Knotted surfaces (with knot group $\mathbb{Z}$).

I. Reminders:

- I.1 History of exotic surfaces (survey)
- I.2 Rim surgery.

II Topological equivalence

- II.1 History and methods (survey).
- II.2 Some examples. (proof sketches).

III Knotted surfaces with knot group $\mathbb{Z}$ and ribbon discs.

Emphasis on surgery theory. (Building bridges?).

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I.1 History of exotic surfaces

I.2 Rim surgery.

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- Manifolds are smooth, compact and connected
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NB: · You might prefer I require F_1 and F_2 be topologically ambient isotopic.
(ie that Φ be isotopic to id).

· For simplicity let me use equivalence.

· (Quinn: $\pi_0(\mathrm{Top}(M)) \xrightarrow{\cong} \mathrm{Aut}(H_2(M), Q_M)$).

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See also:
- Finashin 2002
- Hoffman-Sunukjian 2012

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Given a closed surface $\Sigma \subset X$ and a knotted arc $K \subset B^3$, rim surgery outputs a new surface $\Sigma(K)$.

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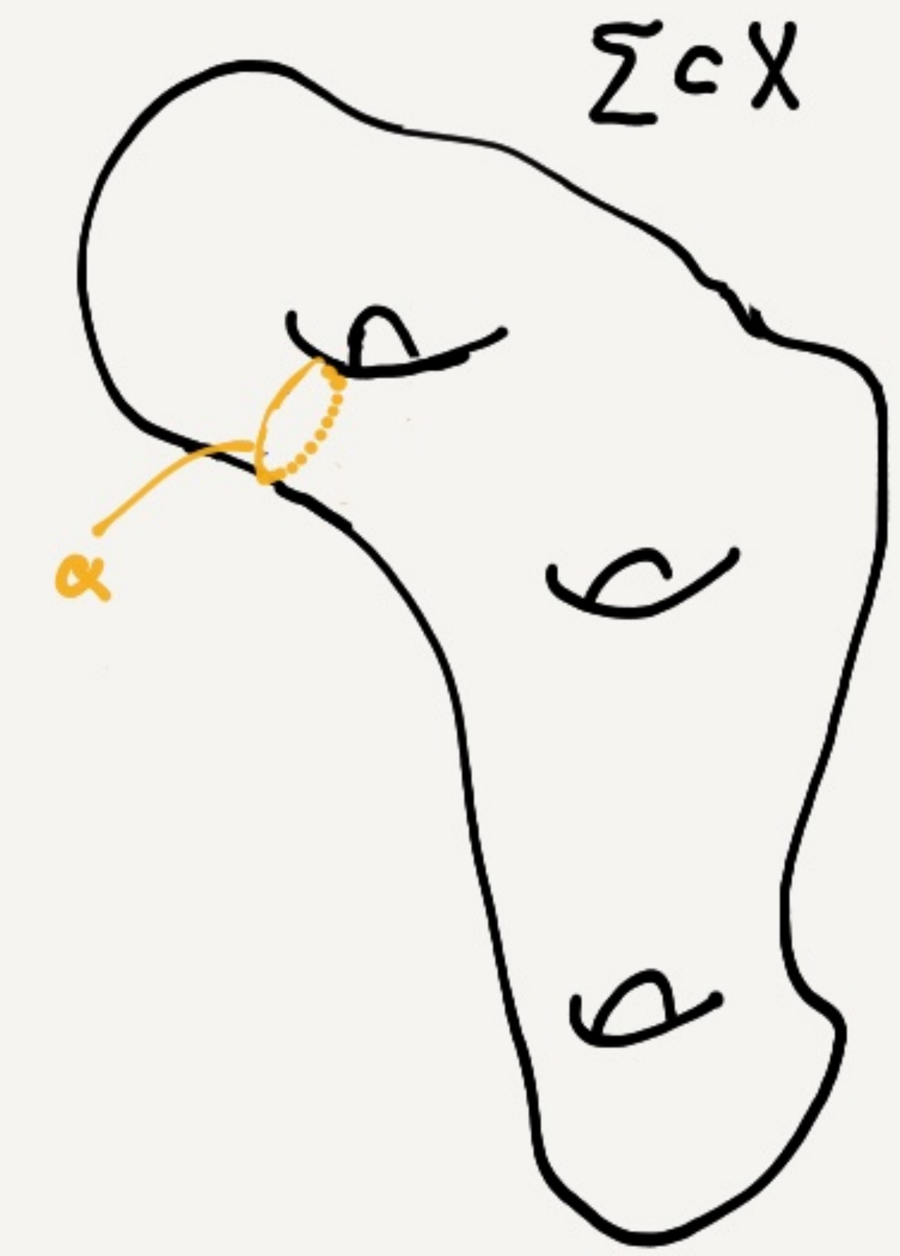

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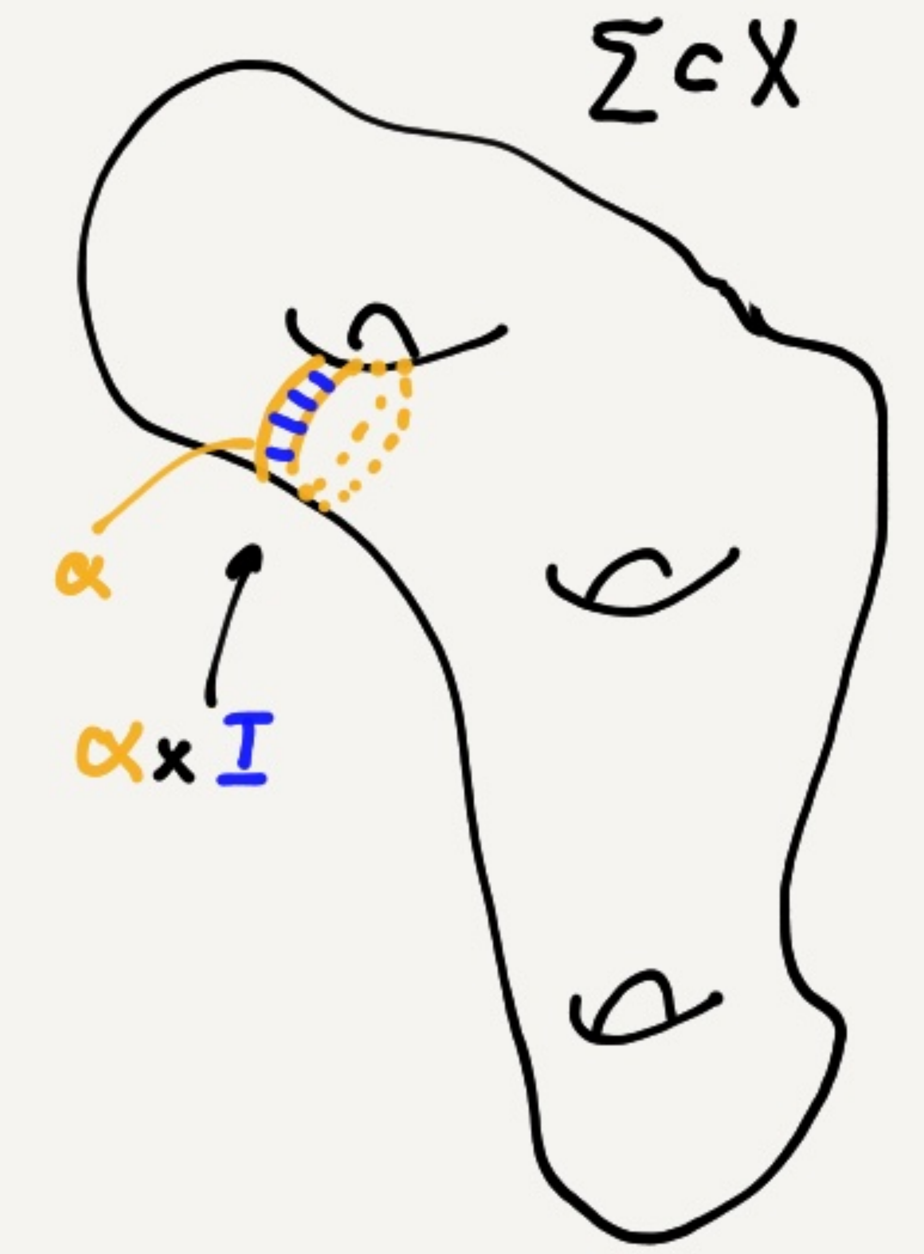

The construction:

$$(X, \Sigma(K,\alpha)) = (X, \Sigma \setminus (\alpha \times I) \qquad\qquad).$$

(I am ignoring framings).

