Scanning diffeomorphisms

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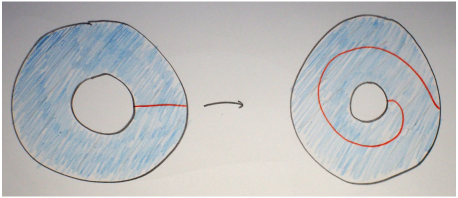
Notation: If M is a manifold $\mathrm{Diff}(M)$ denotes the group of diffeomorphisms of M that restrict to $Id_{\partial M}$.

Consider the projection $\pi : S^1 \times D^1 \to D^1$. The subgroup of fiber-preserving diffeomorphisms in $\mathrm{Diff}(S^1 \times D^1)$ has the homotopy-type of $\Omega\mathrm{Diff}^+(S^1)$, i.e.

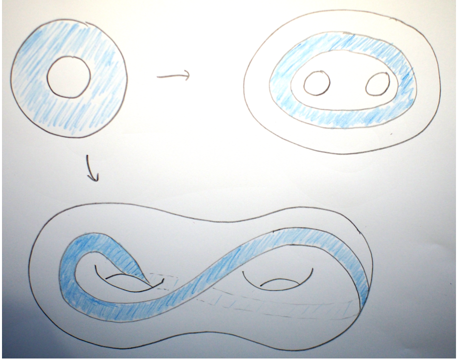
$$\mathrm{Diff}^\pi(S^1 \times D^1) \simeq \Omega S^1 \simeq \mathbb{Z}$$



These are called model Dehn twists.

An analogy with Dehn's work

Given an annulus A embedded in a surface Σ one can extend a model Dehn twist on A to all Σ via the identity map. These are called Dehn twists in Σ.



Dehn proved $\pi_0 \mathrm{Diff}^+(\Sigma)$ is finitely generated by such Dehn twists, and that isotopy-classes can be distinguished by their actions on $\pi_1\Sigma$.

Our work is roughly modelled on Dehn's.

(1) Provided $n \geq 3$ we construct a model family of fiber-preserving diffeomorphisms

$$\Omega^2 S^{n-1} \to \mathrm{Diff}(\mathcal{B}_{n+1})$$

where $\mathcal{B}_{n+1}$ is a handlebody we call a **barbell manifold** – the exterior of a 2-component string-link in D^{n+1}.

Where Dehn detected and distinguished his twists in $\pi_0 \mathrm{Diff}(\Sigma)$ using the action on $\pi_1 \Sigma$, we will detect our diffeomorphisms using variations of Cerf's **scanning map** $\mathrm{Diff}(D^n) \xrightarrow{\simeq} \Omega \mathrm{Emb}(D^{n-1}, D^n)$.

(2) We embed the barbell manifolds $\mathcal{B}_{n+1} \to S^1 \times D^n$ and detect the inclusion $\Omega^2 S^{n-1} \to \mathrm{Diff}(S^1 \times D^n)$ using a scanning map of the form

$$\mathrm{Diff}(S^1 \times D^n) \to \Omega^{n-1} \mathrm{Emb}(I, S^1 \times D^n).$$

(3) We show that $\pi_{n-3} \mathrm{Diff}(S^1 \times D^n)$ is a not finitely-generated abelian group for all $n \geq 3$.

Points of emphasis

(1) These results are essentially classical in nature, using only elementary algebraic and differential topology, such as homotopy groups of wedges of spheres, the Pontriagin-Thom construction, isotopy extension and transversality.

(2) These are high-dimensional topology results that just happen to begin in dimension four, i.e. we are proving theorems about $\pi_{n-3}\mathrm{Diff}(S^1 \times D^n)$ for $n \geq 3$.

(3) We find some 'guidance' in the Embedding Calculus, in that it tells us where to look for things. But we do not require any theorems from the subject in our proofs.

(4) Most of these arguments appear in our paper **Knotted 3-balls in S^4** (arXiv v3). The remaining argument will appear in an upcoming paper titled **Scanning Diffeomorphisms**.

(0) Cerf's Scanning Theorem

Theorem: (Cerf) There is a homotopy-equivalence

$$\mathrm{Diff}(D^n) \to \Omega \mathrm{Emb}(D^{n-1}, D^n)$$

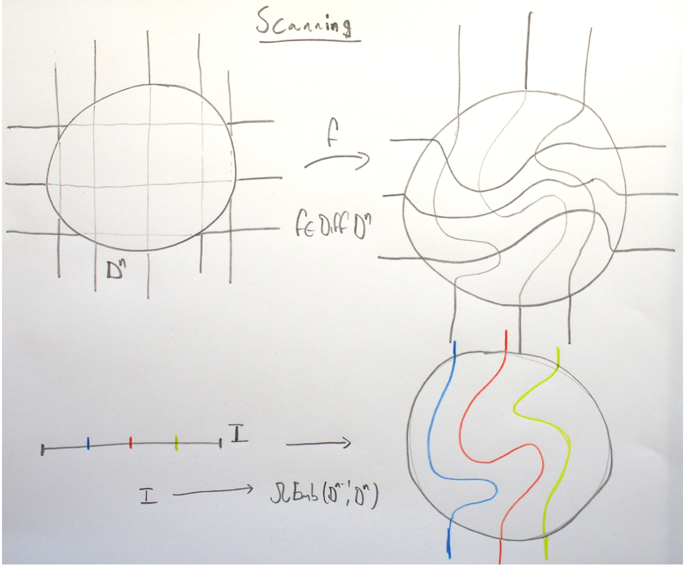
Proof sketch: Consider the restriction fibre-bundle (Palais)

$$\mathrm{Diff}(D^n) \to \mathrm{Emb}(D^{n-1}, D^n).$$

- The fiber has the homotopy-type of $\mathrm{Diff}(D^n)^2$.
- The fiber-bundle is null homotopic.
- The fiber inclusion $\mathrm{Diff}(D^n)^2 \to \mathrm{Diff}(D^n)$ is homotopic to group multiplication.
- Conclude the homotopy-fiber of the inclusion $\mathrm{Diff}(D^n)^2 \to \mathrm{Diff}(D^n)$ is homotopy-equivalent to both $\Omega \mathrm{Emb}(D^{n-1}, D^n)$ and $\mathrm{Diff}(D^n)$.

Chasing through Serre models for homotopy-fibers tells us the homotopy-equivalence is the scanning map.

(0) Cerf's Scanning Theorem



Definition: A (n, k)-handlebody of genus g is obtained by attaching g disjoint k-handles to the n-disc, D^n.

Example:

- An $(n, 0)$-handlebody is a disjoint union of n-discs.
- A $(3, 1)$-handlebody is the traditional 3-manifold handlebody.

Definition: A trivial $(n+1, n-1)$-handlebody of genus 2 is what we call a barbell manifold.

