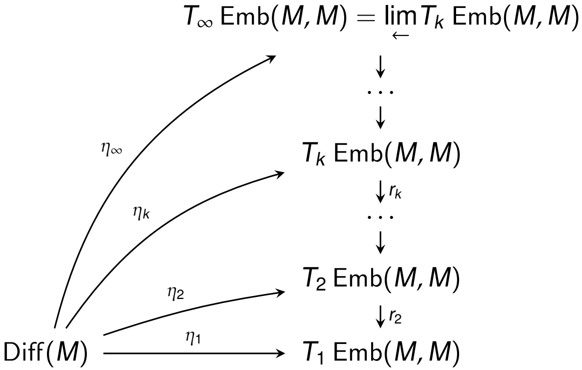
Self-embedding calculus and tautological classes

Nils Prigge

Self-embedding calculus

Let M^d be a smooth closed manifold, then $\mathrm{Diff}(M) = \mathrm{Emb}(M, M)$. Hence, we can study the space of diffeomorphisms through the Taylor approximations $\eta_k : \mathrm{Diff}(M) \to T_k\,\mathrm{Emb}(M, M)$.



Self-embedding calculus

Three themes from Jens' talk:

(i) Instead of $\mathrm{Diff}(M)$ we can study the classifying space $\mathrm{B\,Diff}(M)$;

(ii) Tautological classes: For smooth oriented fibre bundle $\pi : E \to B$ with fibre M^d define for $c \in H^{|c|}(\mathrm{B\,SO}(d))$

$$\kappa_c(\pi) := \int_\pi c(T_\pi E) \in H^{|c|-d}(B);$$

Any such construction that is natural under pullback determines a characteristic class in $H^*(\mathrm{B\,Diff}(M))$.

(iii) If $\kappa_c(\pi) \neq 0$ for any fibre bundle $\pi : E \to B$ with fibre M, then $0 \neq \kappa_c \in H^*(\mathrm{B\,Diff}^+(M))$.

Goal for this talk:

(i) Construct a delooping of the self-embedding tower;

(ii) Extend the construction of tautological classes over the delooping of the self-embedding tower;

(iii) Obtain information about $(\mathrm{B})T_k\,\mathrm{Emb}(M,M)$ similar as in (iii) by showing that certain tautological classes are non-trivial.

Question

How good is the approximation $\eta_k : \mathrm{Diff}(M) \to T_k\,\mathrm{Emb}(M,M)$?

Part I - Delooping the self-embedding tower

Classifying spaces

For a topological monoid/E_1-algebra M one can construct a space $\mathrm{B}\,M$. If $\pi_0(M)$ is a group, then $\Omega\,\mathrm{B}\,M \simeq M$.

- Recall one possible description of the Taylor tower

$$T_k\,\mathrm{Emb}(M,N) = \mathbb{R}\,\mathrm{Map}_{\mathrm{PSh}(\mathrm{Disk}_k)}(\mathrm{Emb}(-,M),\mathrm{Emb}(-,N))$$

  due to Boavida de Brito and Weiss.
- $T_k\,\mathrm{Emb}(M,M)$ is a derived endomorphism space and thus a monoid under compositions — for a suitable choice of derived mapping spaces.

  Remark: Such a description is very good if you want to study the delooping with tools from homotopy theory.
- Goal: Give a more geometric/concrete description.

The Haefliger model of embedding calculus

The following model is due to Goodwillie-Klein-Weiss inspired by work of Haefliger and Dax. All maps are smooth and mapping spaces have C^∞-topology.