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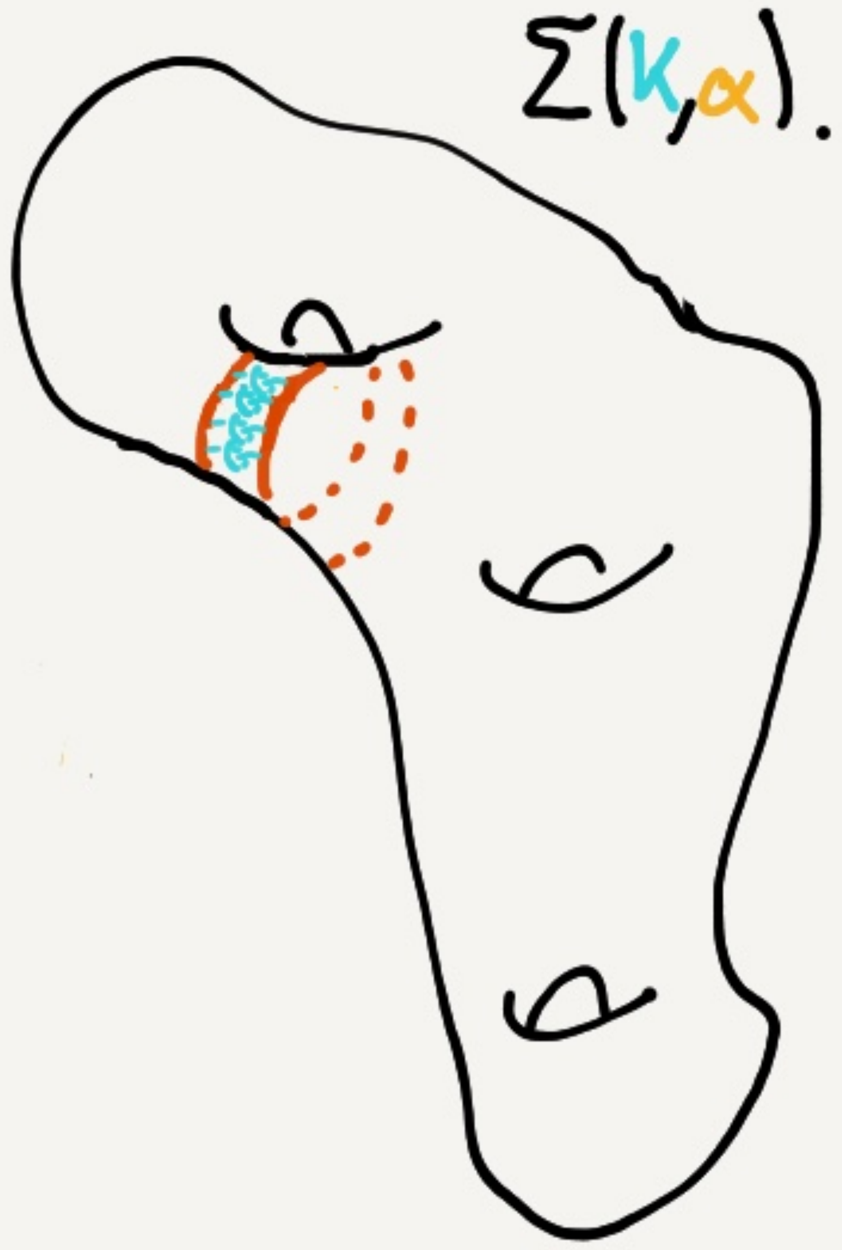

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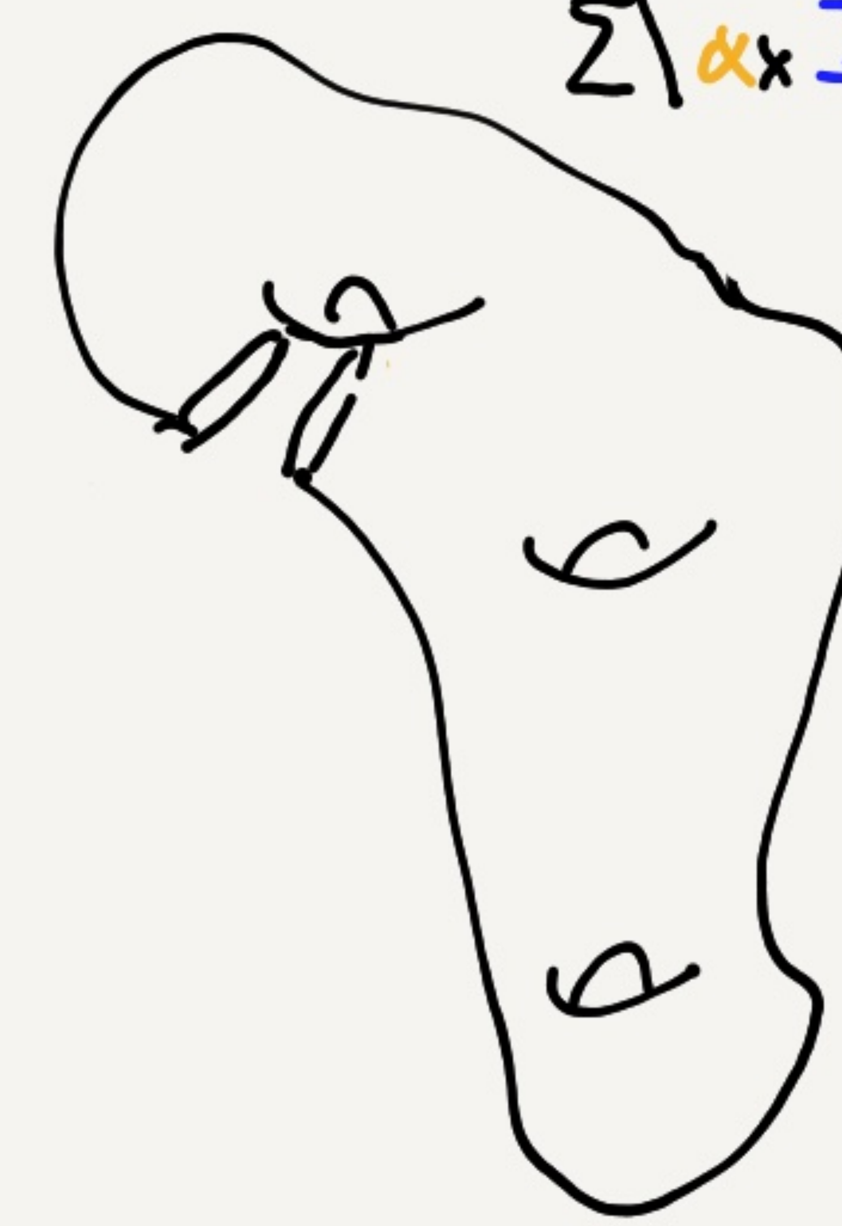

