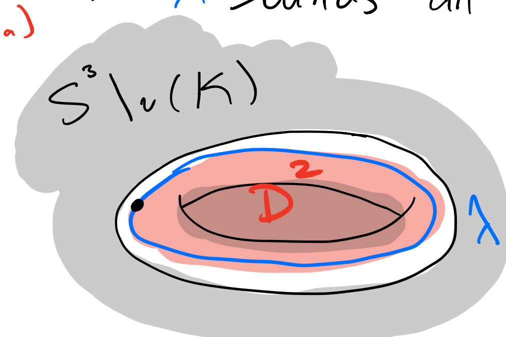
Motivation (?)

If $K \subset S^3$ is a knot with $\pi_1(S^3 \setminus K) = \mathbb{Z}$, then K is unknotted.

<u>Pf.</u> longitude λ is nullhomotopic in $S^3 \setminus \nu(K)$



Papakyriakopoulos
loop thm (Dehn's Lemma) $\rightarrow$ so λ bounds an <u>embedded</u> D^2



i.e. K is the unknot

In 4D,

Freedman

If $S^2 \xhookrightarrow[\text{locally flat}]{\text{top}} S^4$ with $\pi_1(S^4 \setminus S^2) \cong \mathbb{Z}$,

then S^2 is top locally flat unknotted

(Harder bit)

i.e. $S^2 = \partial B^3$

where $B^3 \xhookrightarrow[\text{locally flat}]{\text{top}} S^4$

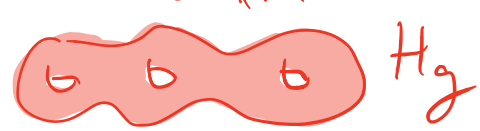
Conway-Powell 2020

If $\Sigma_g \hookrightarrow S^4$ orientable surface of genus $g > 2$ with

$\pi_1(S^4 \setminus \Sigma_g) \cong \mathbb{Z}$ then Σ_g is top locally flat unknotted

i.e. $\Sigma_g = \partial H_g$

where $H_g \xhookrightarrow[\text{locally flat}]{\text{top}} S^4$



But for $g = 1, 2$ or in smooth category: <u>unknown</u>

Surfaces Σ_1, Σ_2 are <u>exotic</u> if they are top locally flat isotopic but not smoothly isotopic.

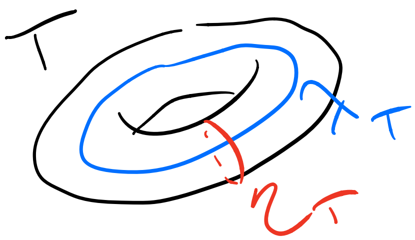
Most common construction of exotic oriented surfaces in other 4-mfds comes from <u>knot surgery.</u>

(Fintushel - Stern)

Knot surgery:

inputs: • 4-mfd Z^4

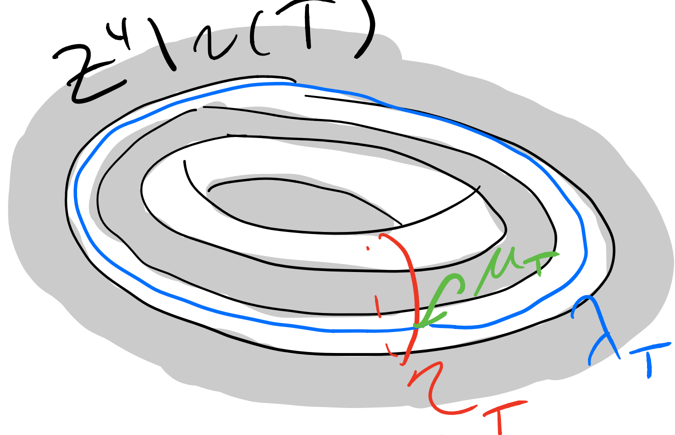
(all smooth) • $T \subset Z^4$ a torus with $[T]\cdot[T] = 0$



• a knot $K \subset S^3$

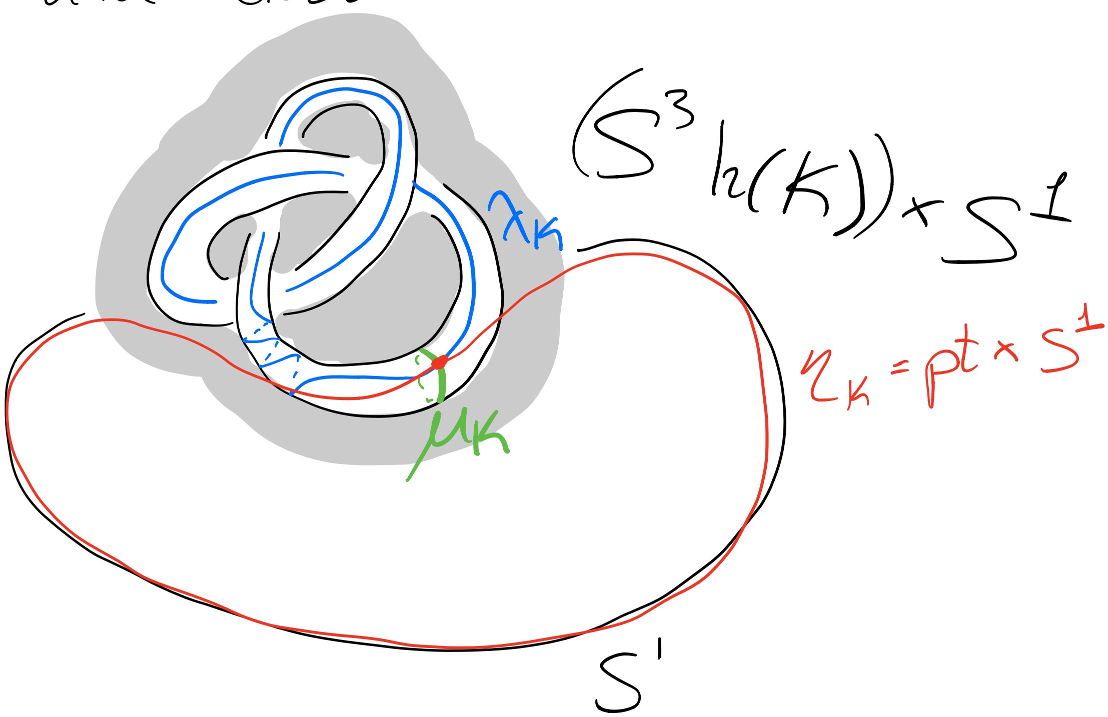
To perform surgery:

delete $\nu(T)$ from Z^4.



