On Budney-Gabai

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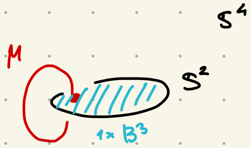
by Danica Kosanović

- general outline of the proof TODAY
- computations with emb. calculus.

people.mpim-bonn.mpg.de/danica

$S^2 \subseteq S^4$ standard



$\Rightarrow \quad \overline{S^4 \setminus \nu S^2} \cong S^1 \times B^3$

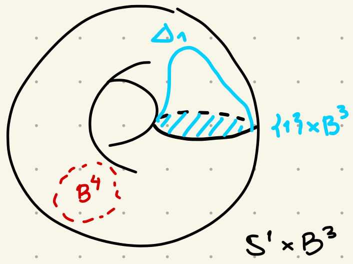
Thm. $\exists\, B^3 \overset{\Delta}{\hookrightarrow} S^1 \times B^3$ and $\Delta \underset{\text{rel}\,\partial}{\not\simeq} \{1\} \times B^3$ i.e. KNOTTED 3-BALL $\longrightarrow$ get a knotted 3-ball for $S^2 \subseteq S^4$

$$\begin{array}{ccc} B^3 & \overset{\Delta}{\hookrightarrow} & S^1 \times B^3 \\ \cup\!\!| & & \cup\!\!| \\ \partial & \xmapsto{\ id\ } & \{1\} \times \partial \end{array}$$

fixes ∂ pointwise

Observation: Given $\phi \in \mathrm{Diff}(S^1 \times B^3, \partial)$ get $\Delta_1 := \phi(\{1\} \times B^3)$

then: $\Delta_1 \underset{\text{rel}\,\partial}{\simeq} \{1\} \times B^3$ iff $\phi \simeq \phi'$ with $\mathrm{supp}(\phi') \subseteq B^4$ since move support away:



$\Rightarrow$ Want to study the fibre bundle: [Palais]

$$\mathrm{Diff}(B^4, \partial) \xrightarrow[\text{Id}]{\text{extend by}} \mathrm{Diff}(S^1 \times B^3, \partial) \xrightarrow{\text{restr}} \mathrm{Emb}_\partial(B^3, S^1 \times B^3)$$

those ϕ which fix $\{1\} \times B^3$ $\qquad \phi \mapsto \phi|_{\{1\} \times B^3}$

Since on π_0 we get:

elts are isotopy classes of diffeo

elts are knotted 3-balls # $[\partial \times I]$

$\Rightarrow$ For Thm need a non-trivial elt in $\pi_0 \mathrm{Diff}(S^1 \times B^3, \partial) / \pi_0 \mathrm{Diff}(B^4, \partial)$

THEY CONSTRUCT ∞-LY MANY.

OUTLINE:

STEP 1. $\exists$ diagram:

STEP 2. Compute

$$\begin{tikzcd}
 & & \pi_1 \mathrm{Diff}_0(S^1\times S^3) \arrow[d, "r_*"] & \\
 & \Lambda^1_3 \arrow[r, hook] \arrow[dr] \arrow[drr] & \pi_1 \mathrm{Emb}_u(S^1, S^1\times S^3) \arrow[r, two heads, "W_1"] \arrow[d, "\delta"] & \mathbb{Z} \\
\pi_0 \mathrm{Diff}(B^4, \partial) \arrow[rr, hook, "\iota"] & & \pi_0 \mathrm{Diff}_0(S^1\times B^3, \partial) \arrow[r, "\mathrm{restr}"] \arrow[ll, bend left, dashed, "s"] & \pi_0 \mathrm{Emb}_\partial(B^3, S^1\times B^3)
\end{tikzcd}$$

x

STEP 3.

This is injective!

Strategy:

$$\boxed{\begin{array}{l} \text{Prove} \\ \quad \mathrm{im}(r_*) \cap \Lambda^1_3 = \{0\} \\ (\text{Thm } 3.1 \text{ and Prop } 3.2) \end{array}}$$

STEP 4.

Injective as well? $\Rightarrow$ Thm

Strategy:

$$\boxed{\begin{array}{l} \exists s \text{ with } s\circ\iota \simeq \mathrm{id} \\ \text{Hence } x \notin \mathrm{im}(\iota) \text{ iff } s(x) = 0 \\ \text{So enough to check:} \\ \qquad s(\delta(x)) = 0 \text{ for } x \in \Lambda^1_3. \\ \qquad\qquad (\text{Cor } 3.6) \end{array}}$$

FUN FACTS: $\mathrm{Diff}(B^4, \partial) \simeq \Omega^5(\ldots)$, $\mathrm{Diff}(S^1\times B^3, \partial) \simeq \Omega^4(\ldots)$, $\pi_0 \mathrm{Emb}(B^3, S^1\times B^3)$ is a group.