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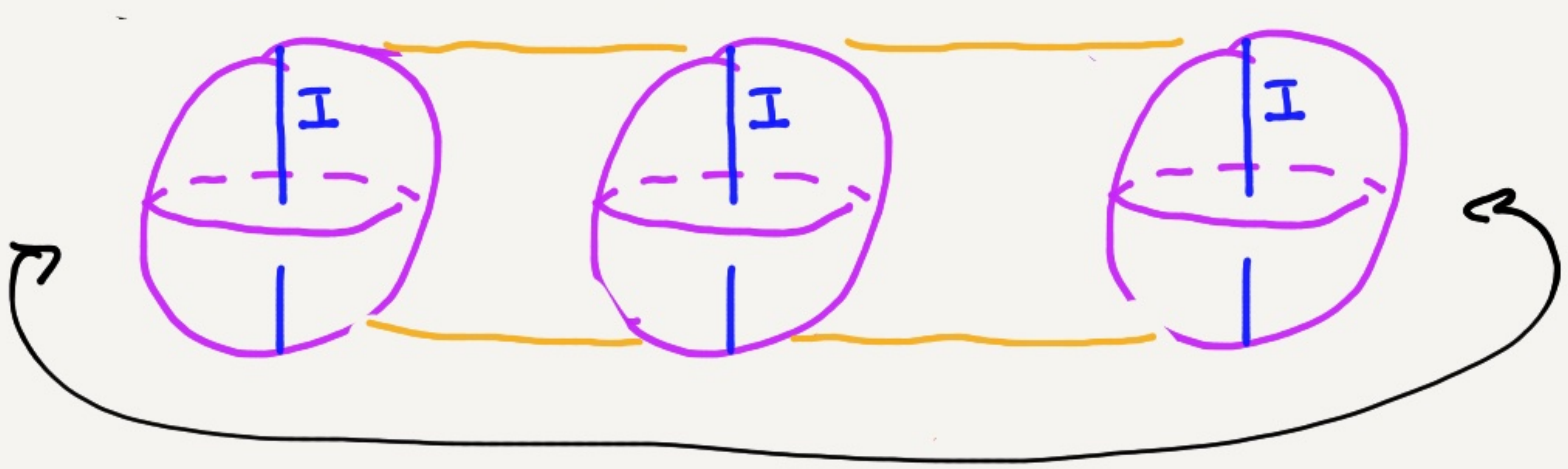

framings!

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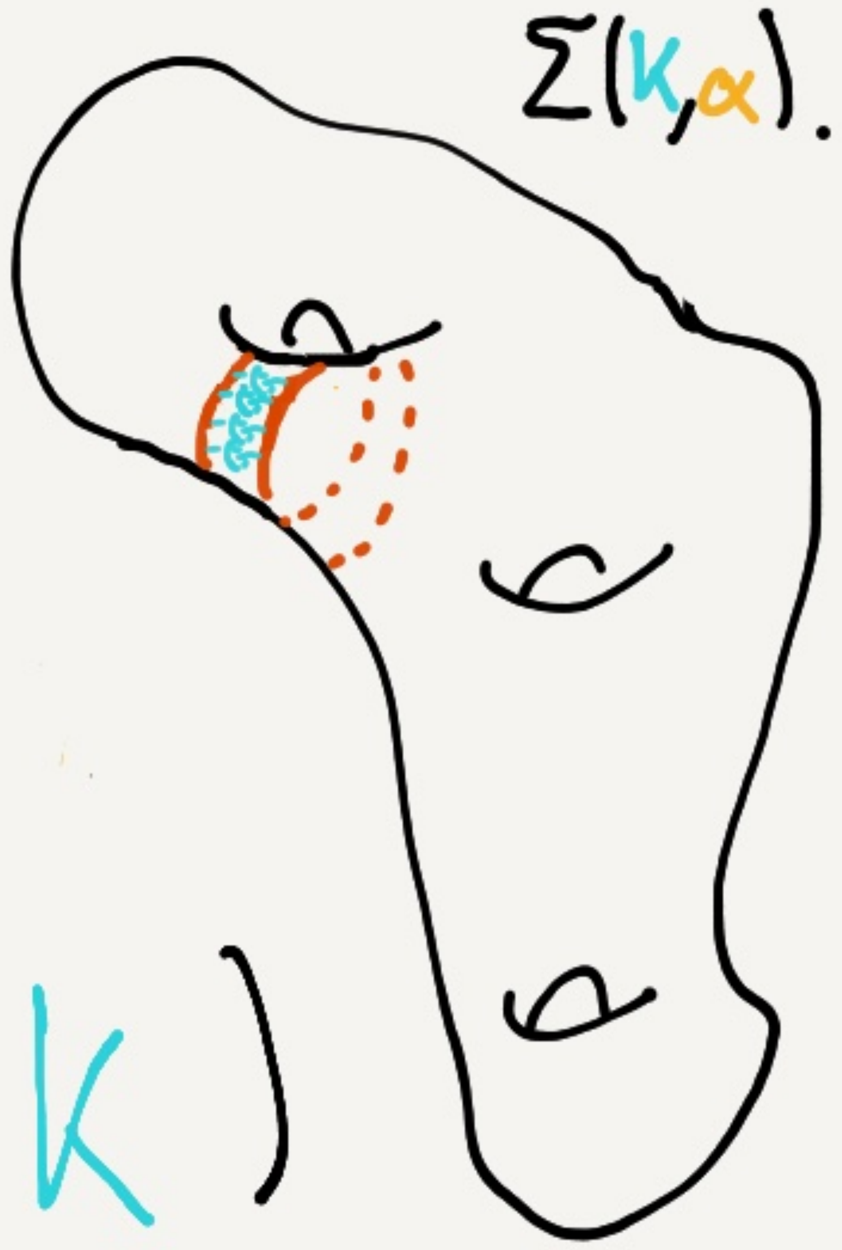