Alternatively, the barbell manifold can be thought of as the result of drilling neighbourhoods of two properly-embedded arcs from D^{n+1}, or as the boundary connect-sum of two copies of $S^{n-1} \times D^2$.

The standard barbell we denote $\mathcal{B}_{n+1} \equiv S^{n-1} \times D^2 \#_\partial S^{n-1} \times D^2$.

The barbell manifold $\mathcal{B}_{n+1}$ fibers (trivially) over the interval I

$$\mathcal{B}_{n+1} \simeq \left(S^{n-1} \times D^1 \#_\partial S^{n-1} \times D^1\right) \times I.$$

Consider fiber-preserving subgroup $\mathrm{Diff}^\pi(\mathcal{B}_{n+1})$ of diffeomorphisms of $\mathcal{B}_{n+1}$. By design,

$$\mathrm{Diff}^\pi(\mathcal{B}_{n+1}) \simeq \Omega \mathrm{Diff}(S^{n-1} \times D^1 \#_\partial S^{n-1} \times D^1).$$

$S^{n-1} \times D^1 \#_\partial S^{n-1} \times D^1$ is the twice-punctured ball. So there is a fiber sequence

$$\mathrm{Diff}(S^{n-1} \times D^1 \#_\partial S^{n-1} \times D^1) \longrightarrow \mathrm{Diff}(D^n) \xrightarrow[\text{null}]{} \mathrm{Emb}(\sqcup_2 B^n, D^n).$$

$$\mathrm{Diff}(S^{n-1} \times D^1 \#_\partial S^{n-1} \times D^1) \to \mathrm{Diff}(D^n) \to \mathrm{Emb}(\sqcup_2 B^n, D^n).$$

The above bundle map is null-homotopic, and the induced fiber sequence

$$\Omega\mathrm{Emb}(\sqcup_2 B^n, D^n) \to \mathrm{Diff}(S^{n-1} \times D^1 \#_\partial S^{n-1} \times D^1) \to \mathrm{Diff}(D^n)$$

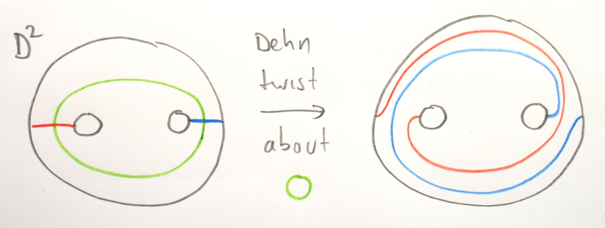
is trivial.

Proposition: (BG)

$$\mathrm{Diff}^\pi(\mathcal{B}_{n+1}) \simeq \Omega^2 S^{n-1} \times \Omega^2 SO_n^2 \times \Omega\mathrm{Diff}(D^n)$$

The map $\Omega^2 S^{n-1} \to \mathrm{Diff}(\mathcal{B}_{n+1})$ is the barbell diffeomorphism family.

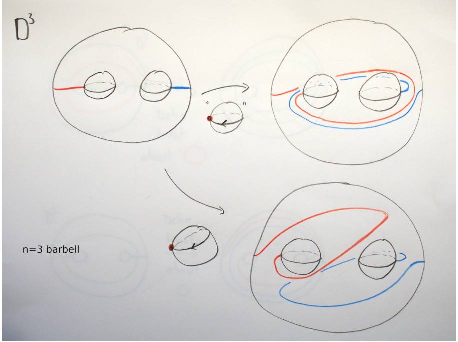
Step back – a Dehn twist.



Dehn twists could be viewed as being parametrized by $\Omega S^1 \equiv \mathbb{Z}$.

(1) Barbell - visualizing diffeomorphisms

Visualizing the barbell diffeomorphism. Parametrized by $\Omega^2 S^{n-1}$.



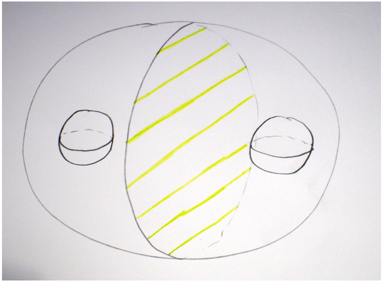
The mid-ball of the barbell $\mathcal{B}_{n+1}$ is the linearly embedded D^n splitting $\mathcal{B}_{n+1}$ into two copies of $S^{n-1} \times D^2$.
The standard cocores E_1 and E_2 are the cocores of our $(n-1)$-handle attachments, i.e. if we puncture $\mathcal{B}_{n+1}$ at the cocores it becomes a D^{n+1}.

Proposition: (BG) There is a homomorphism $\pi_{n-3}\mathrm{Diff}(\mathcal{B}_{n+1}) \to \mathbb{Z}$ making the composite

$$\mathbb{Z} \equiv \pi_{n-3}\Omega^2 S^{n-1} \to \pi_{n-3}\mathrm{Diff}(\mathcal{B}_{n+1}) \to \mathbb{Z}$$

an isomorphism.

Sketch: Fiber the mid-ball by parallel oriented intervals. The homomorphism is signed count of of pairs of points $t_1 < t_2$ on a common mid-ball interval such that $f(t_1) \in E_1$ and $f(t_2) \in E_2$.

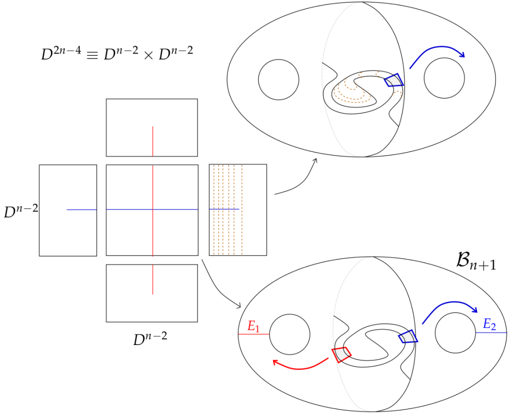


(1) Barbells - visualizing diffeomorphisms

Conclusion: The fibering of the mid-ball by parallel intervals induces a scanning map $\mathrm{Diff}(\mathcal{B}_{n+1}) \to \Omega^{n-1}\mathrm{Emb}(I, \mathcal{B}_{n+1})$. The image of the induced map

$$\pi_{n-3}\mathrm{Diff}(\mathcal{B}_{n+1}) \to \pi_{2n-4}\mathrm{Emb}(I, \mathcal{B}_{n+1})$$

contains a split infinite-cyclic subgroup of $\pi_{2n-4}\mathrm{Emb}(I, \mathcal{B}_{n+1})$.