$\partial(Z^4 \setminus \nu(T)) \cong T^3 =$
(meridian μ_T) $\times$ (λ_T) $\times$ (η_T)

Also delete $\nu(K)$ from S^3 and cross with S^1



$$\partial(S^3 \setminus \nu(K)) \times S^1 \cong T^2 \times S^1 \cong T^3$$

$$= (\text{meridian } \mu_K) \times (\lambda_K) \times (\eta_K)$$

Now glue

$$Z(T,K) := (Z^4 \setminus \nu(T)) \cup ((S^3 \setminus \nu(K)) \times S^1)$$

indentifying

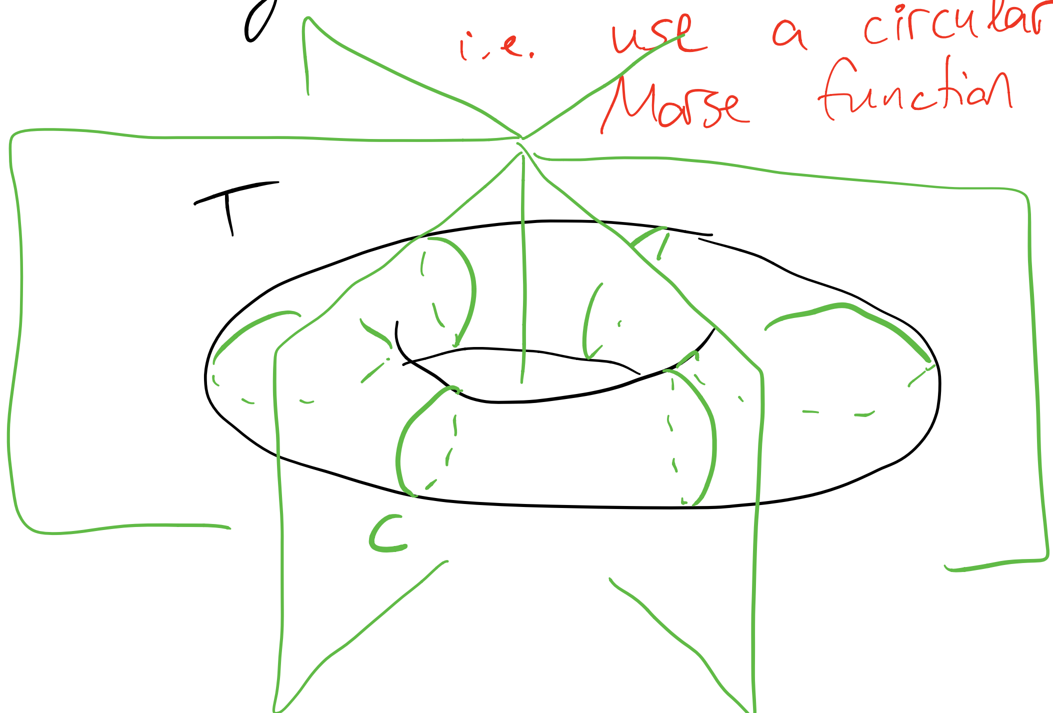
$$[\lambda_K] \to [\mu_T]$$

(technically have to specify more, but we don't really care that much)

One way to think about this:

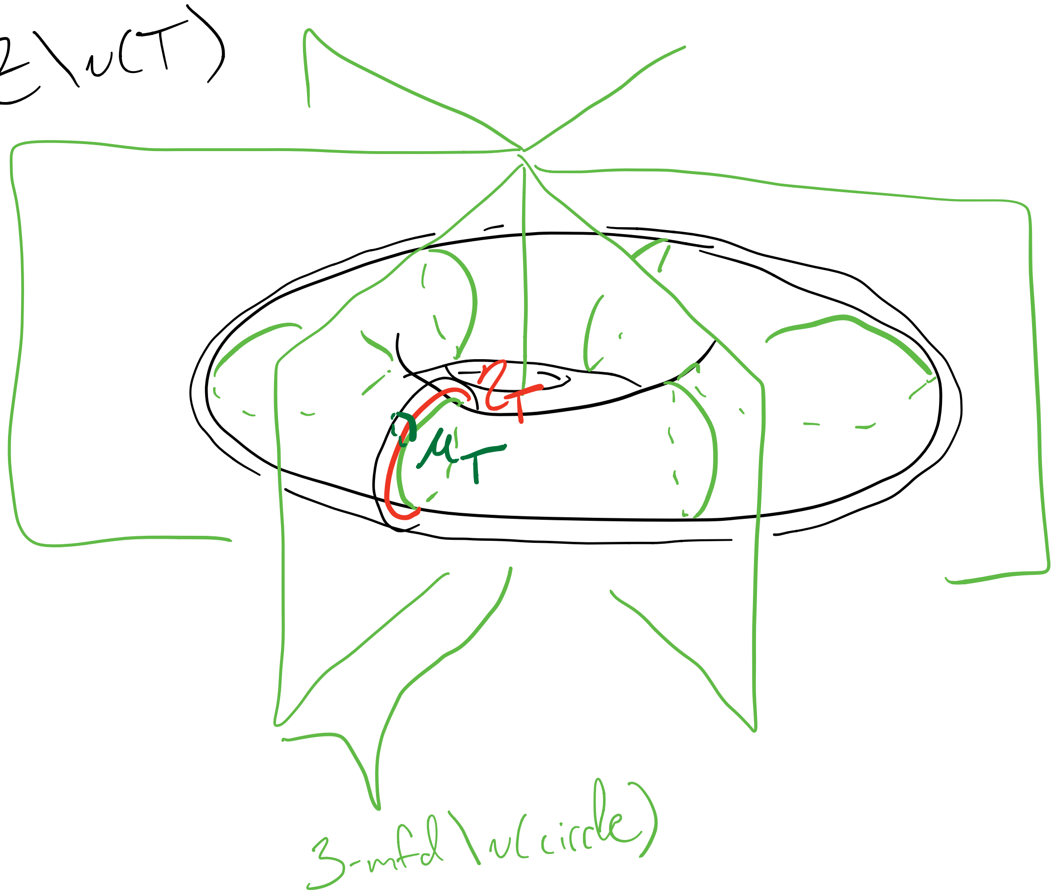
Consider 3D cross-sections of Z^4 meeting T in circles

i.e. use a circular Morse function



When we reglue the $(S^3 \setminus \nu(K)) \times S^1$, have $(S^3 \setminus \nu(K)) \times pt$ glued into each 3D piece.

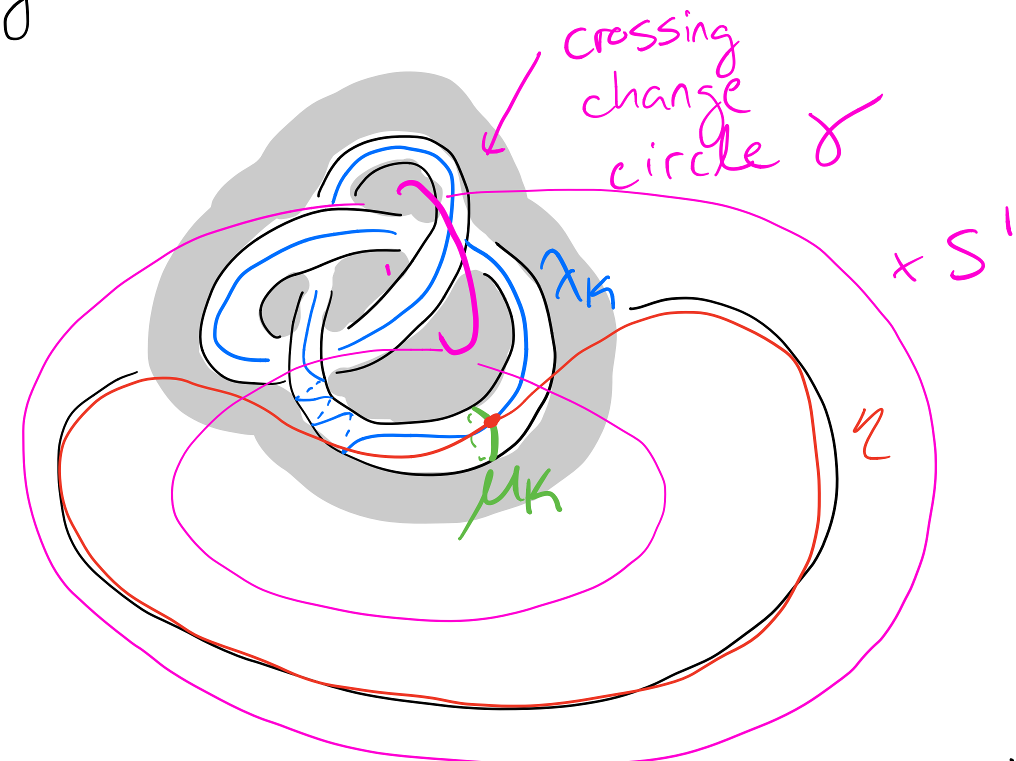
$Z \setminus \nu(T)$



So when we glue, we are deleting a circle from each 3D cross-section and regluing a copy of $S^3 \setminus \nu(K)$.