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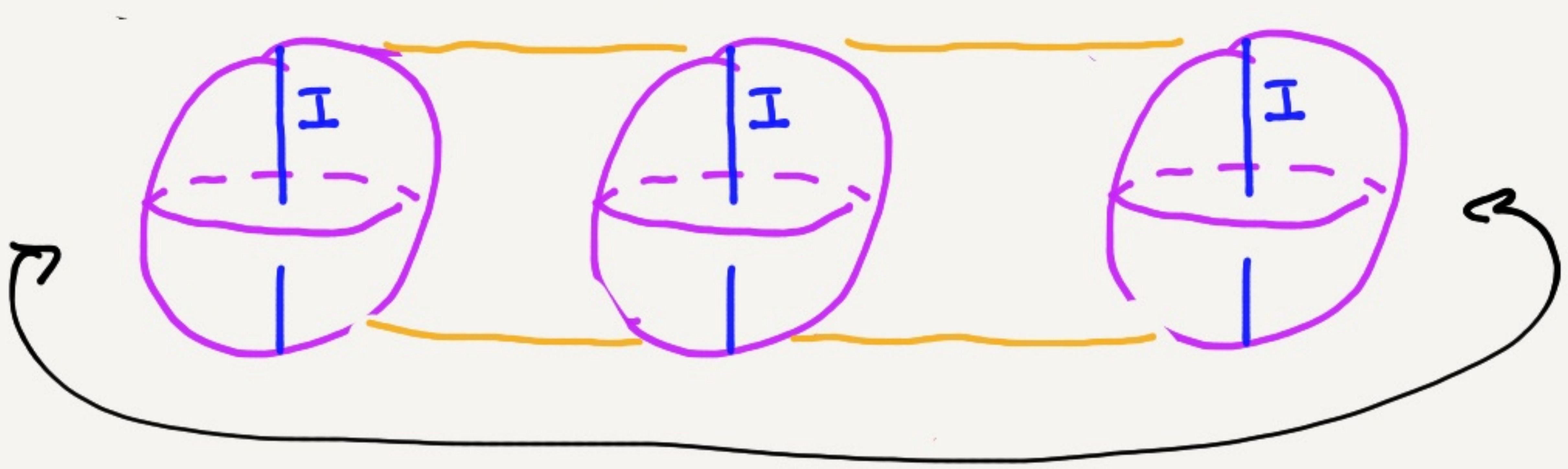
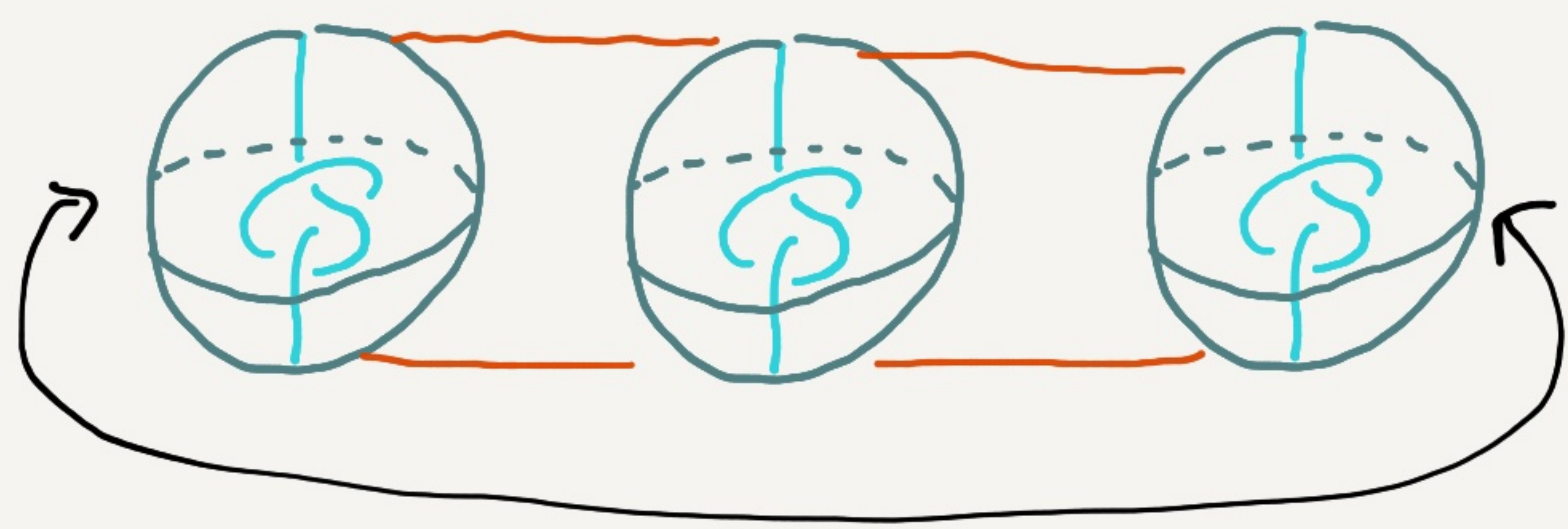


$S^1 \times (B^3, K)$



m-twist R.m Surgery.

m-twist Rim Surgery.

$$\left(X, \Sigma_m(K,\alpha)\right)$$

$$[0,1]\times(B^3,K)\Big/_{(0,x)\sim(1,\tau^m(x))} =: S^1_{\tau^m}\times(B^3,K)$$

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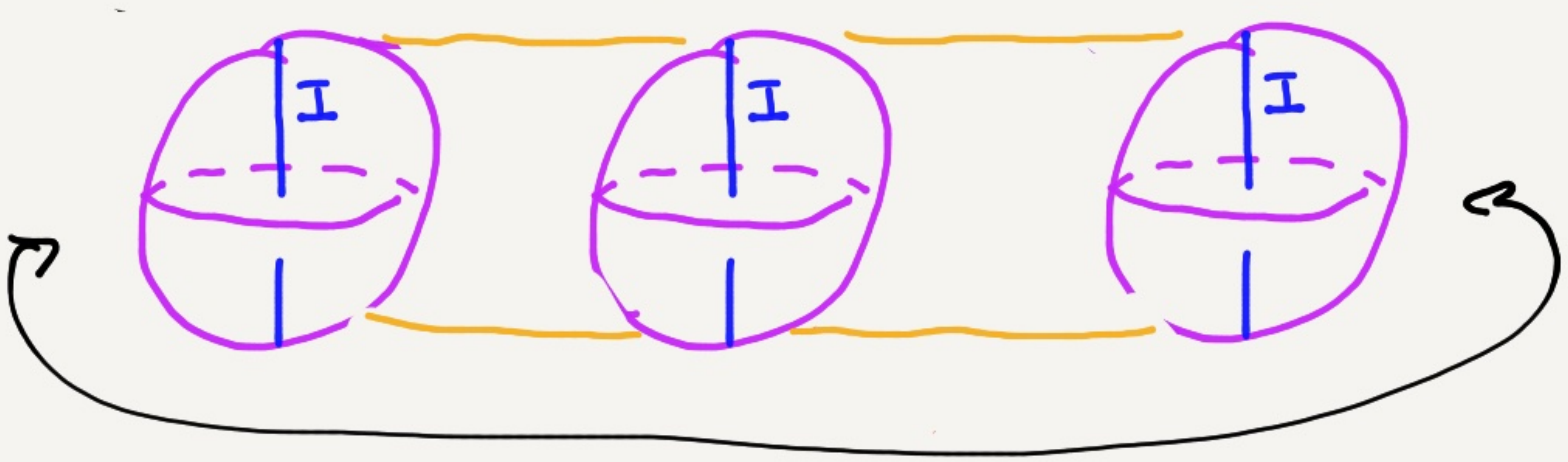
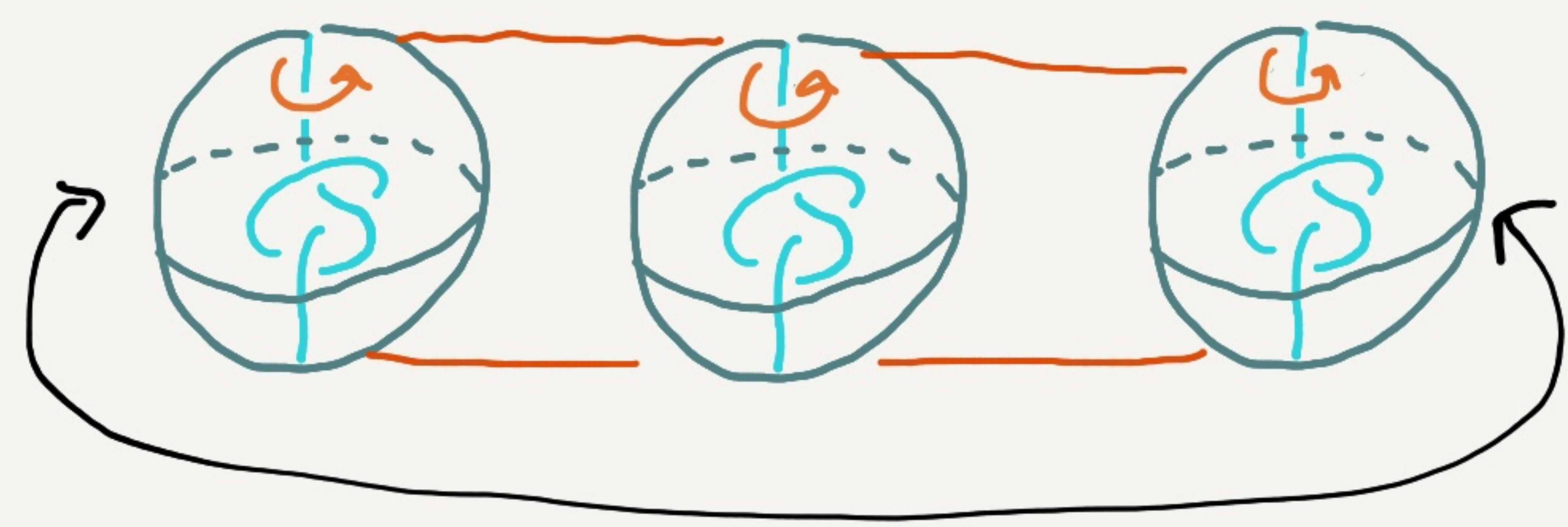