- The first Taylor approximation is

$$T_1 \operatorname{Emb}(M,N) = \left\{ \begin{array}{ccc} TM & \xrightarrow{\bar{f}} & TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array} \middle| \bar{f} \text{ linear vb. monomorphism} \right\}$$

The Haefliger model of embedding calculus

- The second Taylor approximation is the homotopy pullback

$$\begin{array}{ccc} T_2\,\mathrm{Emb}(M,N) & \longrightarrow & \mathrm{Map}(M,N) \\ \downarrow & & \downarrow \\ \mathrm{IvMap}(M^2,N^2) & \longrightarrow & \mathrm{Map}^{S_2}(M^2,N^2) \end{array}$$

where

$$\mathrm{IvMap}(M^2,N^2) := \{F \in \mathrm{Map}^{S_2}(M^2,N^2) \,|\, (DF)^{-1}(T\Delta_N) = T\Delta_M\}$$

is the space of strongly isovariant maps. Define

$$T_2\,\mathrm{Emb}(M,N) := \left\{ H \in \mathrm{Map}^{S_2}(M^2,N^2)^I \,\middle|\, \begin{array}{l} H_0 \in \mathrm{Map}(M,N) \\ H_1 \in \mathrm{IvMap}(M^2,N^2) \end{array} \right\}$$

The Haefliger model of sembedding calculus

- For $k \geq 3$ there is a similar description

$$T_k \operatorname{Emb}(M, N) \subset \operatorname{Map}(\Delta^{k-1}, \operatorname{Map}(M^k, N))$$

  such that the restriction of $F \in T_k \operatorname{Emb}(M, N)$ to the faces $\sigma \subset \Delta^{k-1}$ have image in certain subspaces of $\operatorname{Map}(M^k, N)$.
- For $k \geq 2$ the map $\eta_k : \operatorname{Emb}(M, N) \to T_k \operatorname{Emb}(M, N)$ assigns an embedding $i : M \to N$ the constant map $\operatorname{const}_{i \circ \pi_1}$. For $k = 1$ it is defined as $\eta_1(i) = Di$.
- The restriction map $T_2 \operatorname{Emb}(M, N) \to T_1 \operatorname{Emb}(M, N)$ assigns to a strongly isovariant map $F : M^2 \to N^2$ the induced bundle monomorphism of normal bundles $\nu(\Delta_M) \to \nu(\Delta_N)$.

Advantages

- Very concrete and potentially amenable to geometric arguments;
- $\operatorname{Diff}(M)$ acts continuously on $T_k \operatorname{Emb}(M, N)$ by pre-composition;

The Haefliger model of self-embeddings

...is obviously a monoid for $k \leq 2$

For $M = N$, the Haefliger model for $k = 1, 2$ is

$$T_1 \operatorname{Emb}(M, M) = \left\{ \begin{array}{ccc} TM & \xrightarrow{\bar{f}} & TM \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M \end{array} \middle| \bar{f} \text{ linear vb. monomorphism} \right\}$$

$$T_2 \operatorname{Emb}(M, M) = \left\{ H \in \operatorname{Map}^{S_2}(M^2, M^2)^I \middle| \begin{array}{l} H_0 \in \operatorname{Map}(M, M) \\ H_1 \in \operatorname{IvMap}(M^2, M^2) \end{array} \right\}$$

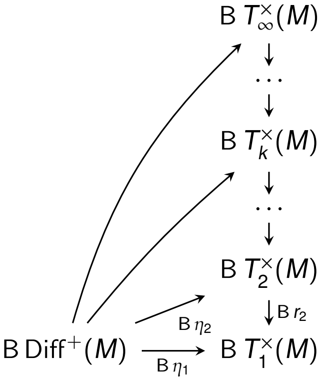
are monoids under composition.

Definition

$T_k^\times(M) \subset T_k \operatorname{Emb}(M, M)$ is the union of path-components that are homotopy invertible under composition. If M is oriented, then we further impose that the image in $T_1^\times(M)$ is contained in the orientation preserving tangential homotopy equivalences.

$$\eta_k : \operatorname{Diff}^+(M) \longrightarrow T_k^\times(M) \subset T_k \operatorname{Emb}(M, M)$$

Delooping the self-embedding tower



Definition

A TM-fibration is a fibration $\pi : E \to B$ with fibre M and a vector bundle $T_\pi E \to E$ such that the restriction to the fibres $T_\pi E|_{\pi^{-1}(b)}$ is equivalent to the tangent bundle TM.

