Characteristic classes of manifold bundles

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Diffeomorphism groups and their classifying spaces

All manifolds M in this talk are *smooth* and *connected* (unless explicitly specified otherwise).

We allow $\partial M \neq \emptyset$. Issues with orientations will be mostly neglected; if they play a role will be mentioned occasionally.

We denote the group of diffeomorphisms of M with $\mathrm{Diff}(M)$. It has a natural topology (due to Whitney, often called C^∞) that takes all derivatives into account. A sequence $(f_j \in \mathrm{Diff}(M))_{j=1}^{\infty}$ converges iff all derivatives converge.

It thus makes sense to consider the classifying space $B\mathrm{Diff}(M)$. Caveat: in general, $B\mathrm{Diff}(M) \not\simeq B\pi_0\mathrm{Diff}(M), B\mathrm{Diff}^\delta(M)$.

We are interested in the homotopy type of $B\mathrm{Diff}(M)$; it carries more information than $\mathrm{Diff}(M)$ seen as a space or as a group. One can say it combines both structures into one.

Naive guess: if M is endowed with a "good" Riemannian metric g, then $\mathrm{Isom}(M,g) \to \mathrm{Diff}(M)$ is a weak equivalence.

Look at this for spheres:

$$\begin{array}{ccccc} & & & \swarrow & SO(d+1) \\ & & & & \downarrow \sim \\ \mathrm{Diff}_\partial(D^d) & \longrightarrow & \mathrm{Diff}^+(S^d) & \longrightarrow & \mathrm{Fr}^+(S^d) \end{array}$$

Hence we get

$$SO(d+1) = \mathrm{Isom}^+(S^d, g_{\mathrm{round}}) \xrightarrow{\sim} \mathrm{Diff}^+(S^d) \Leftrightarrow \mathrm{Diff}_\partial(D^d) \sim *.$$

This is satisfied $d = 1, 2, 3$ (folklore, Smale, Hatcher) but fails for all $d \geq 4$ (Kervaire–Milnor, Novikov, Burghelea-Lashof,..., in higher dimensions; Watanabe in dimension 4).

By an old theorem of Ehresmann, any proper submersion $\pi\colon E \to B$ is a *smooth fiber bundle*, i.e., for $M = \pi^{-1}(b)$ a/the fiber, we can identify π with $M \times_{\mathrm{Diff}(M)} P \to B$ for $P \to B$ a principal $\mathrm{Diff}(M)$-bundle.

We thus get (for all reasonable spaces B):

$$\begin{aligned} & \{\text{Smooth } M\text{-bundles } E \to B\}/\text{concordance} \\ \cong\; & \{B \to B\mathrm{Diff}(M)\}/\text{homotopy}, \end{aligned}$$

and $\mathrm{H}^*(B\mathrm{Diff}(M))$ is the ring of characteristic classes of smooth M-bundles.

Our goal is to understand the homotopy type of $B\mathrm{Diff}(M)$ and the ring of characteristic classes $\mathrm{H}^*(B\mathrm{Diff}(M))$ as well as possible.

The 1st (classical) approach: surgery and K-theory

This approach tries to compare $\mathrm{Diff}(M)$ to $\mathrm{Aut}(M)$, the homotopy automorphisms of M which form a "group up to homotopy".

Introduce $B\widetilde{\mathrm{Diff}}(M)$, the classifying space for *block bundles*. Then we have a factorization $B\mathrm{Diff}(M) \to B\widetilde{\mathrm{Diff}}(M) \to B\mathrm{Aut}(M)$.