rotate m times



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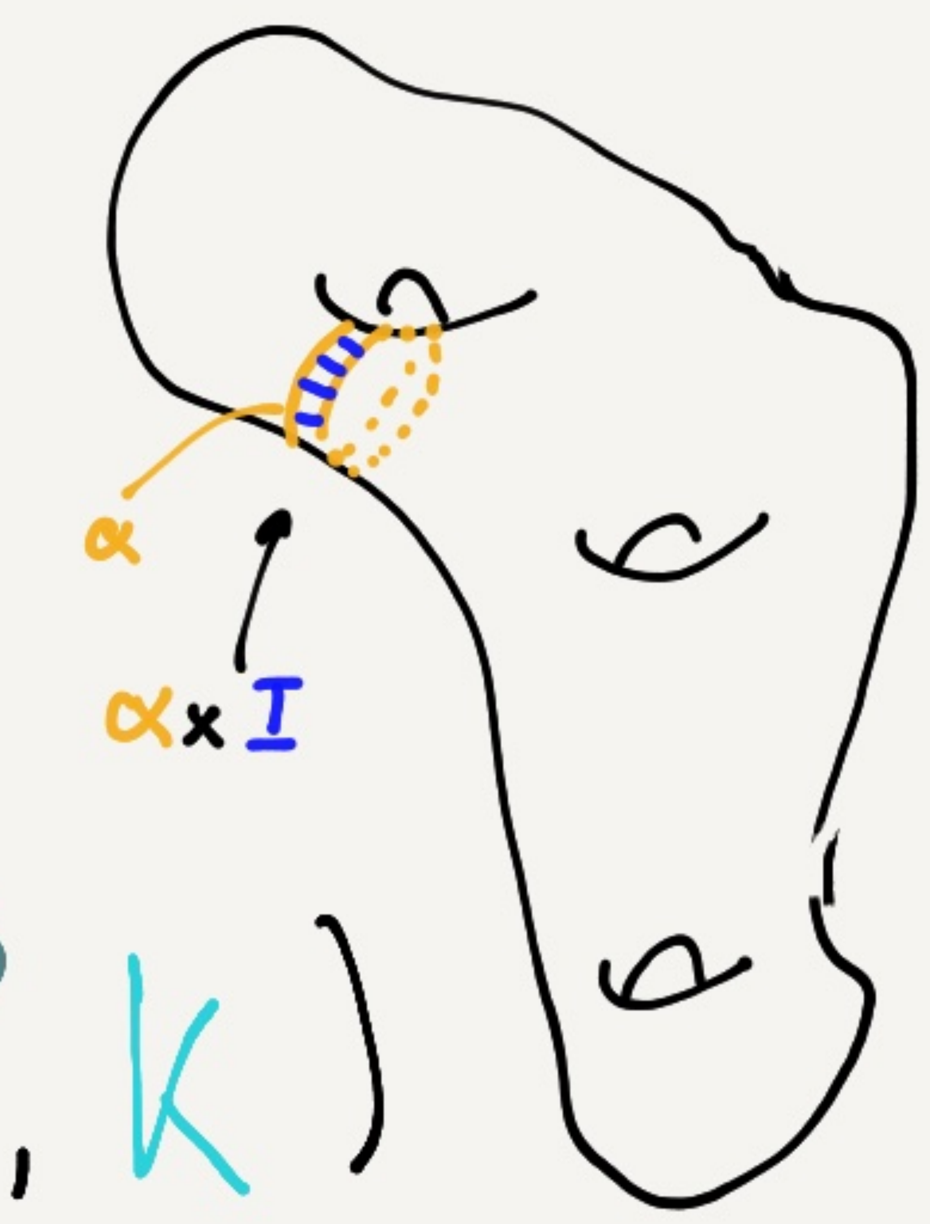

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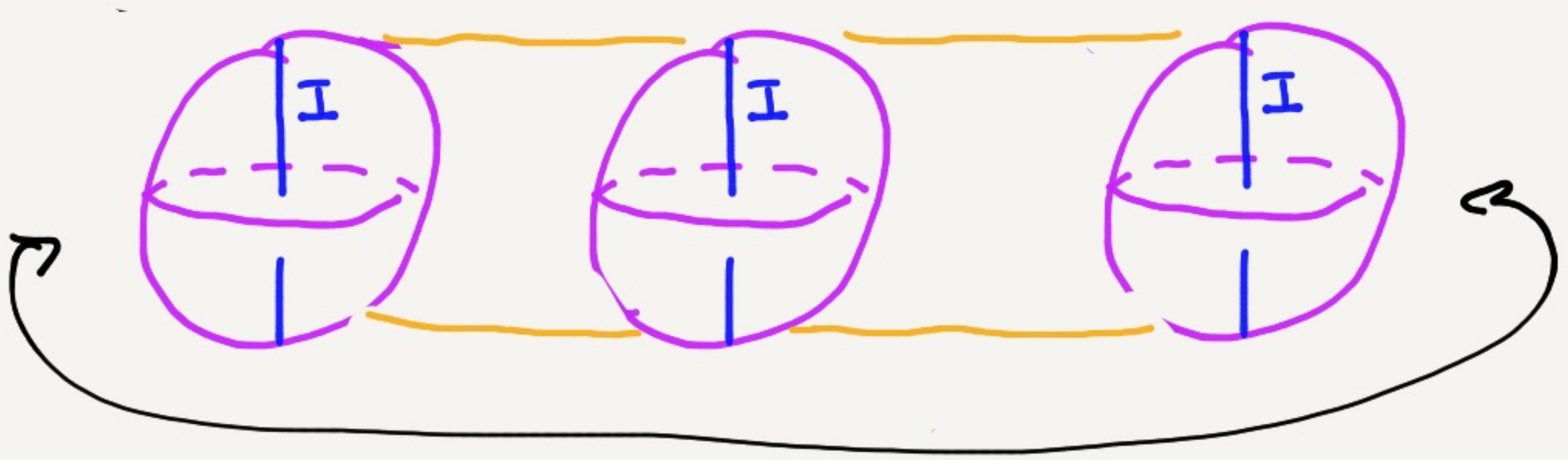