Theorem (Berglund, May)

$\mathrm{B}\, T_1^\times(M)$ *classifies oriented* TM*-fibrations.*

Consequence: We can define tautological classes/generalized MMM-classes $\kappa_c \in H^*(\mathrm{B}\, T_k^\times(M))$ for all $k \geq 1$.

Part II - Studying $\mathrm{B}\,T_k^\times(M)$ through tautological classes

Question: Can we detect the difference between $\mathrm{B}\,\eta_k : \mathrm{B}\,\mathrm{Diff}^+(M) \to \mathrm{B}\,T_k^\times(M)$ with tautological classes (ideally for $k = \infty$)?

- Reformulation: Is there a relation among tautological classes that holds for fibre bundles but not over $\mathrm{B}\,T_k^\times(M)$?
- Most relations among tautological classes (that we know) depend only on the underlying fibration or even just on $\dim H^*(M)$. Hence, these also hold on $\mathrm{B}\,T_k^\times(M)$.
- One of the few relations that uses the manifold structure is based on the signature theorem.

Theorem. (Hirzebruch) Let M^{4k} be a closed oriented manifold, then the signature is given by $\mathrm{sgn}(M) = \langle L_k(TM), [M] \rangle$ where $L_k \in H^{4k}(\mathrm{B}\,\mathrm{SO}; \mathbb{Q})$.

$$L_1 = \frac{p_1}{3} \qquad L_2 = \frac{7p_2 - 4p_1^2}{45} \qquad L_3 = \frac{1}{945}(62p_3 - 13p_1p_2 + 2p_1^3) \quad \ldots$$

The family signature theorem

Trick: Since $H_*(X;\mathbb{Q}) \cong \Omega^{\mathrm{fr}}(X) \otimes \mathbb{Q}$ we can define a cohomology class $H^i(X;\mathbb{Q})$ by defining its evaluation on elements $[f : N^i \to X, \xi] \in \Omega_i^{\mathrm{fr}}(X) \otimes \mathbb{Q}$.

Definition (Signature classes)

For d even define for all $i + d \equiv 0 \bmod 4$ classes $\sigma_i \in H^i(\mathrm{B\,Diff}^+(M^d);\mathbb{Q})$ which assign to $[f : N^i \to \mathrm{B\,Diff}^+(M), \xi] \in \Omega^{\mathrm{fr}}(\mathrm{B\,Diff}^+(M)) \otimes \mathbb{Q}$ the signature of the pullback bundle $\mathrm{sgn}(f^*E)$.

Theorem (Family signature theorem)

Let $\pi : E \to B$ be a fibre bundle with fibre M^d a closed, oriented manifold. Then

$$\kappa_{L_i} = \begin{cases} \sigma_{4i-d} & \text{if } d \text{ is even} \\ 0 & \text{if } d \text{ is odd} \end{cases}$$

Studying $\mathrm{B}\, T_k^\times(M)$ through tautological classes

- The signature of the total space of an oriented M-fibration $\pi : E \to B$ only depends on the local coefficient system $H^{d/2}(\pi^{-1}(b); \mathbb{Z})$ over B (due to Meyer).
- Hence, σ_i is pulled back from a class in $H^i(\mathrm{B}\,\mathrm{Aut}(H^{d/2}(M), \langle , \rangle); \mathbb{Q})$.
- In particular, it can be pulled back to $\sigma_i \in H^i(\mathrm{B}\, T_k^\times(M); \mathbb{Q})$.

Question

Does the family signature theorem hold on $\mathrm{B}\, T_k^\times(M)$ (ideally for $k = \infty$), i.e. is $\kappa_{L_i} = \sigma_{4i-2d} \in H^{4i-2d}(\mathrm{B}\, T_k^\times(M^{2d}); \mathbb{Q})$?

Theorem (P.)

The family signature theorem does not hold on $\mathrm{B}\, T_2^\times(M^{2d})$*. For* M^{2d} *smooth, closed oriented* $0 \neq \kappa_{L_i} - \sigma_{4i-2d} \in H^{4i-2d}(\mathrm{B}\, T_2^\times(M); \mathbb{Q})$ *for* $d < 2i \leq 2d - 2$.