How do you undo a knot surgery?

Need to "unknot" the glued in $(S^3 \setminus \nu(K)) \times S^1$



Do surgery on $\nu(\gamma \times S^1) \cong S^1 \times D^2 \times S^1$

make sure to reglue so

$\partial D^2 \longrightarrow \pm[\gamma] + [\text{meridian } \gamma \times S^1]$

to achieve ± 1 surgery on each $\gamma \times pt$.

Repeat on some unknotting sequence for K to transform

$$(S^3 \setminus \nu(K)) \times S^1 \rightsquigarrow (S^3 \setminus \text{unknot}) \times S^1$$

$$= D^2 \times S^1 \times S^1$$

$$Z(T, K) \rightsquigarrow Z$$

$\therefore$ Knot surgery is equivalent to a sequence of usual surgeries on nullhomologous tori $T_1, \dots, T_n$

Even better:

If $\pi_1 Z \setminus T = 1$, then

$$\pi_1 Z \setminus T_i = \mathbb{Z}.$$

Why is this surgery useful?

<u>Thm</u>, (Fintushel - Stern)

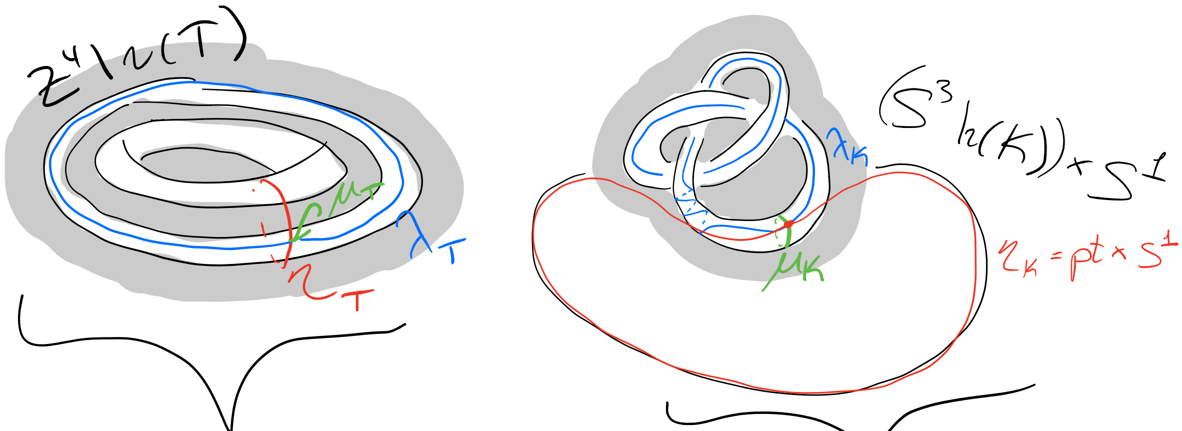
If $\pi_1(Z \setminus T) = 1$, then $Z(T,K)$ is homeomorphic to Z.

<u>Pf</u> $\pi_1(Z \setminus T) \twoheadrightarrow \pi_1(Z)$, so Z is simply connected.

Note $\pi_1 \partial((S^3 \setminus K) \times S^1) \twoheadrightarrow \pi_1((S^3 \setminus K) \times S^1)$

so $Z(T,K)$ is also simply connected.

Moreover, $(S^3 \setminus K) \times S^1$ is a homology $S^1 \times S^1$. We glue it into $Z^4 \setminus \nu(T)$ specifically to respect homology.



$\nu(T)$ is a homology $S^1 \times S^1$ generated by $[\eta_T]$ $[\lambda_T]$

replaced by

$(S^3 \setminus \nu(K)) \times S^1$ homology S^1 generated by $[\mu_K]$ $[\eta_K]$

(we glued remaining curves μ_T to λ_K)

Conclude:

Z, $Z(T, K)$ have same homology, including pairing on H_2.

$\rightsquigarrow$ So if $\pi_1(Z \setminus T)$, then Z, $Z(T, K)$ are smooth, $\pi_1 = 0$ 4-mfds with same H_2 intersection pairing

$\xRightarrow{\text{Freedman}}$ Z, $Z(T, K)$ are homeomorphic.

$\blacksquare$

Another key fact:

Fintushel-Stern understood how knot surgery affects Seiberg-Witten invariants.

Z a 4-mfd with $b_2^+(Z) \geq 2$

$SW_Z : \{\text{spin}^c \text{ structures on } Z\} \to \mathbb{Z}$

$\updownarrow$ ($H_1(Z;\mathbb{Z})$ has no 2-torsion)

$SW_Z : \left\{\begin{array}{c}\text{characteristic elements} \\ \text{of } H^2(Z;\mathbb{Z})\end{array}\right\} \to \mathbb{Z}$

Fact: SW_Z is a diffeomorphism invariant, but not a homeomorphism invariant.

Thm (Fintushel - Stern)

If $\cdot\ \pi_1(Z \setminus T) = 1$

$\cdot\ [T] \neq 0$

$\cdot\ b^+(Z) \geq 2$

$$SW_{Z(T,K)} = SW_Z \cdot \underbrace{\Delta_K}_{\text{Alexander Polynomial of } K}(2[T])$$

This proof uses the unknotting sequence observation from before

$\Rightarrow$ So if $\Delta_K \neq 1$ and $SW_Z \neq 0$ then

$Z, Z(T,K)$ are exotic!

Illustrative Application

<u>Hoffman - Sunukjian</u>

There exists a torus $T \xhookrightarrow{\text{smooth}} X^4$ in a 4-manifold X with $\pi_1 X = 1$ so that T is top locally flat unknotted but not smoothly unknotted. (but $X \not\cong S^4$)

<u>Pf</u> Let K be unknotting $= 1$ knot

- with $\Delta_K \neq 1$ e.g. $K =$ trefoil

- Let F be torus in 4-mfd X with $F \cdot F = 0$, $\pi_1 X = 1$, $b_2^+(X) \geq 6$, $SW_X \neq 0$

K-knot surgery on F is equivalent to regular surgery on some nullhomologous torus T_K with $\pi_1 X \setminus T_K \cong \mathbb{Z}$.

Sunukjian $\Rightarrow$ T_K is top locally flat unknot.

But surgery on T_K gives $X(F, K)$.

If T_K is smooth unknot,

then $X(F,K) \cong X$.

