Problems in 4-mfld topology (that I don't mind if you working

The following central problems are

1) True and known in TOP in all dims $n \geq 2$,
2) Unknown for PL/smooth and $n = 4$.

i) Pé conj $\quad S^n \simeq \Sigma^n_{CAT} \Rightarrow \Sigma \cong_{CAT} S^n$. $\quad$ codim 0

ii) Schoenflies conj $\quad S^{n-1} \subseteq_{CAT} S^n$ is CAT unknotted $\quad$ codim 1

(iii) Unknotting $\quad S^{n-2} \subseteq_{CAT} S^n$, $S^n \setminus S^{n-2} \simeq S^1$ $\quad$ codim 2
$\Rightarrow$ CAT unknotted.

(iv) $f: S^n \xrightarrow{\cong_{CAT}} S^n$ o.p $\Rightarrow f \simeq_{CAT} Id$

v) $M^n \simeq_{CAT} T^n \Rightarrow M^n \cong_{CAT} T^n$.

Key names: $\boxed{\text{Newman}}$ Brown, Lenino, Kirby-Siebenmann
$(n \geq 5)$ Hsiang-Shaneson, BNSW.

In PL/smooth $n \geq 4$, not generally true (i, iv, v), (ii), (iii) CAT indep.
$n < 4$, all Qs CAT indep.
must try to track names... many

Who proved (i)
$n \geq 5$, CAT = TOP ?

1, 2, 3, ?, 5, 6, 12, 56, 61 $\quad$ Maybe 126 ??
(no more odd dimensions)

The rest of the talk is on this

Chart of equivalent statements relating to 4D surgery conj. for TOP 4-mflds class^n.

All are open

Surgery exact seq (known for π_1 good)

Surgery for all stable Lagrangians, for every π_1

easy

surgery on some stable Lagrangian, $\pi_1 = F_2$

easy

Exactness at $N(X)$, $\pi_1 = F_2$

Surgery

Good boundary links are freely slice

obv

The family of links $\mathcal{L}$ is freely slice

Kirby diagrams

Good
Boundary link: $L = L_1 \cup \dots \cup L_m \subseteq S^3$. link

is a bdy link if $\exists$ disjointly embedded Seifert surfaces

$\Sigma_1 \cup \dots \cup \Sigma_m \subseteq S^3$ $\partial F_i = L_i$.

meridians

iff $\exists\ \phi : \pi_1(S^3 \setminus L) \twoheadrightarrow F_m$ with $\{\phi(\mu(L_i))\}$ generators.
Hm.

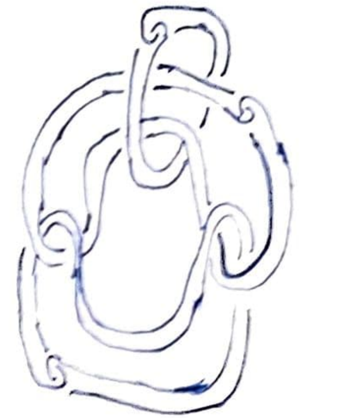
A boundary link is good if $\exists$ such a map $\phi : \pi_1(S^3 \setminus L) \twoheadrightarrow F_m$
(SBL) with ker ϕ perfect.

ie $H_1(S^3 \setminus L ; \mathbb{Z}[F_m]) = 0$.

Most general known result on SBLs

eg

Wh(J) where J is a lk no 0 link.



Thm (Cha-Kim-P.)
Building on work of Freedman + Krushkal

Every SBL with a boundary link Seifert surface admitting a HT$^+$ good basis is freely slice

FK - same without + $\exists$ surgery cond.

Freely slice

$\exists\ \sqcup g_i : \bigsqcup_{i=1}^{m} D^2 \hookrightarrow D^4$ lf emb

$g_i(\partial D^2) = L_i$ $\quad \pi_1(D^4 \setminus \cup g_i(\cup D^2)) \cong F_m$

gen by $\mu(L_i)$.

Surgery "exact sequence" This can help to classify 4-mfds. cf Kreck surgery has a similar reformulation.

$\mathcal{N}(X \times [0,1], X \times \{0,1\}) \to L_5(\mathbb{Z}\pi_1 X) \to \mathcal{S}^{TOP}(X) \to \mathcal{N}(X) \xrightarrow{\sigma} L_4(\mathbb{Z}\pi_1(X))$

image = stabiliser of action

action

structure set

DONM

L groups

surgery obstr map.