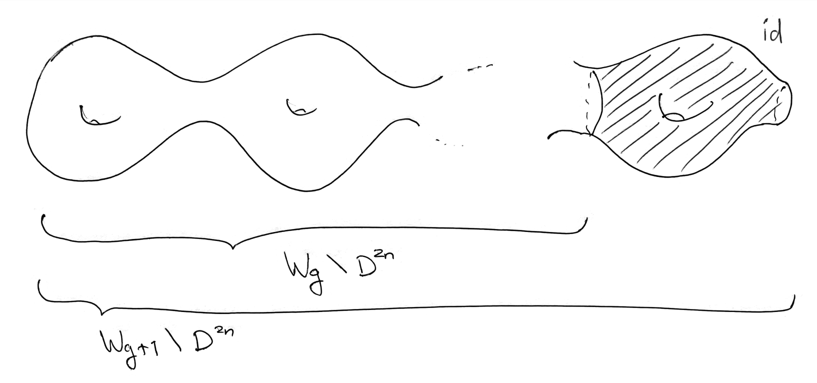
- The homotopy fiber of $B\widetilde{\mathrm{Diff}}(M) \to B\mathrm{Aut}(M)$ can be accessed via surgery theory.
(Quinn's space-level version of the surgery exact sequence)

- The homotopy fiber of $B\mathrm{Diff}(M) \to B\widetilde{\mathrm{Diff}}(M)$ can be accessed via Waldhausen's algebraic K-theory of spaces.
(Hatcher, Weiss–Williams)

The 2nd approach: homological stability and moduli spaces

This approach was coined by Galatius and Randal-Williams, following the seminal work of Madsen–Weiss. We denote $W_g := \#^g S^n \times S^n$. In dimension 2, this is a surface of genus g.

We have maps $\mathrm{Diff}_\partial(W_g \backslash D^{2n}) \to \mathrm{Diff}_\partial(W_{g+1} \backslash D^{2n})$.



Theorem (Harer in dimension 2, G–R-W in high dimensions)
$B\mathrm{Diff}_\partial(W_g \backslash D^{2n}) \to B\mathrm{Diff}_\partial(W_{g+1} \backslash D^{2n})$ *induces and isomorphism in homology in degrees* $\leq\sim g/2$ *(*$\leq\sim 3/2g$ *for surfaces)*.

The 0-dimensional analogue is due to Nakaoka and states that $\mathrm{H}_* B\Sigma_g \xrightarrow{\sim} \mathrm{H}_* B\Sigma_{g+1}$ in degrees $\leq\sim 1/2g$

N. Perlmutter proved analogues in odd dimensions.

This phenomenon is called *homological stability*.

Once homological stability is established, we need to understand the stable homology.

In dimension 0, this follows from Barratt–Priddy–Quillen which we can be sated as $\Omega B\left(\coprod_{g \geq 0} B\Sigma_g\right) \sim \mathbf{Z} \times \Omega^\infty_0 \mathbf{S} = \Omega^\infty \mathbf{S}$.

We will later sketch what happens in higher dimensions. Key: specific model for $B\mathrm{Diff}(M)$ deserving the name "moduli space":

$B\mathrm{Diff}(M) \sim \{N \subset (0,1)^\infty \mid N \text{ smooth manifold}, N \cong M\}$

Tautological classes

What classes in $\mathrm{H}^*(B\mathrm{Diff}(M))$ can we define?

For an oriented smooth M^{2n}-bundle $\pi\colon E \to B$, we have:

• The vertical tangent bundle $T_\pi := \ker(TE \to \pi^* TB)$ over E, classified by a map $E \to BSO(2n)$.

• The fiber integration map $\int_\pi\colon H^*E \to H^{*-d}B$. If B is a smooth manifold, this is $\mathrm{PD}_B \circ \pi_* \circ \mathrm{PD}_E$.

Thus for $c \in \mathrm{H}^*(BSO(2n); \mathbf{Q}) = \mathbf{Q}[p_1, p_2, \ldots, p_n, e]/\langle e^2 - p_n \rangle$ we can define $\kappa_c := \int_\pi c(T_\pi)$.

This is called *tautological* or *generalized Miller–Morita–Mumford* or simply κ-class.

For surface bundles ($n = 1$) we only have $\kappa_i := \kappa_{e^{i+1}}$ of degree $2i$.

Madsen–Weiss

Unlike in higher dimensions, we have that the components of $\mathrm{Diff}_\partial(S)$ are contractible for S a surface ($\partial S \neq \emptyset$ allowed) other than S^2, T^2 (Earle–Eells).