But

$$SW_{X(F,K)} = SW_X \cdot \Delta_K(2[F])$$

so

$$SW_{X(F,K)} \neq SW_X$$

$$\Rightarrow X(F,K) \not\cong X$$

$\Rightarrow T_K$ not the smooth unknot.

Problems with this strategy:

- Need $b_2^+(X) \geq 2$ for SW to be defined (Smooth isotopy obstruction)

- Also need $\pi_1(X \setminus T) = 1$, so need $b_2 > 0$ just to fit T (for SW formula)

- Used $b_2^+(X) \geq 6$ to apply thm of Sunukjian

$[T] = [T']$, $\pi_1(X \setminus T) = \pi_1(X \setminus T') = \mathbb{Z}$, $b_2^+ X \geq 6 \Rightarrow T, T'$ top isotopic

So basically no hope for S^4.

Another approach:

Compromise and allow boundary.

Thm (Juhasz-M-Zemke 2020)
For any $g > 0$, there are ∞-many neatly embedded genus-g surfaces in B^4 that are pairwise
- top isotopic rel ∂
- not smoothly equivalent

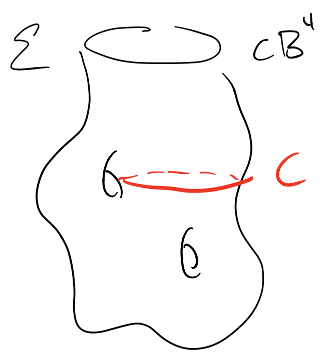
(No diffeomorphism $(B^4, \Sigma) \cong (B^4, \Sigma')$)

(Hayden has similar paper constructing similar finite families of ribbon surfaces, including disks)

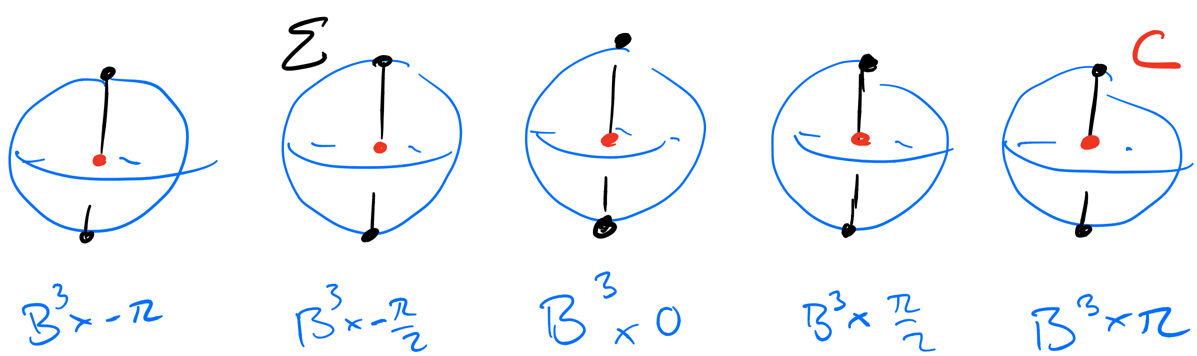
The surfaces are obtained from <u>rim surgery</u> (Fintushel-Stern) which is related to knot surgery.

Rim surgery on a surface Σ:

Choose a curve C in Σ



$\nu_{B^4}(C) = S^1 \times B^3$, intersecting Σ in $S^1 \times I$

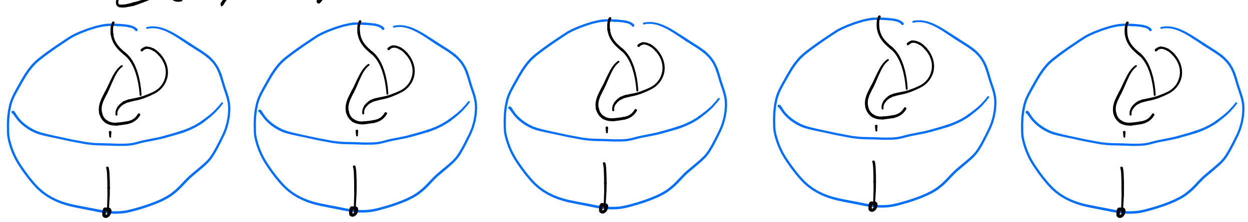


choose a knot K and an integer n.

Replace each $pt \times I$ with a tangle for K, twisting n times.

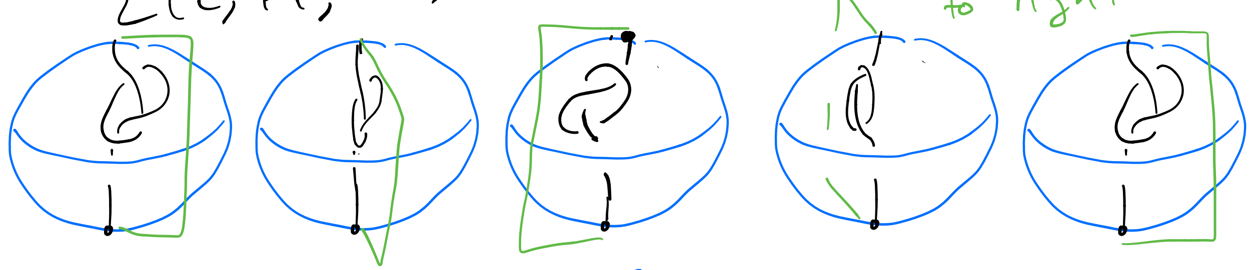
Call the result $\Sigma(C, K, n)$.

$\Sigma(C, K, 0)$

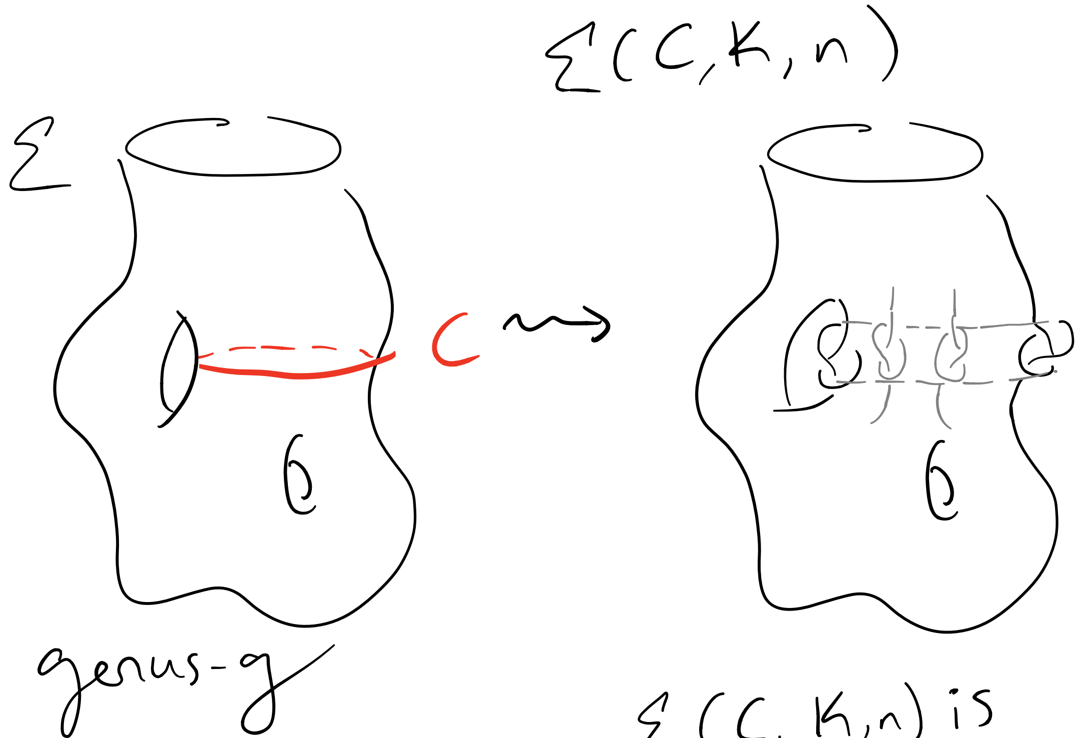


$B^3 \times -\pi$ $\quad$ $B^3 \times -\frac{\pi}{2}$ $\quad$ $B^3 \times 0$ $\quad$ $B^3 \times \frac{\pi}{2}$ $\quad$ $B^3 \times \pi$