(2) Implantations

An embedding $\mathcal{B}_{n+1} \to M$ to an $(n+1)$-manifold M gives an induced homomorphism

$$\pi_{n-3}\mathrm{Diff}(\mathcal{B}_{n+1}) \to \pi_{n-3}\mathrm{Diff}(M).$$

Provided M has a suitable fiber structure compatible with the barbell's mid-ball fibering, this gives hope one can show the image is non-trivial.

For $M = S^1 \times D^n$ we use the fibering of $\{1\} \times D^n$ by intervals, i.e. $\mathrm{Diff}(S^1 \times D^n) \to \Omega^{n-1}\mathrm{Emb}(I, S^1 \times D^n)$ giving

$$\pi_{n-3}\mathrm{Diff}(S^1 \times D^n) \to \pi_{2n-4}\mathrm{Emb}(I, S^1 \times D^n).$$

Think of this as a variation of Cerf's scanning map $\mathrm{Diff}(D^n) \to \Omega\mathrm{Emb}(D^{n-1}, D^n)$.

(2) Implantations - detection, calc. of embeddings

Tools to study embedding spaces such as $\mathrm{Emb}(I, S^1 \times D^n)$ were developed in the 60's and 70's by Cerf, Hatcher-Quinn, Haefliger and Dax. I will describe a later refinement due to Goodwillie, Weiss and Klein called the embedding calculus.

Associated to two compact manifolds M, N there is a **tower of maps**

$$\begin{array}{ccc} \mathrm{Emb}(M,N) & \xrightarrow{ev_k} & T_k\mathrm{Emb}(M,N) \\ & \searrow^{ev_{k-1}} & \downarrow \\ & & T_{k-1}\mathrm{Emb}(M,N) \end{array}$$

$T_k\mathrm{Emb}(M,N)$ is called the k-th stage of the tower. The map ev_k is called the k-th evaluation map.

Theorem: (GWK) The map ev_k is $k(n-m-2)+1-m$-connected, i.e. an isomorphism on π_j for $j < k(n-m-2)+1-m$ and an epimorphism for $j = k(n-m-2)+1-m$.

The *Mapping Space Model* (due to Dev Sinha) for the Taylor Tower in Embedding calculus is the mapping space

$$T_k\mathrm{Emb}(I, M) \simeq \mathrm{Map}(C_n[I], C_n'[M]).$$

- $C_n[M]$ indicates the Fulton-Macpherson compactified configuration space of n-tuples of distinct points in M.
- $C_n'[M]$ is the pull-back of UTM^n to $C_n[M]$, i.e. the points are decorated with unit tangent vectors.
- The maps are required to be stratum-preserving.
- The maps are aligned.
- $ev_k(f)$ is the map $(t_1, \cdots, t_k) \longmapsto (f'(t_1), \cdots, f'(t_k))$.

We will be computing Whitehead products in homotopy groups, using the Pontriagin-Thom construction.

The Whitehead product of two maps $f_i : S^{k_i} \to X$ $i = 1, 2$ is the obstruction to the map $f_1 \vee f_2 : S^{k_1} \vee S^{k_2} \to X$ extending over $S^{k_1} \times S^{k_2}$, and denoted $[f_1, f_2] \in \pi_{k_1+k_2-1} X$.

The Whitehead product $[\cdot, \cdot] : \pi_n X \times \pi_m X \to \pi_{m+n-1} X$ satisfies:

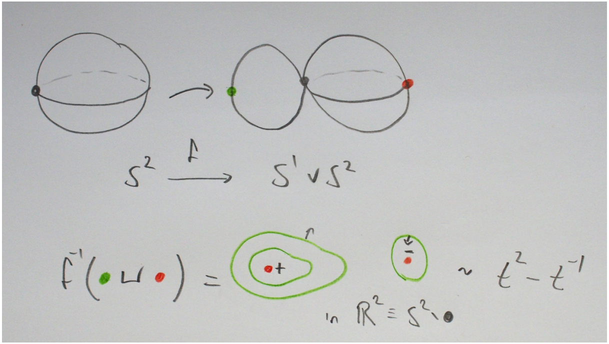
- it is bilinear,
- graded symmetric, i.e. $[y, x] = (-1)^{nm}[x, y]$,
- the Jacobi identity

$$(-1)^{pr}[[f, g], h] + (-1)^{pq}[[g, h], f] + (-1)^{rq}[[h, f], g] = 0, \text{ where}$$

$$f \in \pi_p X, g \in \pi_q X, h \in \pi_r X \text{ with } p, q, r \geq 2.$$

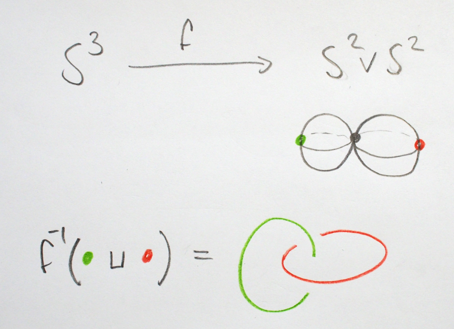
The **Pontriagin-Thom construction** tells us that maps of spheres into wedges of spheres, taken up to homotopy, corresponds with the framed cobordism classes of disjoint manifolds in the domain sphere (remove its basepoint).

Example:



$$\pi_2(S^1 \vee S^2) \simeq \mathbb{Z}[t^{\pm 1}]$$

Example:



$$\pi_3(S^2 \vee S^2) \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

Theorem: (Hilton-Milnor) The rational homotopy groups of a wedge of spheres is freely generated by the rational homotopy groups of the wedge summands, with respect to the Whitehead bracket structure.

There is the (split) fiber-sequence

$$S^1 \vee S^n \vee \cdots \vee S^n \to C_k(S^1 \times D^n) \to C_{k-1}(S^1 \times D^n),$$

thus the homotopy-group $\pi_m C_k(S^1 \times D^n)$ is isomorphic to

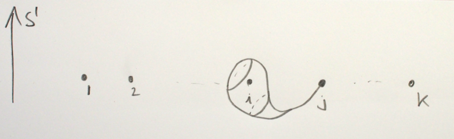
$$\bigoplus_{0 \le j < k} \pi_m \left(S^1 \vee \bigvee_j S^n \right)$$

which are themselves rationally generated by Whitehead products.

(2) Implantations - detection, $C_k[S^1 \times D^n]$

Denote the generators of $\pi_1 C_k[S^1 \times D^n] \simeq \mathbb{Z}^k$ by $\{t_i : i = 1, 2, \cdots, n\}$.

The class $w_{ij} \in \pi_n C_k[S^1 \times D^n]$ has all k points stationary, with the exception of point j that orbits around point i.



$\pi_n C_k[S^1 \times D^n]$ is generated by the set $\{t_l^q.w_{ij} \ \forall i, j, l, q\}$, with the relations

- $w_{ii} = 0 \ \forall i$
- $w_{ij} = (-1)^{n+1} w_{ji} \ \forall i \neq j$.
- $t_l.w_{ij} = w_{ij}$ provided $l \notin \{i, j\}$.
- $t_j.w_{ij} = t_i^{-1}.w_{ij} \ \forall i, j$.