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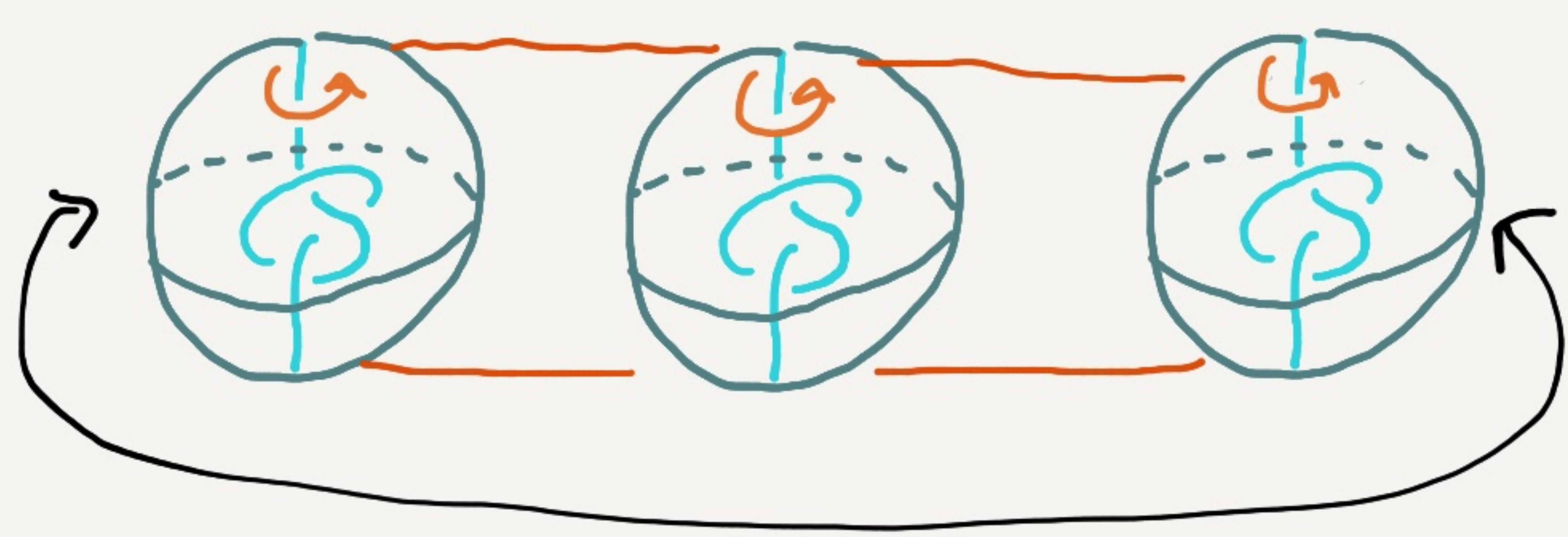

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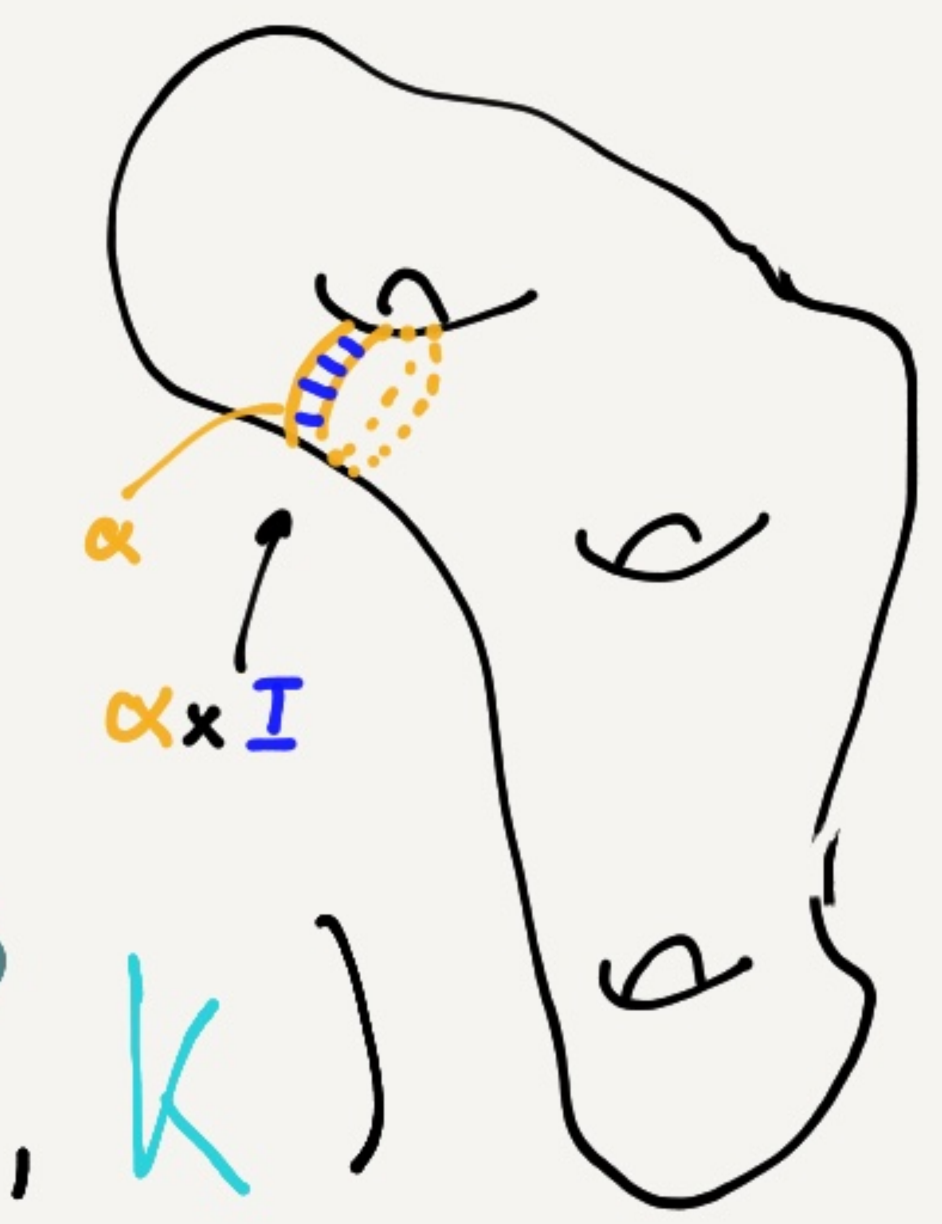