$\Sigma(C, K, 1)$ $\quad$ twist once, from left to right



$B^3 \times -\pi$ $\quad$ $B^3 \times -\frac{\pi}{2}$ $\quad$ $B^3 \times 0$ $\quad$ $B^3 \times \frac{\pi}{2}$ $\quad$ $B^3 \times \pi$



$\Sigma(C, K, n)$ is another genus-g surface.

Thm (Zeeman)

If C bounds a framed (top local flat / smooth) disk into $B^4 \setminus \nu(C)$,

then $\Sigma(C, K, 1)$ is (top local flat / smooth) isotopic to Σ for any K.

On the other hand,

a surface in B^4 with $\partial = J$ induces some preferred element of

$\alpha_\Sigma \in \widehat{HFK}(S^3, J) \otimes \mathbb{F}[\mathbb{R}^n]$ (cobordism map evaluated on 1)

and

Juhasz-Zemke: $\alpha_{\Sigma(K,C,n)} = \alpha_\Sigma \cdot \Delta_K$

TL;DR: If $\alpha_\Sigma \neq 0$

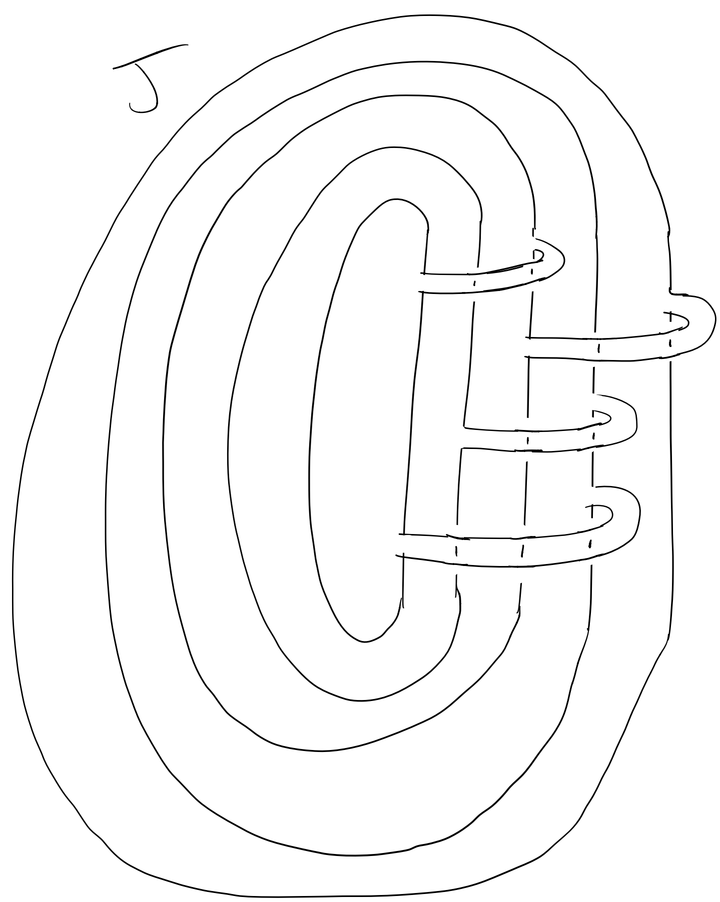
e.g. Σ a SQP surface for a strongly quasipositive knot J

and $\Delta_K \neq 1$, then

$\Sigma(K, C, n)$ is <u>not</u> smoothly isotopic to Σ.

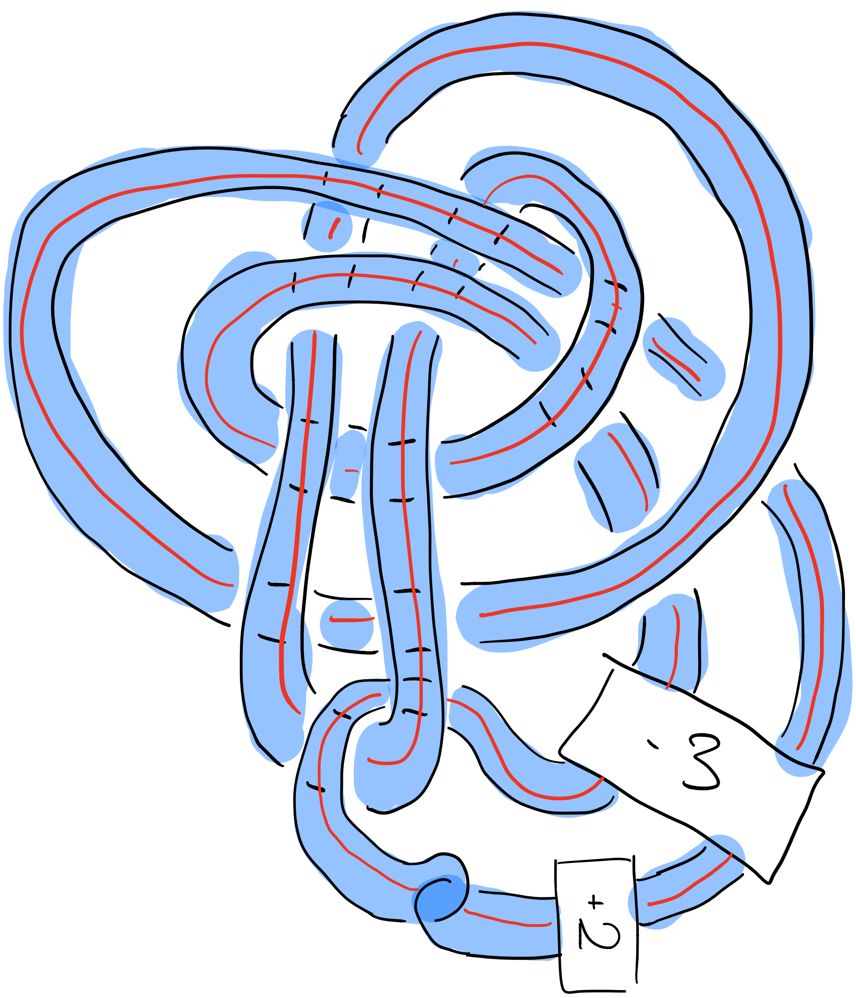
J = closure of braid factors of form $(\sigma_i \cdots \sigma_{j-1})\sigma_j(\sigma_i \cdots \sigma_{j-1})^{-1}$

i.e.