The homotopy-group $\pi_{2n-1}C_k[S^1 \times D^n]$ is rationally generated by Whitehead products of the π_n generators, and they satisfy the relations:

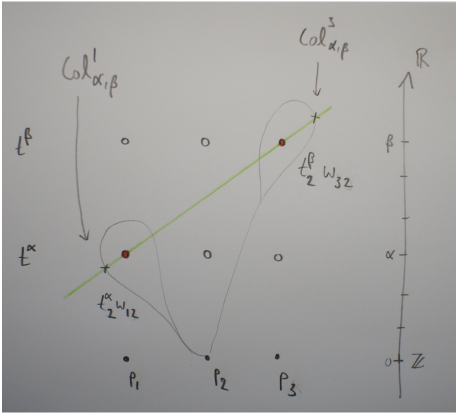
- $t_p.[f,g] = [t_p.f, t_p.g]\ \forall p \in \mathbb{Z}, f, g \in \pi_n C_k[S^1 \times D^n]$.
- $[w_{ij}, w_{lm}] = 0$ if $\{i,j\} \cap \{l,m\} = \emptyset$
- $[w_{ij}, w_{jl}] = [w_{jl}, w_{li}] = [w_{li}, w_{ij}]$

To show there are no further relations we construct submanifolds of $C_3[S^1 \times D^n]$ such that they intersect the above homotopy-classes in non-trivial framed cobordism classes.

- The collinear submanifolds of $C_3[\mathbb{R}^1 \times D^n]$ are defined as $Col^1_{\alpha,\beta}$ consists of triples $(p_2, t^\alpha.p_1, t^\beta.p_3)$ that sit on a straight line with this linear ordering.
- Similarly $Col^3_{\alpha,\beta}$ are the triples $(t^\alpha.p_1, t^\beta.p_3, p_2)$ that sit on a straight line with this linear ordering.

These two manifolds are disjoint, and detect the homotopy class $[t_2^\alpha w_{12}, t_2^\beta w_{32}]$.

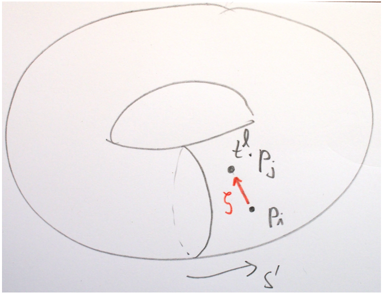
Example:



$Col^1_{\alpha,\beta}$ detects $t_2^\alpha w_{12}$, and $Col^3_{\alpha,\beta}$ detects $t_2^\beta w_{32}$. These manifolds are disjoint, and their preimage via the map $[t_2^\alpha w_{12}, t_2^\beta w_{32}] : S^{2n-1} \to C_3(S^1 \times D^n)$ is a Hopf link, therefore not null cobordant.

Brackets of the form $[t_i^{\alpha} w_{ij}, t_i^{\beta} w_{ij}]$ are detected by pairs of **cohorizontal manifolds** in $C_k[\mathbb{R} \times D^n]$.

The cohorizontal submanifold $t^l Co_i^j$ of $C_2[\mathbb{R} \times D^n]$ is the submanifold where $t^l.p_j = p_i + \epsilon\zeta$ where $\zeta \in \{0\} \times \partial D^n$ is some fixed direction, and $\epsilon > 0$.



(3) Closure operations

Proposition: (BG) Given an element of $[f] \in \pi_{n-2}\mathrm{Emb}(I, S^1 \times D^n)$, consider the 2nd stage of the Taylor tower

$$ev_2(f) : S^{n-2} \times C_2[I] \to C_2'[S^1 \times D^n].$$

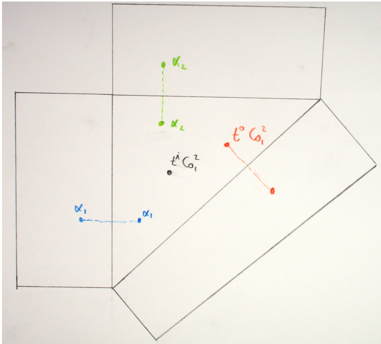
There is a **canonical null homotopy** of $ev_2(f)$ restricted to $S^{n-2} \times \partial C_2[I]$, giving us a **closure map**

$$\overline{ev_2}(f) : S^n \to C_2'[S^1 \times D^n]$$

Sketch: Along the $t_1 = t_2$ facet this map is giving derivative of f, in the sense of grade-school calculus. Use the lift of ev_2 to the universal cover to construct the extension $S^{n-2} \times C_2[I] \to S^n$. Further observe this extension restricted to $t_1 = 0$ and $t_2 = 1$ can be straight-line homotoped to the constant map.

The homotopy group $\pi_n C_2'[S^1 \times D^n] \simeq \mathbb{Z}[t^{\pm 1}] \oplus \mathbb{Z}^2$, where the isomorphism is given by the cohorizontal count, plus the degree of the velocity vector maps. If we denote the generators by t^k, α_1, α_2, the figure below depicts the allowable cobordisms, giving us the isomorphism

$$W_2 : \pi_{n-2}\mathrm{Emb}(I, S^1 \times D^n) \to \mathbb{Z}[t^{\pm 1}]/\langle t^0 \rangle.$$



Given an element of $[f] \in \pi_{2n-4}\mathrm{Emb}(I, S^1 \times D^n)$ such that $ev_2(f) \in \pi_{2n-4} T_2\mathrm{Emb}(I, S^1 \times D^n)$ is null, we form the closure of the 3^{rd} evaluation map $ev_3(f) : S^{2n-4} \times C_3[I] \to C_3'[S^1 \times D^n]$ by attaching null-homotopies to all four faces, giving us a based map

$$\overline{ev_3}(f) : S^{2n-1} \to C_3'[S^1 \times D^n].$$

Proposition: (BG) The homotopy-class of $\overline{ev_3}(f)$ is well-defined modulo a subgroup we call R, generated by the torsion subgroup plus

$$[t_2^\alpha w_{23}, t_2^\beta w_{23}] \text{ on } t_1 = 0 \text{ face},$$

$$[t_1^\alpha w_{13} + t_2^\alpha w_{23} + a_1 w_{21}, t_1^\beta w_{13} + t_2^\beta w_{23} + a_1 w_{21}] \text{ on } t_1 = t_2 \text{ face},$$

$$[t_1^\alpha w_{12} + t_1^\alpha w_{13} + a_2 w_{32}, t_1^\beta w_{12} + t_1^\beta w_{13} + a_2 w_{32}] \text{ on } t_2 = t_3 \text{ face},$$

$$[t_1^\alpha w_{12}, t_1^\beta w_{12}] \text{ on } t_3 = 1 \text{ face}.$$

Given an element of $[f] \in \pi_{2n-4}\mathrm{Emb}(I, S^1 \times D^n)$ such that $ev_2(f) \in \pi_{2n-4} T_2\mathrm{Emb}(I, S^1 \times D^n)$ is null, we form the closure of the 3^{rd} evaluation map $ev_3(f) : S^{2n-4} \times C_3[I] \to C_3'[S^1 \times D^n]$ by attaching null-homotopies to all four faces, giving us a based map