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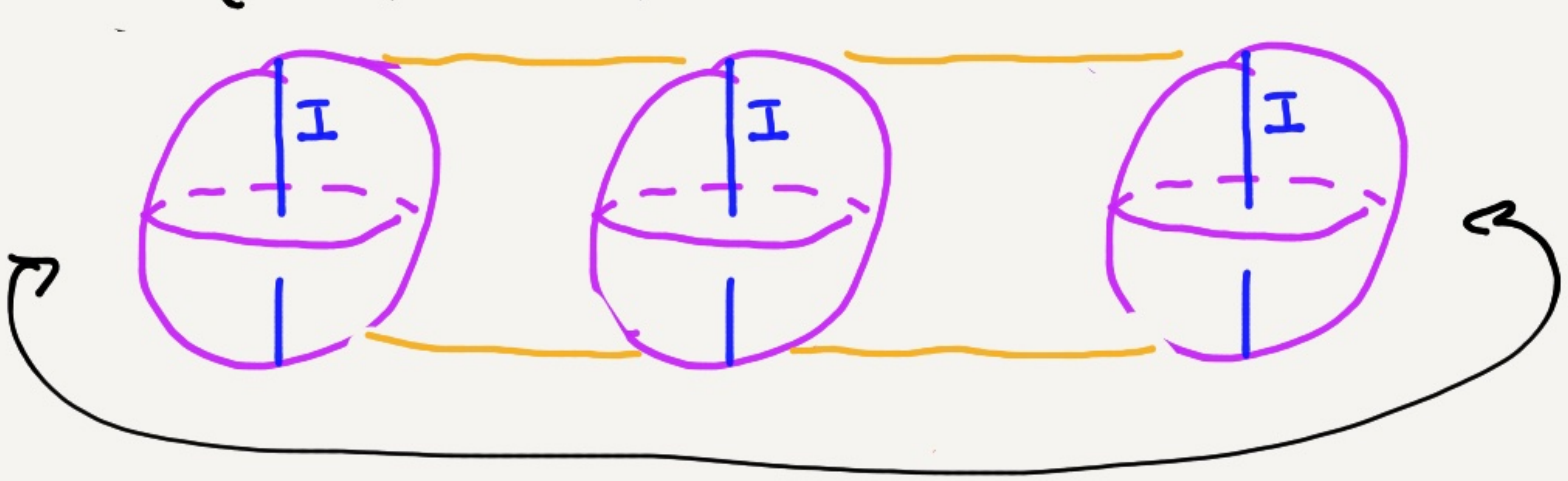

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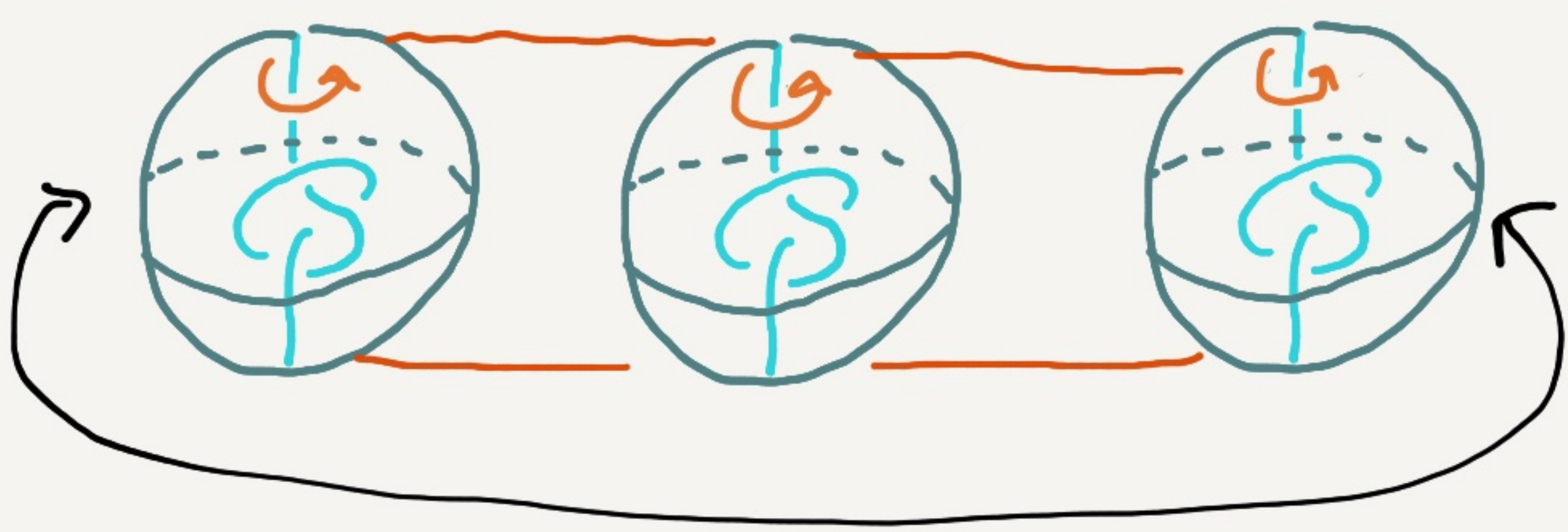

Knotted surfaces (with knot group $\mathbb{Z}$).

I. Reminders:

I.1 History of exotic surfaces

I.2 Rim surgery.

II Topological equivalence

II.1 History and methods

II.2 Some examples. (Emphasis on surgery theory).

II.1 TOP-equivalence

<u>Conventions</u>: From now on, we work in TOP.

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Take away: (nearly) All of the following results depend on Freedman's 5-d relative s-cobordism theorem.

recall (W^5, M_0^4, M_1^4) is a **relative s-cobordism** if

- $\partial W \cong M_0 \cup M_1 \cup \partial M_0 \times [0,1]$.
- $M_i \to W$ is a simple homotopy equivalence $i=0,1$.

Freedman: **If** $\pi_1(M_0)$ is "good", then W is homeomorphic to $M_0 \times [0,1]$.
In particular M_0 and M_1 are homeomorphic.

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- Kreck 1990 : If $\#_{i=1}^{10} \mathbb{RP}^2 \xhookrightarrow{F_0, F_1} S^4$ have $\pi_1(S^4 \setminus F_i) \cong \mathbb{Z}/2$ and $e(F_i) = 16$, then F_0 and F_1 are equivalent.

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<u>conventions</u>: Top manifolds are compact, connected, oriented.

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Apply Freedman's 5-d s-cobordism theorem to deduce that $M_0 \cong M_1$.

provided $\pi_1(M_0)$ is good.

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- It is enough to show that $S^4 \setminus S_1 \overset{\Phi}{\cong} S^4 \setminus S_2$ rel ∂:

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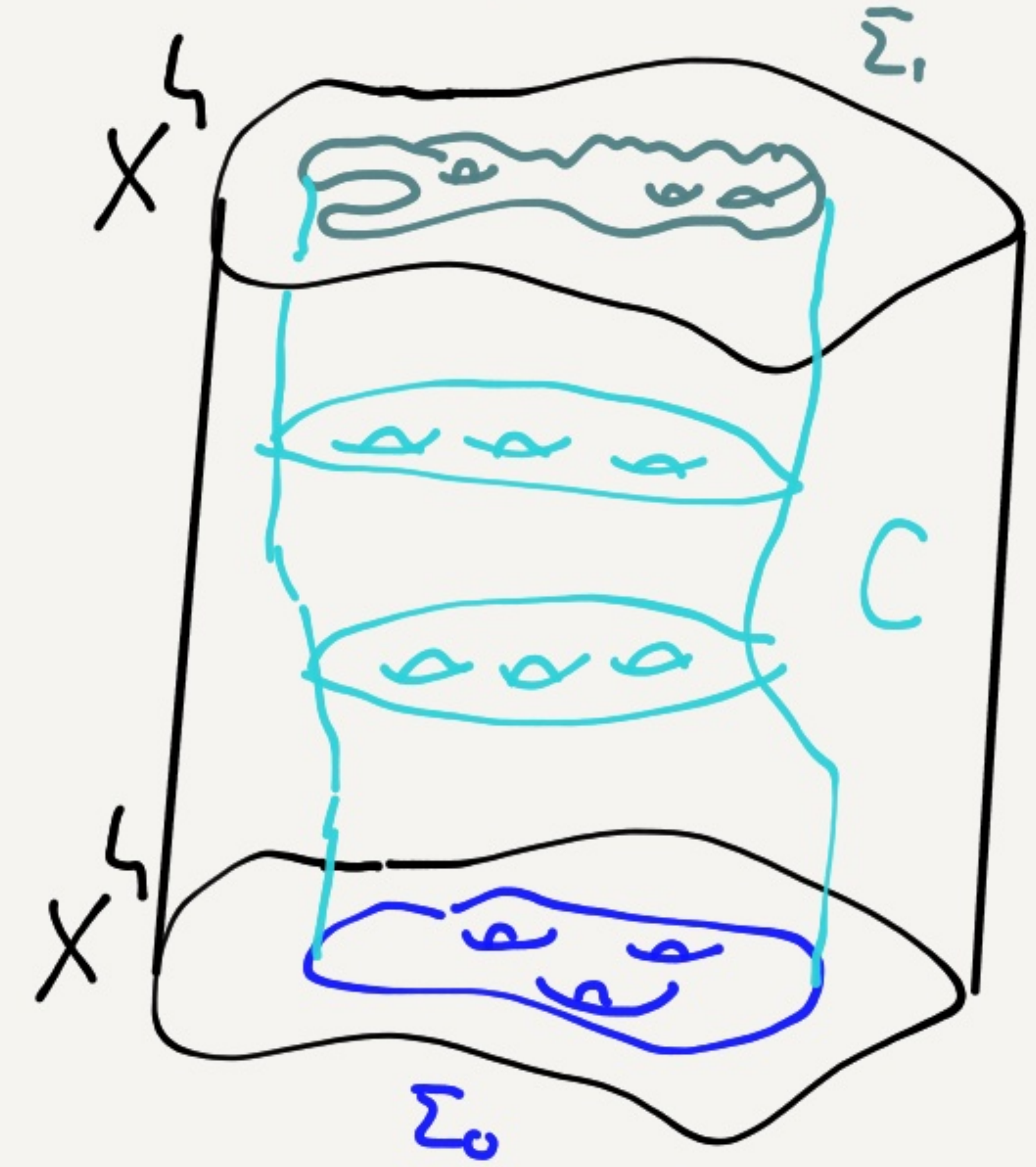