So e.g.

Take $Wh_0^+(Wh_0^+(J))$
for some SQP knot J ← e.g. trefoil



C bounds a top locally flat disk

Σ (or add genus)

and consider $\Sigma(C, K, 1)$
for different K.

Rmk

Important takeaways are the knot surgery and rim surgery operations, which underly most constructions of oriented exotic surfaces.

A different direction:

How do you get rid of exotic behavior?

i.e. Given exotic $\Sigma, \Sigma' \hookrightarrow X^4$,

what operation makes Σ and Σ' smoothly isotopic?

Perspective 1: The operation should change X^4

Perspective 2: The operation should change Σ and Σ'.

Thm (Auckly-Kim-Melvin-Ruberman-Schwartz 2017)

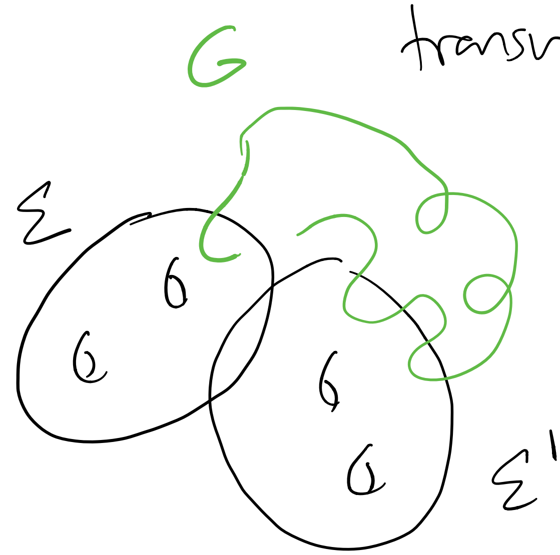
Let Σ, Σ' be homologous genus-g surfaces in X with $\pi_1 X \setminus \Sigma = \pi_1 X \setminus \Sigma' = 1$.

Then Σ and Σ' are smoothly isotopic in

$$\begin{cases} X \# S^2 \tilde{\times} S^2 & \text{if } [\Sigma] \text{ characteristic} \\ X \# S^2 \times S^2 & \text{else} \end{cases}$$

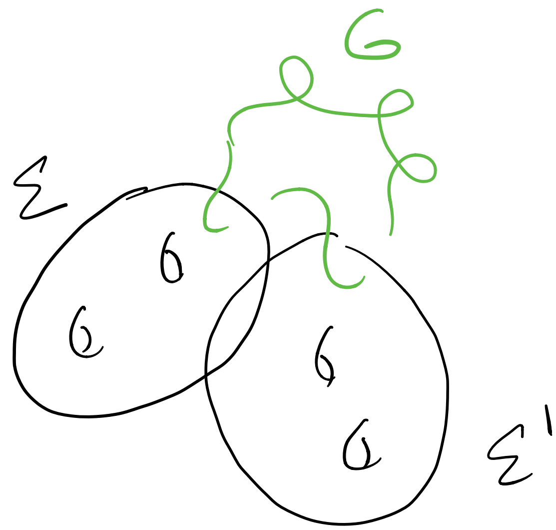
Pf

$\pi_1 X \setminus \Sigma \Rightarrow 1 \Rightarrow$ there is an immersed sphere G intersecting Σ transversely geometrically once.



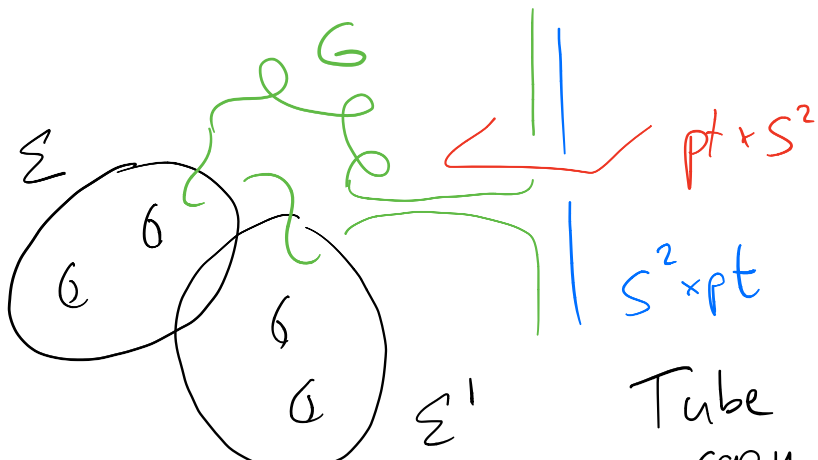
Then $\langle G, \Sigma' \rangle = 1$ but $|G \cap \Sigma'|$ might be > 1.

Homotope G, maybe making more self-intersections, until G also intersects Σ' only once.



4D Light bulb thm (Gabai 2017) says that if G were <u>embedded</u> and <u>$G \cdot G = 0$</u> then Σ and Σ' would be smoothly isotopic.

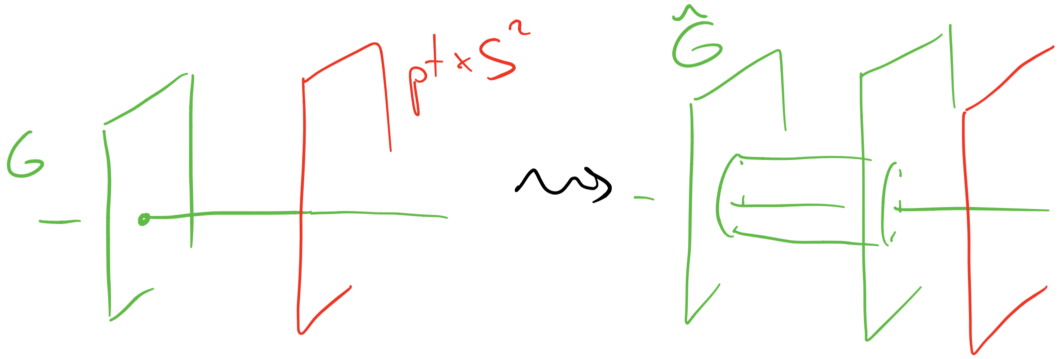
So consider $X \# (S^2 \times S^2)$

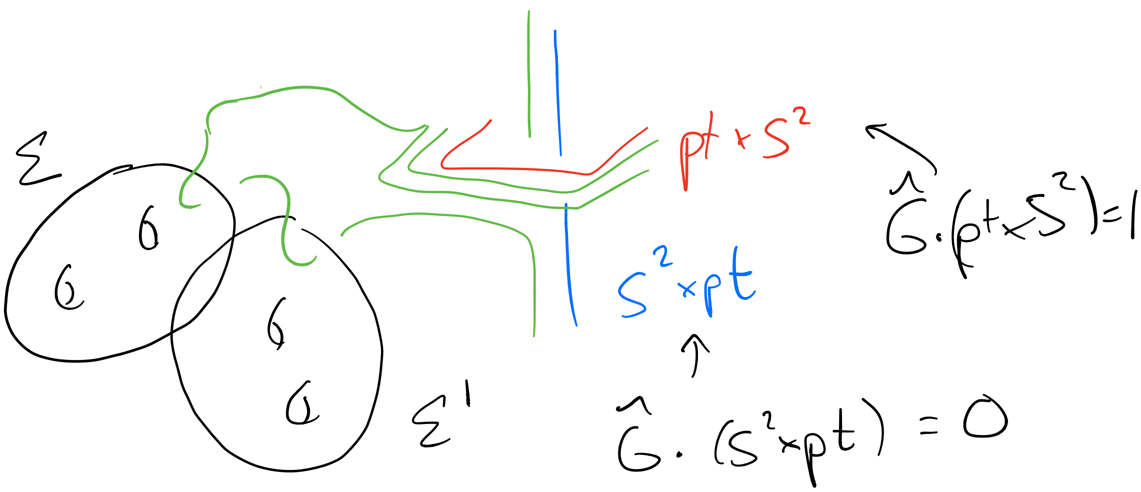


Tube G to a copy of $S^2 \times pt$.