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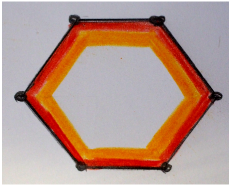
Proposition: (BG) The homotopy-class of $\overline{ev_3}(f)$ is well-defined modulo a subgroup we call R, generated by the torsion subgroup plus

$$\left(t_1^{\alpha-\beta} t_3^{-\beta} - t_1^{\alpha} t_3^{\alpha-\beta} + (-1)^{n-1}\left(t_1^{\beta} t_3^{\beta-\alpha} - t_1^{\beta-\alpha} t_3^{-\alpha}\right)\right)[w_{12}, w_{23}],$$

$$[t_2^{\alpha} w_{23}, t_2^{\beta} w_{23}], \quad [t_1^{\alpha} w_{12}, t_1^{\beta} w_{12}],$$

$$[t_1^{\alpha} w_{13}, t_1^{\beta} w_{13}] + \left(t_1^{\alpha-\beta} t_3^{-\beta} + (-1)^n t_1^{\beta-\alpha} t_3^{-\alpha}\right)[w_{12}, w_{23}].$$



The relator

$$t_1^{\alpha-\beta} t_3^{-\beta} - t_1^{\alpha} t_3^{\alpha-\beta} + (-1)^{n-1} \left(t_1^{\beta} t_3^{\beta-\alpha} - t_1^{\beta-\alpha} t_3^{-\alpha} \right)$$

we call the **hexagon relation** as the subgroup of $GL_2\mathbb{Z}$ generated by the exponent-mapping automorphisms, i.e.

$$\begin{pmatrix} \alpha-\beta \\ -\beta \end{pmatrix} \longmapsto \begin{pmatrix} \alpha \\ \alpha-\beta \end{pmatrix}, \quad \begin{pmatrix} \alpha-\beta \\ -\beta \end{pmatrix} \longmapsto \begin{pmatrix} \beta \\ \beta-\alpha \end{pmatrix}, \quad \begin{pmatrix} \alpha-\beta \\ -\beta \end{pmatrix} \longmapsto \begin{pmatrix} \beta-\alpha \\ -\alpha \end{pmatrix}$$

is isomorphic to the dihedral group of the hexagon. There is a partition of the integer lattice $\mathbb{Z}^2$ into orbits of the dihedral group action. Modulo this relator, the subgroup generated by the 12-element orbits have rank 7.

Since $\pi_{2n-4} T_2 \mathrm{Emb}(I, S^1 \times D^n)$ is torsion (exponent $|\pi_{2n-2} S^n|$) there is a well-defined map

$$\pi_{2n-4}\mathrm{Emb}(I, S^1 \times D^n) \to \mathbb{Q} \otimes \pi_{2n-1} C_3'[S^1 \times D^n]/R$$

extending the $f \longmapsto \overline{ev_3}(f)$ construction.

Definition: We call the above extension W_3 (the coarse closure)

$$W_3 : \pi_{2n-4}\mathrm{Emb}(I, S^1 \times D^n) \to \mathbb{Q} \otimes \pi_{2n-1} C_3'[S^1 \times D^n]/R.$$

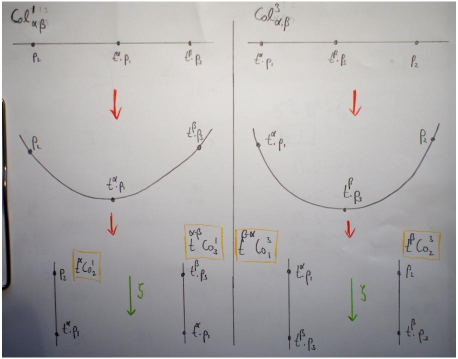
To compute W_3 we use the collinear manifolds.

Computational Device: There is a (singular) 'cobordism' of the manifold pairs

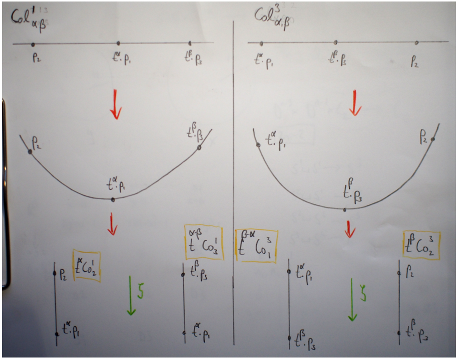
$$(Col^1_{\alpha,\beta}, Col^3_{\alpha,\beta})$$

(which detected $[t_2^\alpha w_{12}, t_3^\beta w_{23}]$) and

$$(t^\alpha Co_2^1 - t^{\alpha-\beta} Co_3^1, t^{\beta-\alpha} Co_1^3 - t^\beta Co_2^3)$$

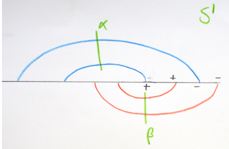


Computational Device:

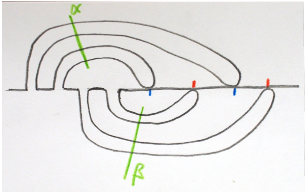


Warning! The manifold pair $(t^{\alpha}Co^1_2 - t^{\alpha-\beta}Co^1_3, t^{\beta-\alpha}Co^3_1 - t^{\beta}Co^3_2)$ is not disjoint. *But this is not a problem for us.* This argument was inspired by Misha Polyak.

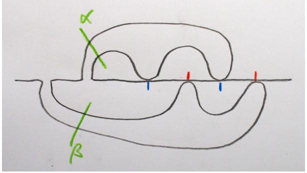
To define the family $G(\alpha, \beta)$ we begin with a 'chord diagram'.



The purpose of this diagram is to describe an immersion $I \to S^1 \times D^n$ with four double-point pairs.



α, β indicate the homotopy-class of the 'short-cut loop' in the S^1 factor, $\pi_1 S^1 \simeq \mathbb{Z}$.