A sketch of the proof

- Find a space X with a map $X \to \mathrm{B}\, T_2^{\times}(M)$ for which we can compute the pullback of the MMM-classes.
-

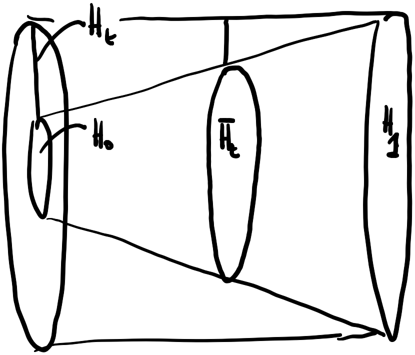
$$\mathcal{G}(TM) = \left\{ \begin{array}{ccc} TM & \xrightarrow{\bar{f}} & TM \\ \downarrow & & \downarrow \\ M & \xrightarrow{\mathrm{Id}} & M \end{array} \middle| \bar{f} \text{ vb. isomorphism} \right\}$$

$$h\mathcal{G}^{S_2}(S(TM)) = \left\{ \begin{array}{ccc} S(TM) & \xrightarrow{\bar{f}} & S(TM) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\mathrm{Id}} & M \end{array} \middle| \bar{f} \text{ fibrewise } S_2\text{-htpy eq.} \right\}$$

$$F(M,2) := \mathrm{hofib}_{\mathrm{Id}}(\mathcal{G}(TM) \longrightarrow h\mathcal{G}^{S_2}(S(TM)))$$

A sketch of the proof

Claim: There is a map $F(M,2) \longrightarrow T_2^\times(M)$.



Recall:

- An element $H \in F(M,2)$ is a map $H : S(TM) \times I \to S(TM)$ such that H_t is S_2-equivariant over Id_M, H_0 is linear and $H_1 = \mathrm{Id}$.
- An element $F \in T_2^\times(M)$ is a map $F : M^2 \times I \to M^2$ such that F_t is S_2-equivariant, H_0 is strongly isovariant and $H_1 = f \times f$.

Observation: This map deloops $\mathrm{B}\, F(M,2) \longrightarrow \mathrm{B}\, T_2^\times(M)$.

A sketch of the proof

Easier to study $\mathrm{B}\,F(M,2)$:

1.) $B\mathcal{G}(TM) = \mathrm{Map}(M, \mathrm{B}\,\mathrm{SO}(d))_{TM}$
2.) $Bh\mathcal{G}^{S_2}(S(TM)) = \mathrm{Map}(M, \mathrm{B}\,\mathrm{hAut}^{S_2}(S^{d-1}))_{S(TM)}$

For d even:

- $\mathrm{B}\,\mathrm{SO}(d) \simeq_{\mathbb{Q}} K(\mathbb{Q}, d) \times \prod_{i=1}^{d/2-1} K(\mathbb{Q}, 4i)$
- $\mathrm{B}\,\mathrm{hAut}^{S_2}(S^{d-1}) \simeq_{\mathbb{Q}} K(\mathbb{Q}, d)$

Theorem (Thom)
$\mathrm{Map}(X, K(G,n))_f \simeq \prod_{i=1}^{n} K(H^{n-i}(X;G), i)$

Corollary $\pi_*(\mathrm{B}\,F(M,2)) \otimes \mathbb{Q} \neq 0$ "often"

Sidenote: $[B, \mathrm{B}\,F(M,2)]$ classifies the following data

1.) A vector bundle $T_\pi E \to B \times M$ such that $T_\pi E|_{b\times M} \cong TM$,
2.) $S(T_\pi E)$ is S_2-htpy. eq. to $B \times S(TM)$ over B.

If $c(T_\pi E) = x \otimes [M] + \ldots \in H^*(B) \otimes H^*(M)$, then $\kappa_c = x$.