At each self-intersection of G, tube G to $pt \times S^2$.

Get an <u>embedded</u> sphere $\hat{G}$





If $\hat{G} \cdot \hat{G}$ even, can tube to more copies of $S^2 \times pt$ until

$$\hat{G} \cdot \hat{G} = 0$$

$\rightsquigarrow \Sigma, \Sigma'$ smoothly isotopic

Case 1

If $[\Sigma]$ not characteristic,
WLOG $G \cdot G$ even

Case 2

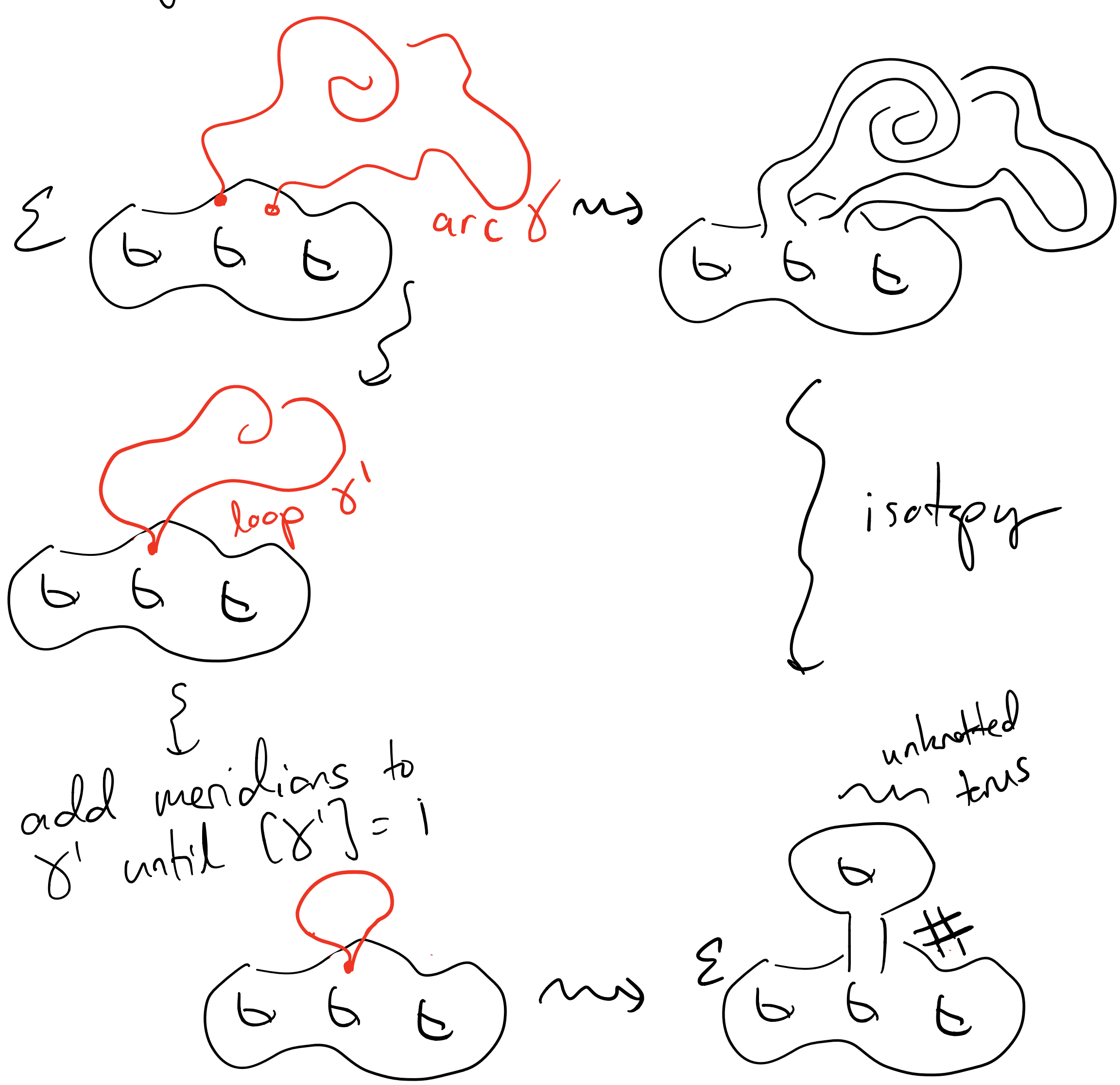
If $[\Sigma]$ characteristic,
then $G \cdot G$ odd so
use $S^2 \tilde{\times} S^2$
instead.

Other perspective:
leave 4-mfd alone

Thm (Baykur-Sunukjian)
Basically every strategy we have for creating top. isotopic surfaces creates surfaces that are smoothly isotopic if you add one extra tube (genus) to each.

Observe:

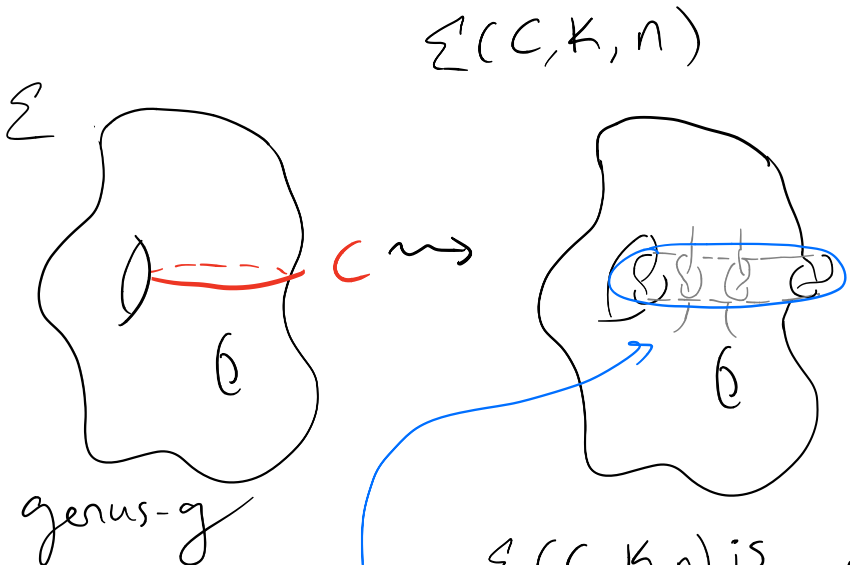
If $\pi_1(X \setminus \Sigma)$ is cyclic, then adding genus to Σ always yields Σ # (unknotted torus)