Resolving this immersion would give us a map

$$\hat{G}(\alpha,\beta) : S^{n-2} \times S^{n-2} \times S^{n-2} \times S^{n-2} \to \mathrm{Emb}(I, S^1 \times D^n).$$

We pre-compose $\hat{G}(p,q)$ with the map

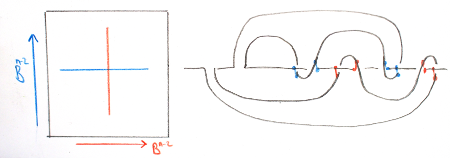
$$\Delta : S^{n-2} \times S^{n-2} \to S^{n-2} \times S^{n-2} \times S^{n-2} \times S^{n-2}$$

given by $\Delta(v,w) = (v, M(v), w, M(w))$ where $M : S^{n-2} \to S^{n-2}$ is a map with $deg(M) = -1$. This corresponds to the signs in the initial chord diagram.

The composite $\hat{G}(\alpha,\beta)\circ\Delta$, when restricted to $S^{n-2}\vee S^{n-2}$ is null, giving us a commutative diagram

$$
\begin{array}{ccc}
S^{n-2}\times S^{n-2} & \xrightarrow{\hat{G}(p,q)\circ\Delta} & \mathrm{Emb}(I, S^1\times D^n)\ . \\
\searrow & & \nearrow_{G(p,q)} \\
& S^{n-2}\times S^{n-2}/S^{n-2}\vee S^{n-2}\equiv S^{2n-4} &
\end{array}
$$

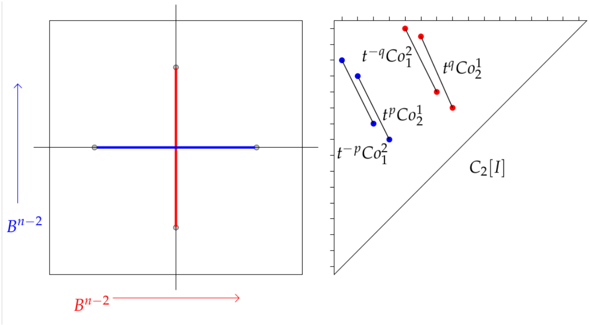
The homotopy-class of $G(\alpha,\beta)\in\pi_{2n-4}\mathrm{Emb}(I,S^1\times D^n)$ is uniquely defined.

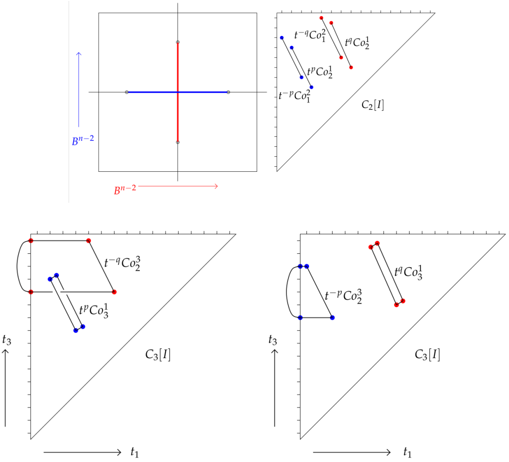


Proposition: (BG)

$$W_3(G(p,q)) = t_1^{p-q} t_3^{-q}[w_{12}, w_{23}]$$

$ev_2(f)$ and its null-homotopy.

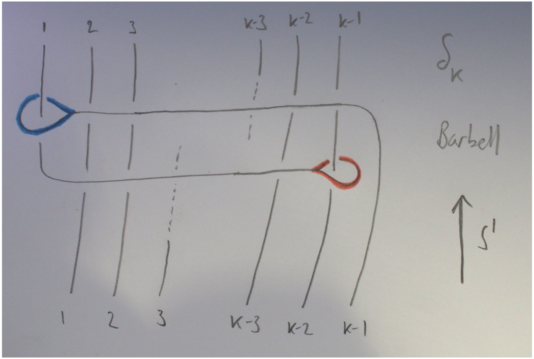




$lk(Co_3^1, Co_2^3)$ $\quad$ $lk(Co_3^1, Co_2^3)$

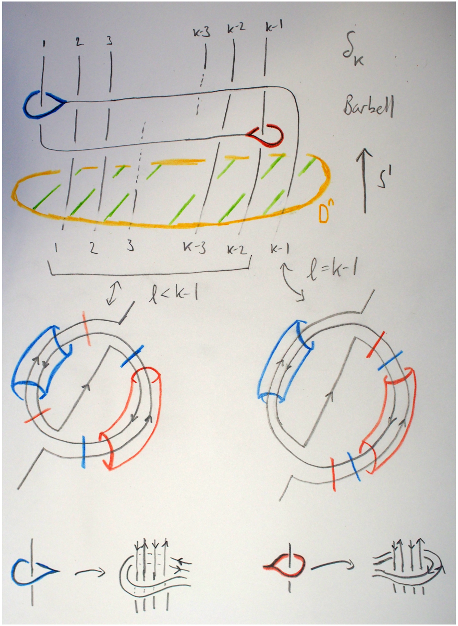
Coefficient of $t_1^p t_3^q$ is $lk(\overline{ev_3}(f)^{-1}(t^p Co_2^1 - t^{p-q} Co_3^1, t^{q-p} Co_1^3 - t^q Co_2^3))$.

(4) The δ_k computation



Theorem: (BG) The set $\{\delta_k : k \geq 4\}$ is $\mathbb{Z}$-linearly independent in $\pi_{n-3}\mathrm{Diff}(S^1 \times D^n)$.

(4) The δ_k computation



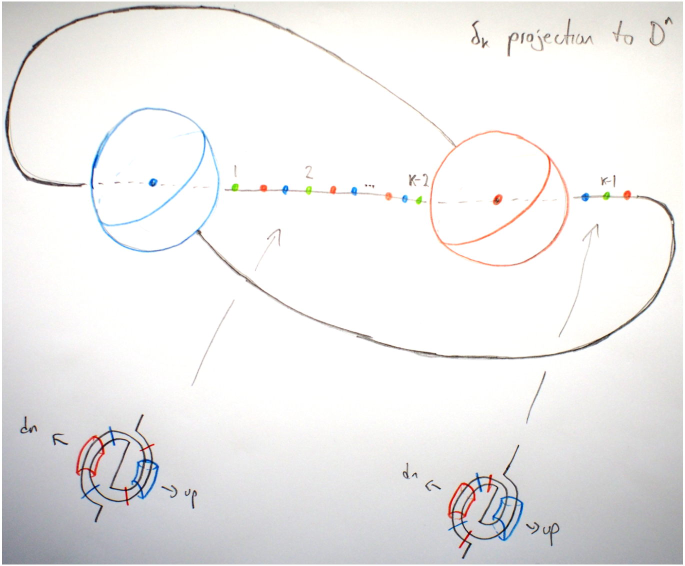
Our approach is to consider the map

$$\pi_{n-3}\mathrm{Diff}(S^1 \times D^n) \to \pi_{2n-4}\mathrm{Emb}(I, S^1 \times D^n)$$

for the elements δ_k.