Exactness at $\mathcal{N}(X)$

Suppose $\sigma\begin{pmatrix} \nu_M & \xrightarrow{b} & ? \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{pmatrix} = 0 \in L_4$.

ie

$\lambda_f : K_2(f) \times K_2(f) \to \mathbb{Z}\pi_1 X$. int pairing. We

Then suppose _ some $P \subseteq K_2(f) \oplus \mathbb{Z}\pi_1^{2k}$, Lagrangian of

$\lambda_f \oplus H_+^k(\mathbb{Z}\pi)$,

Suppose $\exists k, l$: $\lambda_f \oplus H_+^k \cong H_+^l$

Can represent generators of P by a disjointly emb S^2s with trivial normal bundle. Then do surgery to change f to a $f' : M' \xrightarrow{\simeq} X$ by a normal bordism

$\Rightarrow$ $\exists$ exactness at $\mathcal{N}(X)$

Rmk Surgery on a given stable Lagrangian is used to define the L_5 action via Wall realisation / plumbing.

The family $\mathcal{L}$

Iterated, ramified Bing doubling on  $=: H$

followed by ramified Wh doubling on all components.

(at least one BD for some comp of H.)

eg Wh(Bor).

Implication: surgery on a Lagrangian / exactness at normal inv. $\Rightarrow$ SBLs freely slice.

Lemma

L freely slice iff 0-surgery $M_L = \partial W^4$

W top 4-mfld, $\vee \mu_{L_i} \cong \vee S^1 \xrightarrow{\simeq} W$.

Pf ($\Rightarrow$) If freely slice, Alexander duality + Hurewicz + Whitehead.

($\Leftarrow$) If $\exists\, W$, $W \cup_{\vee S^1 \times D^2} \bigcup^m D^2 \times D^2 =:$

Check $V \simeq *$

$\partial V \cong S^3$.

5d TOP h-cob thm (Quinn)

$\Rightarrow V \cong D^4 \Rightarrow L$ freely slice with complement W.

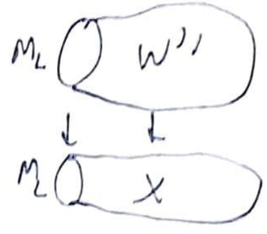
Let $(X, \partial) = (\#^m S^1 \times D^3, M_L)$

$$S(X,\partial) \to N(X,\partial) \xrightarrow{\sigma} L_4(\mathbb{Z}[F_m])$$

$$L_4(\mathbb{Z}[F_m]) \cong L_4(\mathbb{Z}) \oplus \bigoplus^m L_3(\mathbb{Z}) \cong L_4(\mathbb{Z}) \cong \mathbb{Z}$$

sign/8.

1) Find a normal bordism over (X,∂)



Use $\Omega_3^{fr}(X) \cong \bigoplus^m \Omega_2^{fr}$ detected by Arf invariants of components (which $= 0$)

$W'' \rightsquigarrow W'' \#^k \pm E_8 =: W'$ so that $\sigma(W') = 0 \in L_4(\mathbb{Z})$. $\cong \mathbb{Z}$.

Exactness $\Rightarrow$ can surger W' to

$$\begin{array}{ccc} W & \xrightarrow{\simeq} & X \\ \uparrow & & \uparrow \\ M_L & \xrightarrow{\cong} & M_L \end{array}$$

$\Rightarrow L$ freely slice by Lemma.

Implication

$\mathcal{L}$ freely slice $\Rightarrow$ stable surgery on a given Lagrangian.
$\forall L \in \mathcal{L}$.

$$\begin{array}{ccc} \partial M & \to & M \\ \downarrow{\scriptstyle \partial f} & & \downarrow{\scriptstyle f} \\ \partial X & \to & X \end{array}$$

DONM. trivial surgeries on $S^1 \subseteq M$

$M \rightsquigarrow M \# S^2 \times S^2$

stabilise $(K(f), \lambda) \rightsquigarrow (K(f) \oplus \mathbb{Z}[\pi_1 X]^2, \lambda_f \oplus H_+)$

$K(f) \cong P \oplus P^* \oplus Q$

Represent: $P \subseteq K(f)$ by immersed spheres

$f_i : S^2 \looparrowright M$

and P^* dual Lagrangian too

$g_i : S^2 \looparrowright M$.

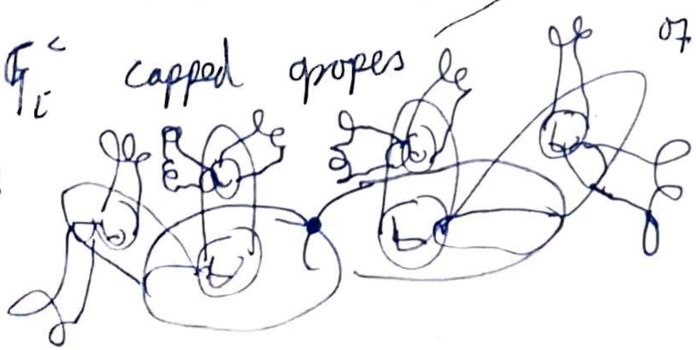
Lemma

Can represent $P \oplus P^*$ by

$\cup \{F_i^c\} \cup \{G_i^c\}$ (4D submanifolds of arbitrary height.)