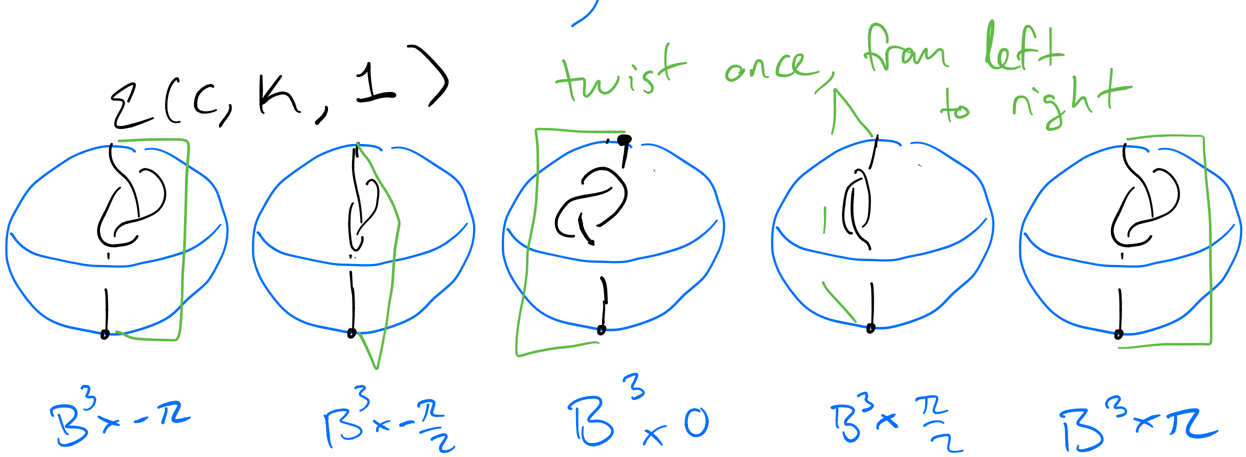
Consider rim surgery:

recall we replace

$\underbrace{C}_{\text{curve in } \Sigma} \times I$ with $S^1 \times$ tangle



$\Sigma(C,K,n)$ is another genus-g surface.



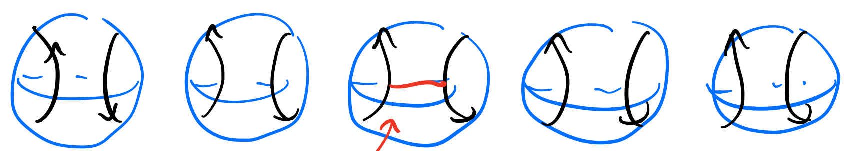
Let's take $\pi_1(X \setminus \Sigma)$ cyclic, which we would need for most theorems about top isotopy.

Kim-Ruberman

If $\pi_1(X \setminus \Sigma)$ cyclic then Σ and $\Sigma(C, K, n)$ are top isotopic

Adding a tube inside the $B^3 \times S^1$ can relate different tangles.

Consider $T_0 \times S^1$

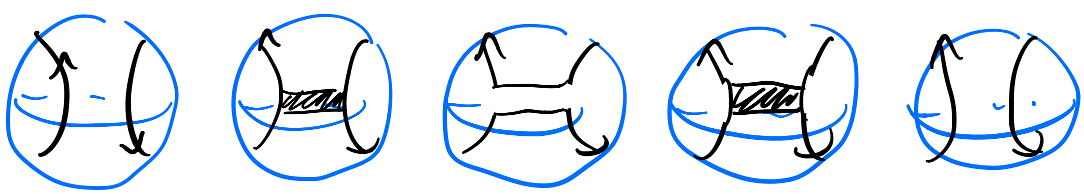


$B^3 \times -\pi$ $\quad$ $B^3 \times -\frac{\pi}{2}$ $\quad$ $B^3 \times 0$ $\quad$ $B^3 \times \frac{\pi}{2}$ $\quad$ $B^3 \times \pi$

Add tube here

Add tube:

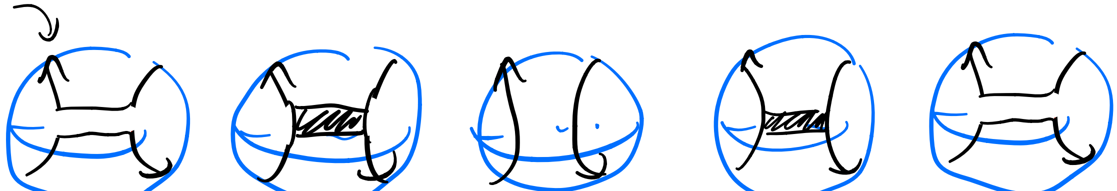
T_0



$B^3 \times -\pi$ $\quad$ $B^3 \times -\frac{\pi}{2}$ $\quad$ $B^3 \times 0$ $\quad$ $B^3 \times \frac{\pi}{2}$ $\quad$ $B^3 \times \pi$

$\downarrow$ isotopy

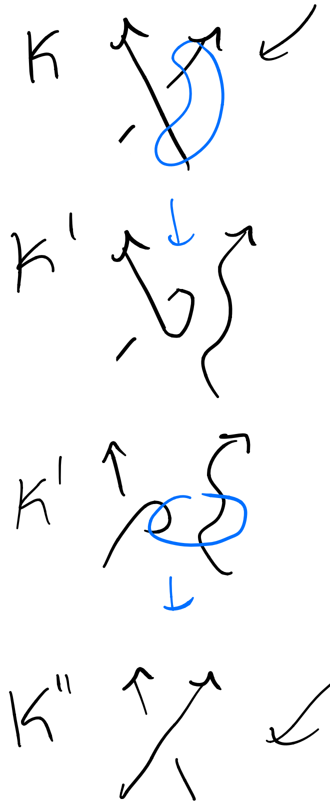
T_∞



$B^3 \times 0$ $\quad$ $B^3 \times \frac{\pi}{2}$ $\quad$ $B^3 \times \pi$ $\quad$ $B^3 \times \frac{3\pi}{2}$ $\quad$ $B^3 \times 2\pi$

Same thing as $T_\infty \times S^1$ + tube!

So by adding $\overbrace{\text{a single tube,}}^{\text{i.e. \# unknotted torus}}$ we can change the tangle used to do rim surgery



$\Sigma(C,K,n) \# T^2$ smoothly isotopic to

$\Sigma(C, K \text{ with crossing change}, n) \# T^2$

i.e. for any knots K, J,
$\Sigma(C,K,n) \# T^2$ is smoothly isotopic to $\Sigma(C,J,n) \# T^2$

And $\Sigma(C, \text{unknot}, m)$ is isotopic to $\Sigma(C, \text{unknot}, n)$, so really get

$$\Sigma(C,K,m) \# T^2$$

smoothly isotopic to

$$\Sigma(C,J,n) \# T$$

for all curves $C \hookrightarrow \Sigma$,

knots J, K

$m, n \in \mathbb{Z}$

when $\pi_1 X \setminus \Sigma$ is cyclic.

(Baykur-Sunukjian also show this for general $\pi_1 X \setminus \Sigma$ but it's a little trickier.)

Big questions

- Are there exotic oriented surfaces in S^4?

- Given an exotic pair of surfaces in a 4-mfd X^4, for what n are the surfaces smoothly isotopic in $X^4 \#_n S^2 \times S^2$ or $X^4 \#_n S^2 \tilde{\times} S^2$?

- Given an exotic unknot Σ in X^4, for what n is $\Sigma \#$ (n unknotted tori) smoothly unknotted?