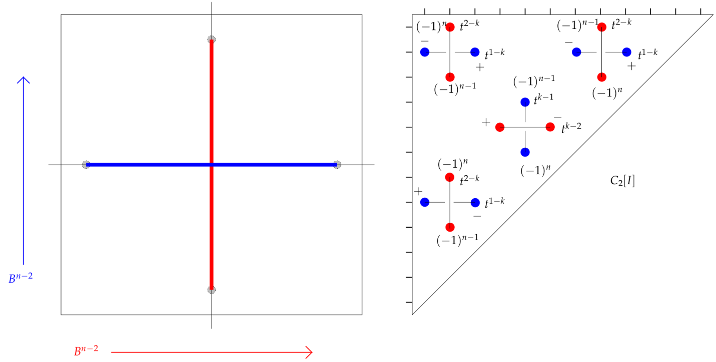
We fiber $\{1\} \times D^n$ by intervals. δ_k leaves these intervals fixed if they do not cross through the barbell. When the interval passes through the l-th strand, the action is depicted in the lower part of the figure.

(4) The δ_k computation



δ_k projected to D^n

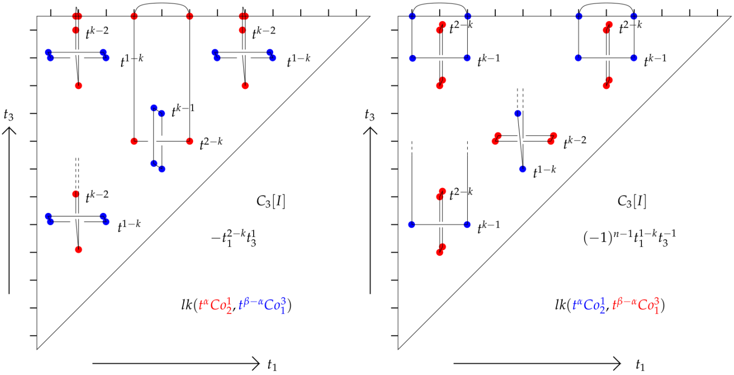
(4) The δ_k computation



Co_1^2 cohorizontal chord diagram with null-homotopy, $l < k - 1$

Think of time flowing into the screen, i.e. we close the red cohorizontal points before the blue.

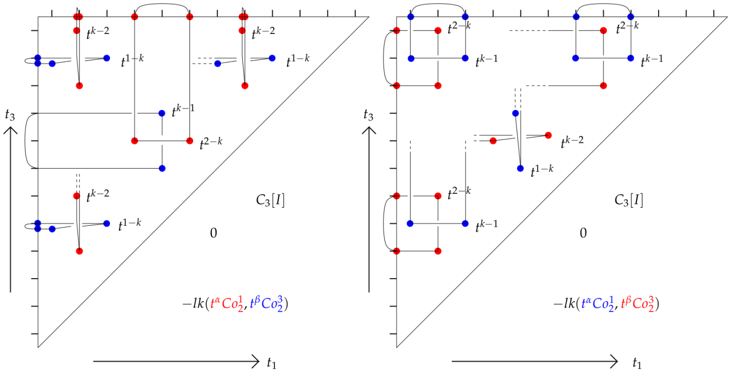
(4) The δ_k computation ($l < k - 1$)



Co_2^1 and Co_1^3 linking in $S^{2n-4} \times C_3[I]$

$t_1^\alpha t_3^\beta$ monomial $lk(\overline{ev_3}(f)^{-1}(t^\alpha Co_2^1 - t^{\alpha-\beta}Co_3^1, t^{\beta-\alpha}Co_1^3 - t^\beta Co_2^3))$

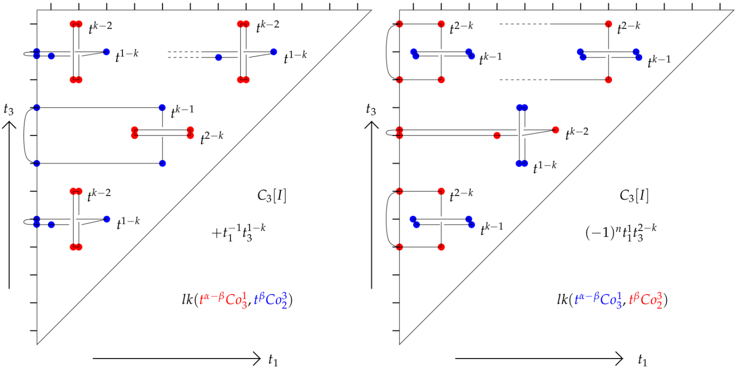
(4) The δ_k computation ($l < k - 1$)



Co_2^1 and Co_2^3 linking in $S^{2n-4} \times C_3[I]$

$t_1^\alpha t_3^\beta$ monomial $lk(\overline{ev_3}(f)^{-1}(t^\alpha Co_2^1 - t^{\alpha-\beta} Co_3^1, t^{\beta-\alpha} Co_1^3 - t^\beta Co_2^3))$

(4) The δ_k computation ($l < k-1$)

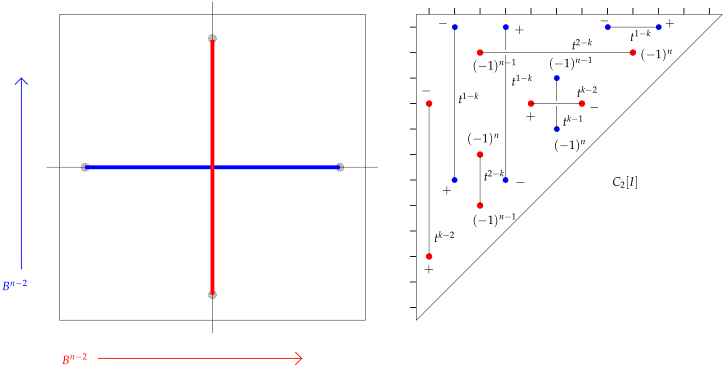


Co_3^1 and Co_2^3 linking in $S^{2n-4} \times C_3[I]$

$t_1^\alpha t_3^\beta$ monomial $lk(\overline{ev_3}(f)^{-1}(t^\alpha Co_2^1 - t^{\alpha-\beta} Co_3^1, t^{\beta-\alpha} Co_1^3 - t^\beta Co_2^3))$

(4) The δ_k computation ($l = k - 1$)

Details skipped!