Thus, $B\mathrm{Diff}_\partial(S_g \backslash D^2) \sim B\Gamma_{g,1}$, with $\Gamma_{g,1}$ the mapping class group.

Theorem (Madsen–Weiss)

$\mathrm{H}^*_{stable}(B\Gamma_{g,1}) = \mathbf{Q}[\kappa_1, \kappa_2, \ldots]$

Theorem (Galatius–Randal-Williams)

For $2n \geq 4$*,*

$\mathrm{H}^*_{stable}(B\mathrm{Diff}_\partial(W_g \backslash D^{2n})) = \mathbf{Q}[\kappa_c, c \textit{ monomial in } p_i, i > \frac{n}{4}, e]$

Cobordism Categories

An important ingredient for the proofs of these results is the concept of cobordism categories: $\mathcal{C}_d$ is a topological category whose

- objects are closed $(d-1)$-manifolds
- morphisms are cobordisms, embedded in $(0,t) \times (0,1)^{\infty-1}$

(Endowed with tangential structures; for instance, orientations).

Theorem (Galatius–Madsen–Tillmann–Weiss)
$B\mathcal{C}_d \sim \Omega^{\infty-1}\mathbf{MTO}(d)$.

In case this is completely new to you, I recommend the lecture notes on talks on the Madsen–Weiss theorem by S. Galatius, available on N. Wahl's website.

We will sketch the idea of the proof for the 0-dimensional case.

The 0-dimensional cobordism category is the E_∞ algebra

$$\{C \subset (0,1)^\infty \mid C \text{ finite}\} \sim \coprod_g B\Sigma_g$$

Introduce the E_{n-k}-algebras (for $0 \leq k \leq n$)

$$\Psi_{n,k} = \{C \subset \mathbf{R}^k \times (0,1)^{n-k} \mid C \text{ discrete in } \mathbf{R}^n\}$$

Points are allowed to "disappear at ∞".

Using simplicial methods, one can prove that $B\Psi_{n,k} \sim \Psi_{n,k+1}$.

For $k \geq 1$, $\Psi_{n,k}$ is connected, hence $\Psi_{n,k} \sim \Omega\Psi_{n,n}$.

We thus get $B\Psi_{n,0} \sim \Psi_{n,1} \sim \Omega\Psi_{n,2} \sim \cdots \sim \Omega^{n-1}\Psi_{n,n}$.

We will explain below that $\Psi_{n,n} \sim S^n$, then
$B\left(\coprod_g B\Sigma_g\right) \sim \text{colim}_n \Omega^{n-1} S^n = \Omega^{\infty-1}\mathbf{S}$. (This is BPQ)

Scanning

We want to understand the homotopy type of

$$\Psi_{n,n} = \{C \subset \mathbf{R}^n \mid C \text{ discrete}\} \overset{\sim}{\hookleftarrow} \{C \subset \mathbf{R}^n \mid |C| \leq 1\} \sim S^n$$

With a similar argument, we can prove

$$\begin{aligned}\{N \subset \mathbf{R}^n \mid N\ d\text{-manifold}\} \overset{\sim}{\hookleftarrow} \{V \subset \mathbf{R}^n \mid V \text{ affine } d\text{-plane}\} \cup \emptyset \\ = \mathrm{Th}(\mathrm{Gr}_d^{\perp}(\mathbf{R}^n))\end{aligned}$$

By definition, $\mathbf{MTO}_n = \Omega^n \mathrm{Th}(\mathrm{Gr}_d^{\perp}(\mathbf{R}^n))$.

Tautological classes & group actions

Recall: the tautological ring of M is sub-ring $\mathcal{R}^*(M) \subset \mathrm{H}^*(B\mathrm{Diff}^+(M); \mathbf{Q})$ generated by κ-classes.

Stable results only describe it in a range. What if we are interested in algebraic properties of $R^*(M)$?

Theorem (Randal-Williams)

If $T^k \curvearrowright M^{2n}$ effectively and such that either

(a) $\chi(M) \neq 0$ *and* M^{T^k} *is connected, or*

(b) M^{T^k} *is discrete and non-empty,*

then K-dim$(R^*(M)) \geq k$.