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<u>Theorem</u> (Sunukjian) Let X^4 be closed with $\pi_1(X) = 1$. If $\Sigma_0, \Sigma_1 \subset X$ are closed genus g surfaces with $[\Sigma_0] = [\Sigma_1] \in H_2(X)$,

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Miller, Klug-Miller also use this "Concordance $\Rightarrow$ isotopy" idea when studying $S^2 \xrightarrow{\Sigma_0, \Sigma_1} X^4$ $\pi_1(X) \neq 1$. Concordance is established "à la lightbulb".

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Knotted surfaces (with knot group $\mathbb{Z}$).

I. Reminders:

I.1 History of exotic surfaces

I.2 Rim surgery.

II Topological equivalence

II.1 History and methods

II.2 Some examples. (Emphasis on surgery theory).

III Knotted surfaces with knot group $\mathbb{Z}$ and ribbon discs.

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$$H_2(\widetilde{X}_\Sigma) \to H_2(\widetilde{X}_\Sigma, \partial \widetilde{X}_\Sigma) \to H^2(\widetilde{X}_\Sigma) \to \mathrm{Hom}_{\mathbb{Z}[t^{\pm 1}]}(H_2(\widetilde{X}_\Sigma), \mathbb{Z}[t^{\pm 1}]).$$

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Recap.

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- I will only discuss $\partial \neq \emptyset$ for $\mathbb{Z}$-discs $(\Sigma, K) \hookrightarrow (D^4, S^3)$.

The proof is not via the surgery programme from before.

Discs in $\mathbb{D}^4$.

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Is the situation for $BS(1,2)$ as for $\mathbb{Z}$?

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Discs in D^4.

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Remark: We have necessary and sufficient criteria for each scenario.

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no time for the definition nor for $|S_K| \leq 2$ today.

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Small lie: $\pi_1(X_K) \leftarrow \pi_1(S^3_0(K))$.

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(✱). If $P_{D_1} = P_{D_2}$, then there is an isomorphism $\pi_1(N_{D_1}) \xrightarrow{\cong} \pi_1(N_{D_2})$

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$N_{D_i} \simeq K(G,1)$.

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3) Analyze the surgery obstruction $\sigma(F) \in L_5^s(\mathbb{Z}[G])$.

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  1) Use (*) to build a rel ∂ homotopy equivalence $f : N_{D_1} \xrightarrow{\simeq} N_{D_2}$.
  2) Build a degree one normal map $(F, f, id) : (W, N_{D_2}, N_{D_1}) \longrightarrow (N_{D_1} \times [0,1], N_{D_1}, N_{D_1})$

$$n(f) = n(id) = \frac{\sigma(N_{D_1}) - \sigma(N_{D_1})}{8} = 0 \in \underbrace{H^4(N_{D_1}, \partial N_{D_1}, \mathbb{Z})}_{=\mathbb{Z}} \oplus \overbrace{H^2(N_{D_1}, \partial N_{D_1}, \mathbb{Z}_2)}^{=0}$$

  3) Show that $L_5^s(\mathbb{Z}[G]) = L_4(\mathbb{Z}[\mathbb{Z}]) = \mathbb{Z}$ and modify W by using $\pm E_8 \times S^1$ to get F' with $\sigma(F') = 0$

Discs in $\mathbb{D}^4$.

Theorem: (C-Powell): A knot K bounds at most two $BS(1,2)$-homotopy ribbon discs up to ambient isotopy rel ∂

proof sketch: $G := BS(1,2)$.

- Idea: Define an injection $\dfrac{\{G\text{-homotopy ribbon discs for } K\}}{\text{Ambient isotopy rel } \partial} \hookrightarrow S_K$. with $|S_K| \leq 2$.

The map is $D \longmapsto P_D := \mathrm{Ker}\left(H_1(\tilde{X}_K) \longrightarrow H_1(\tilde{N}_D)\right)$. (recall $\tilde{X}_K \to \underbrace{X_K}_{= S^3 \setminus K}$, $\tilde{N}_D \to \overbrace{N_D}^{= \mathbb{D}^4 \setminus \nu D}$: $\mathbb{Z}$-covers).

- Goal: If $P_{D_1} = P_{D_2}$, then D_1 and D_2 are isotopic rel ∂.
- Given G-ribbon discs D_1, D_2 for K, we show that

(*). If $P_{D_1} = P_{D_2}$, then there is an isomorphism

$$\begin{tikzcd} & \pi_1(X_K) \arrow[dl] \arrow[dr] & \\ \pi_1(N_{D_1}) \arrow[rr, "\cong"] & & \pi_1(N_{D_2}) \end{tikzcd}$$

- We now run the surgery program to show that $N_{D_1} \cong N_{D_2}$ rel ∂ ($\Rightarrow$ D_1 and D_2 are ambient iso rel ∂)
  1) Use (*) to build a rel ∂ homotopy equivalence $f : N_{D_1} \xrightarrow{\simeq} N_{D_2}$.
  2) Build a degree one normal map $(F, f, \mathrm{id}) : (W, N_{D_0}, N_{D_1}) \longrightarrow (N_{D_1} \times [0,1], N_{D_1}, N_{D_1})$

$$n(f) = n(\mathrm{id}) = \frac{\sigma(N_{D_1}) - \sigma(N_{D_1})}{8} = 0 \in \underbrace{H^4(N_{D_1}, \partial N_{D_1}, \mathbb{Z})}_{= \mathbb{Z}} \oplus \overbrace{H^2(N_{D_1}, \partial N_{D_1}, \mathbb{Z}_2)}^{= 0}$$

  3) Show that $L_5^s(\mathbb{Z}[G]) = \mathbb{Z}$ and modify W by using $\pm E_8 \times S^1$ to get F' with $\sigma(F') = 0$

Since G is good, apply the relative s-cobordism theorem $\Rightarrow N_{D_0} \cong N_{D_1}$ rel ∂.

Thank you.