Co_1^2 cohorizontal chord diagram with null-homotopy, $l = k - 1$

Theorem: (BG) Provided $k \geq 3$

$$W_3(\delta_k) = (k-1)\left(t_1^{-1}t_3^{1-k} + (-1)^{n-1}t_1^{1-k}t_3^{-1} - t_1^{2-k}t_3^{1} + (-1)^n t_1 t_3^{2-k}\right) +$$

$$t_1 t_3^{k-1} + (-1)^{n-1}t_1^{k-1}t_3 - t_1^{1-k}t_3^{2-k} + (-1)^n t_1^{2-k}t_3^{1-k}$$

Corollary: (BG) The group

$$\pi_{n-3}\mathrm{Diff}(S^1 \times D^n)$$

is not finitely generated for $n \geq 3$.

T. Watanabe has an alternative proof of the above for $n = 3$, and has sketched an argument for n odd.

Definition: An n-dimensional **half-disc** is the intersection of the standard n-disc with the half-space

$$HD^n = \{x \in \mathbb{R}^n : \sum x_i^2 \leq 1, x_1 \leq 0\}.$$

We call $(\partial D^n) \cap HD^n$ the round boundary, and $\{x \in HD^n : x_1 = 0\}$ the flat boundary.

Definition: $\mathrm{Emb}(HD^j, D^n)$ denotes the space of embeddings of HD^j into D^n that agree with the standard inclusion $p \longmapsto (p, 0)$ on the round boundary.

These embedding spaces are contractible, using a variant of the uniqueness of collar neighbourhoods argument.

Theorem: (Cerf, BG) There is a locally-trivial fibre bundle

$$\mathrm{Emb}(HD^j, D^n) \xrightarrow[\mathit{flatface}]{\mathit{restr.}} \mathrm{Emb}_u^+(D^{j-1}, D^n) \ .$$

The base space is the space of smooth embeddings of D^{j-1} into D^n that agree with the standard inclusion $p \longmapsto (p, 0)$ on the boundary, equipped with a unit normal vector field, which is also standard on the boundary. The subscript u indicates the embeddings in the base-space are all unknotted.

The fiber of this bundle has the homotopy-type of $\mathrm{Emb}(D^j, S^{n-j} \times D^j)$.

Corollary:

$$\Omega \mathrm{Emb}_u^+(D^{j-1}, D^n) \simeq \mathrm{Emb}(D^j, S^{n-j} \times D^j)$$

Thus $\pi_0 \mathrm{Emb}(D^j, S^{n-j} \times D^j)$ is a group. Both spaces are monoids from the stacking operation, provided $j > 1$. These two operations are the same, under the equivalence.

This is typically called an *Eckmann-Hilton argument*.

Theorem: (Hatcher-Wagoner) Provided $n \geq 6$,

$$\pi_0 \mathrm{Diff}(S^1 \times D^n) \simeq \pi_0 \mathrm{Diff}(D^{n+1}) \oplus \pi_0 \mathrm{Diff}(D^n) \oplus \bigoplus_{\infty} \mathbb{Z}_2.$$

Proposition: (BG) There is a homotopy-equivalence $\mathrm{Diff}(S^1 \times D^n) \simeq \mathrm{Diff}(D^{n+1}) \times \mathrm{Emb}(D^n, S^1 \times D^n)$.

The above allows the reinterpretation of Hatcher-Wagoner as

$$\pi_0 \mathrm{Emb}(D^n, S^1 \times D^n) \simeq \overbrace{\pi_0 \mathrm{Diff}(D^n)}^{\textit{parametrization}} \oplus \overbrace{\bigoplus_{\infty} \mathbb{Z}_2}^{\textit{image}}.$$

Proposition: (BG) The operation of lifting a disc to the *m*-sheeted covering space of $S^1 \times D^n$ induces an endomorphism

$$\pi_0 \mathrm{Emb}(D^n, S^1 \times D^n) \to \pi_0 \mathrm{Emb}(D^n, S^1 \times D^n)$$

provided $n > 1$.

The fixed points, provided $m > 1$, is a subgroup isomorphic to $\pi_0 \mathrm{Emb}(D^n, D^{n+1})$. These are the isotopy-classes of embeddings $D^n \to D^{n+1}$ that agree with the inclusion $p \longmapsto (p, 0)$ on the boundary.

(5) Relations to other problems - Schönflies

Notice $\pi_0\mathrm{Emb}(D^n, D^{n+1}) \simeq \pi_0\mathrm{Emb}(S^n, S^{n+1})$.

The Schönflies Problem asks if every smoothly-embedded S^n in S^{n+1} is isotopic to the round S^n, or equivalently, if

$$\pi_0\mathrm{Diff}(D^n) \to \pi_0\mathrm{Emb}(D^n, D^{n+1})$$

is onto.

Our δ_k examples are in the kernel of the endomorphisms of $\pi_0\mathrm{Emb}(D^3, S^1 \times D^3)$ for $m \geq k$.

Earlier we observed a homotopy-equivalence

$$\Omega\mathrm{Emb}_u^+(D^{n-1}, D^{n+1}) \simeq \mathrm{Emb}(D^n, S^1 \times D^n)$$

where $\mathrm{Emb}_u^+(D^{n-1}, D^{n+1})$ is the space of unknotted co-dimension 2 discs in D^{n+1}. We deduce from that:

- $\pi_{n-3}\mathrm{Emb}(D^n, S^1 \times D^n)$ is not finitely generated,
- $\pi_{n-2}\mathrm{Emb}_u^+(D^{n-1}, D^{n+1})$ is not finitely generated for $n \geq 3$.
- $\pi_{n-2}\mathrm{Emb}_u(D^{n-1}, D^{n+1})$ is not finitely generated for $n \geq 3$.
- $\pi_{n-2}\mathrm{Emb}_u(S^{n-1}, S^{n+1})$ is not finitely generated for $n \geq 3$.

Allen Hatcher has shown that the space

$$\mathrm{Emb}_u(D^1, D^3)$$

is contractible, or equivalently,

$$\mathrm{Emb}_u(S^1, S^3) \simeq V_{4,2} \simeq S^3 \times S^2.$$

This says the space of unknottable embeddings of S^1 in S^3 has the homotopy-type of the subspace of round circles in S^3. The analogue to Hatcher's theorem is **false above dimension** 3.

Hatcher-Wagoner describe an isomorphism ($n \geq 6$)

$$\pi_0 \mathrm{Emb}(D^n, S^1 \times D^n) \to \pi_0 \mathrm{Diff}(D^n) \oplus \bigoplus_{\infty} \mathbb{Z}_2.$$

Conjecture: (BG) The Hatcher-Wagoner diffeomorphisms are detectable by scanning, $\mathrm{Emb}(D^n, S^1 \times D^n) \to \Omega^{n-1} \mathrm{Emb}(I, S^1 \times D^n)$.

$$\pi_0 \mathrm{Diff}(S^1 \times D^n) \to \pi_0 \mathrm{Emb}(D^n, S^1 \times D^n) \to \pi_{n-1} \mathrm{Emb}(I, S^1 \times D^n)$$

Thank-you.

Questions?