F_i^c, G_i^c capped gropes

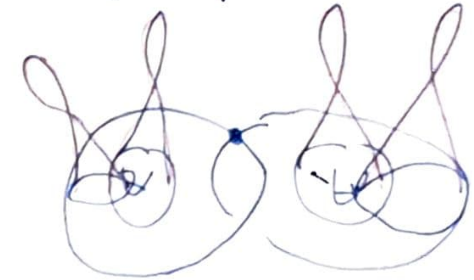
2D spine:



Want embedded sphere in here

Make Kirby diagram

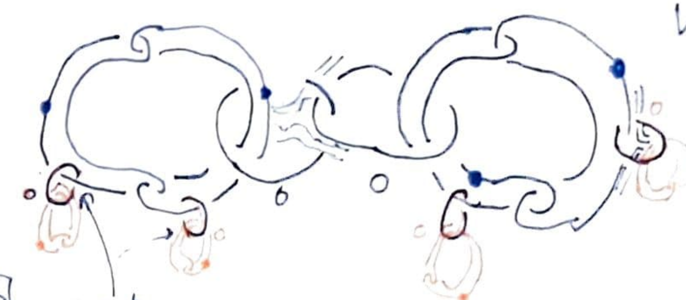
simplest ex;



nbhd of this

=

W



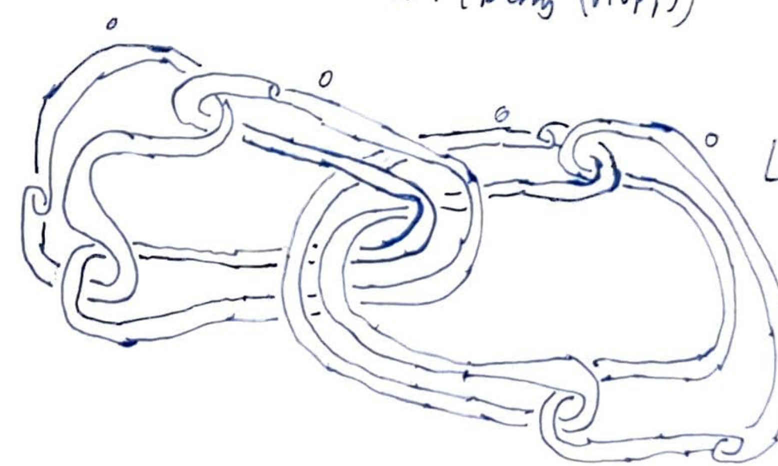
What is? this ∂;

want to cancel.

Slide to produce

$Wh(Bing(Hopf))$

$\partial W = M_L$



$L \in \mathcal{L}$.

L freely slice

$\exists M_L = \partial W'$

$W' \simeq \vee S^1$

$M \setminus W \cup W' = M'$

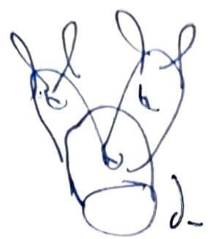
$$W \downarrow \natural S^1 \times D^3 \quad \sim_{\text{normally bordant}} \quad W' \downarrow \natural S^1 \times D^3$$

so can plug this in to get a normal bordism over X.

What about good groups?

π_1 null disc property

A group Γ is **good** if for every disc-like capped grope G_2^c of height 2



and every HM

$$\phi : \pi_1(G_2^c) \to \Gamma,$$

$\exists$ an immersed disc $F: D^2 \looparrowright G_2^c$ with

$$\partial D^2 \mapsto \partial_- G_2^c$$

$$\phi\left(\boxed{\pi_1\text{-null}}\;\pi_1(F(D^2))\right) = \{e\} \subseteq \Gamma.$$

n.b. This is a smooth question

Conjecture Every group is good.

elementary amenable gps are good.

Thm i) Every finite gp is good

(ii) $\mathbb{Z}$ is good

(iii) Good groups are closed under extensions, subgroups, quotients, colimits, $\Gamma \underset{f.i.}{<} \Gamma'$.

Thm Every group is good iff Γ_1 is good

Thm If every group is good then surgery, s-cob. thm holds for top. 4-mflds, $\forall \pi_1$.

Alternative conjecture. Amenable groups are good