$\alpha \in H^{n-i}(X;G)$ –> adjoint map

$f_\alpha : S^i \times X \to K(G,n)$. Then
$f_\alpha^*(\iota_n) = [S^i] \times \alpha + 1 \times f(\iota_n)$

Result: There are elements $\pi_{4i-d}(\mathrm{B}\,F(M,2)) \otimes \mathbb{Q}$ for which $0 \neq \kappa_{L_i}$ but $\sigma_{4i-d} = 0$ by construction.

What about $k > 2$?

Rewrite:

$$\mathcal{G}(TM) = \mathrm{Map}^{\mathrm{SO}(d)}(\mathrm{Fr}^+(M), \mathrm{SO}(d))$$
$$h\mathcal{G}^{S_2}(TM) = \mathrm{Map}^{\mathrm{SO}(d)}(\mathrm{Fr}^+(M), \mathrm{hAut}^{S_2}(S^{d-1}))$$

Observation:

$$\mathrm{hAut}^{S_2}(S^{d-1}) = \mathrm{Aut}^h_{Op^{\leq 2}}(E_d)$$

Definition:

$$F(M,k) := \mathrm{hofib}_{\mathrm{Id}}\left(\mathrm{Map}^{\mathrm{SO}(d)}(\mathrm{Fr}^+(M), \mathrm{SO}(d)) \to \mathrm{Map}^{\mathrm{SO}(d)}(\mathrm{Fr}^+(M), \mathrm{Aut}^h_{Op^{\leq k}}(E_d))\right)$$

Recall: Configuration categories

Due to Boavida de Brito and Weiss — Associate to a manifold M an ∞-category $\mathrm{con}(M)$. There is a homotopy pullback square

$$\begin{array}{ccccc} & & T_k \operatorname{Emb}(M,M) & \longrightarrow & \mathbb{R}\operatorname{Map}_{\mathsf{Fin}}(\mathrm{con}(M,k),\mathrm{con}(M)) \\ & \nearrow & \downarrow & & \downarrow \\ F(M,k) & \longrightarrow & T_1 \operatorname{Emb}(M,M) & \longrightarrow & \mathbb{R}\operatorname{Map}_{\mathsf{Fin}}(\mathrm{con}(M,k)^{\mathrm{loc}},\mathrm{con}^{\mathrm{loc}}(M)) =: Z \end{array}$$

where

$$Z \simeq \Gamma\left(\begin{array}{ccc} X & \hookleftarrow & (m' \in M, f \in \operatorname{Map}^h_{Op}(E_d(T_m M), E_d(T_{m'} M)) \\ \downarrow & & \downarrow \\ M & \hookleftarrow & m \end{array}\right)$$

What about $k > 2$?

One can get some information on $\pi_*(\mathrm{Aut}^h_{Op^{\leq k}}(E_d))$ for finite $k > 2$ to infer that $\pi_*(F(M,k)) \otimes \mathbb{Q} \neq 0$ "sometimes" (work in progress). The signature theorem fails for the same reason for these homotopy classes as before.

Underlying Question

Is $\pi_*(\mathrm{SO}(d)) \otimes \mathbb{Q} \rightarrow \pi_*(\mathrm{Aut}^h_{Op}(E_d)) \otimes \mathbb{Q}$ trivial on Pontrjagin classes?

What about $k = \infty$?

The previous discussion leads one to believe that the family signature theorem does *not* hold on $\mathrm{B}\, T_\infty^\times(M)$.

Conjecture. (Randal-Williams) There is a rational fibration sequence

$$\mathrm{B}\,\mathrm{Diff}_\partial(D^{2n}) \longrightarrow \mathrm{B}\, T_\infty \,\mathrm{Emb}_\partial(D^{2n}, D^{2n}) \longrightarrow \Omega^{\infty+2n} L(\mathbb{Z})$$

$$\mathrm{B}\,\mathrm{Diff}^+(M^{2n}) \longrightarrow \mathrm{B}\, T_\infty^\times \,\mathrm{Emb}(M, M) \longrightarrow \Omega^{\infty+2n} L(\mathbb{Z})$$