Theorem (Galatius–Grigoriev–Randal-Williams)

In dimensions $4m + 2 \geq 6$*, we have*
$R^*(W_g)/\sqrt{0} = \mathbf{Q}[\kappa_{ep_1}, \kappa_{ep_2} \dots, \kappa_{ep_{n-1}}]$ *for* $g > 1$.

G (connected) Lie group. When is the assignment

$$\{\text{Smooth actions } G \curvearrowright M\} \to \{\text{Smooth } M\text{-bundles } E \to BG\}$$

surjective?

Theorem (R.)

For $G = SU(2), M = W_g$, it is not.

I do not know if this can also happen for $G = S^1$.

The 3rd approach: embedding calculus

As this was central to block I, I won't say what embedding calculus à la Weiss *is*.

The reason it is relevant is that we have $\mathrm{Diff}_\partial(M) = \mathrm{Emb}_\partial(M) \to T_\infty \mathrm{Emb}_\partial(M)$.

For $T_\infty \mathrm{Emb}(M, N)$, convergence requires codim ≥ 3.

If we allow the boundary to move, the right dimension count is handle-dim(M), geom-dim(N).

handle-dim$_{\frac{1}{2}\partial}(W_g \backslash D^{2n}) = n$, geom-dim$(W_g \backslash D^{2n}) = 2n$, so $\mathrm{Diff}_{\frac{1}{2}\partial}(W_g \backslash D^{2n})$ is susceptible to embedding calculus.

Playing this off against the results by Galatius–Randal-Williams on $\mathrm{Diff}_\partial(W_g \backslash D^{2n})$ has led to several breakthrough recently.

The most important tool is the *Weiss fiber sequence* that will feature in forthcoming talks.

Diffeomorphisms of discs

We already saw that studying the homotopy type and (co)homology of $B\mathrm{Diff}_\partial(D^d)$ is of fundamental importance.

Morlet: for $d \geq 5$, $B\mathrm{Diff}_\partial(D^d) \sim \Omega^d TOP(d)/O(d)$.

There is a fiber sequence $B\mathrm{Diff}_\partial(D^{d+1}) \to B\mathcal{C}(D^d) \to B\mathrm{Diff}_\partial(D^d)$ with $\mathcal{C}$ meaning concordances.

Waldhausen + Igusa: there is a map $B\mathcal{C}(D^d) \to \Omega A(*) \sim_{\mathbf{Q}} \Omega K(\mathbf{Z})$ that is a homology isomorphism in some range, growing with d. (The range was recently improved significantly by M. Krannich)

This leads to non-trivial classes in $\mathrm{H}^{4j}(B\mathrm{Diff}_\partial(D^{2n+1}); \mathbf{Q})$ (Farrell–Hsiang, known for more than 40 years).

T. Watanabe has discovered unrelated non-trivial classes in $\pi_{\ell(2n-2)} B\mathrm{Diff}^{\mathrm{fr}}_{\partial}(D^{2n+1}) \otimes \mathbf{Q}$ and $\pi_{\ell(4n-6)} B\mathrm{Diff}^{\mathrm{fr}}_{\partial}(D^{2n}) \otimes \mathbf{Q}$ that are related to graph complexes. (about 10 years ago)

M. Weiss's discovery that more Pontryagin classes survive under $BTOP(d) \to BTOP \sim BO$ than in $BO(d)$ gives yet different classes in $\mathrm{H}^{4i-d}(B\mathrm{Diff}^{\mathrm{fr}}_{\partial}(D^{d}); \mathbf{Q})$. (even more recently)

Kupers–Randal-Williams have used deep results on Torelli spaces to prove that 'outside some bands', the cohomology of $B\mathrm{Diff}_{\partial}(D^{2n})$ consists only of K-theory and 'Dalian' classes.

Several people are currently working on better understanding whether and if so, how these different classes in $\mathrm{H}^{*}(B\mathrm{Diff}_{\partial}(D^{d}); \mathbf{Q})$ constitute the whole cohomology.