STEP 1. Consider analogous fibre bundle

$$\mathrm{Diff}(B^4,\partial) \xrightarrow[\mathrm{Id}]{\text{extend by}} \left(\mathrm{Diff}_0(S^1\times B^3,\partial) \times L_0(SO_n)\right) \cong \mathrm{Diff}(S^1\times S^3 \text{ fix } u) \xrightarrow{\text{restr}} \mathrm{Emb}_\partial(B^3, S^1\times B^3)$$

$$\Big\downarrow \text{extend by Id}$$

$$\mathrm{Diff}_0(S^1\times S^3)$$

$$\Big\downarrow r$$

$$\mathrm{Emb}_u(S^1, S^1\times S^3)$$

need to pick a basepoint

Take $u: S^1\times\{pt\} \hookrightarrow S^1\times S^3$

Note:

$$\pi_0\, \mathrm{Emb}(S^1, S^1\times S^3) \xrightarrow[W_0]{\cong} \mathbb{Z}$$

the winding number

get on π_*:

$$\begin{array}{c} \pi_1 \mathrm{Diff}_0(S^1\times S^3) \\ \downarrow r_* \\ \pi_1 \mathrm{Emb}_u(S^1, S^1\times S^3) \\ \downarrow \delta \end{array}$$

$$\pi_0 \mathrm{Diff}(B^4,\partial) \xrightarrow[\mathrm{Id}]{\text{extend by}} \pi_0 \mathrm{Diff}_0(S^1\times B^3,\partial) \xrightarrow{\text{restr}} \pi_0 \mathrm{Emb}_\partial(B^3, S^1\times B^3)$$

$\exists s$

$$\begin{array}{c} \downarrow \\ \pi_0 \mathrm{Diff}_0(S^1\times S^3) \\ \downarrow r \\ \{[u]\} \end{array}$$

STEP 2. Define invariants $W_1 \times W_2 : \pi_1 \mathrm{Emb}_u(S^1, S^1 \times S^3) \longrightarrow \mathbb{Z} \oplus \Lambda^1_3$

"measures rotations"

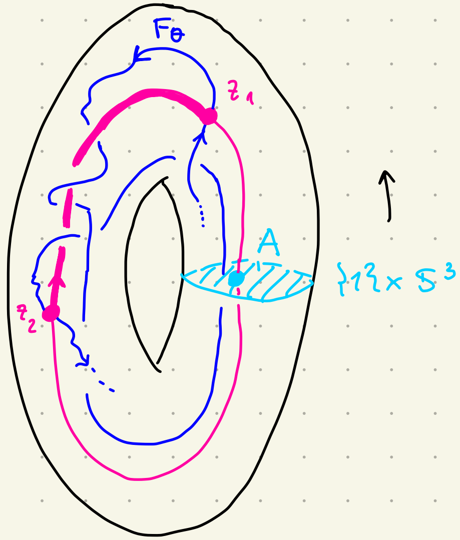
$W_1 : F_\theta,\ \theta \in S^1 \longmapsto \deg\left(\underset{\theta}{S^1} \longrightarrow \underset{F_\theta(1)}{S^1 \times S^3} \xrightarrow{pr} S^1 \right)$

Def^n: $\Lambda^1_3 := \mathbb{Z}\{t^k : k \in \mathbb{Z}\} / \langle t^0, t^{-1}, t^k - t^{-k} : k \in \mathbb{Z} \rangle \cong \mathbb{Z}\{t^2, t^3, \dots\}$

Def^n W_2 "counts the cocircular points in $\mathrm{im}(f) \subseteq S^1 \times S^3$ together w/ their gp elts"

i.e. $W_2(f) = \sum\limits_{\substack{p \in \hat{F}^{-1}(\mathcal{C}) / \Sigma_2 \\ \| \\ (\theta, z_1, z_2)}} \mathcal{L}_p(F) \in \Lambda^1_3$

"cocircular points"



where: $\hat{F} : S^1 \times C_2(S^1) \longrightarrow C_2(S^1 \times S^3)$

$\theta, (z_1, z_2) \quad\quad (F_\theta(z_1), F_\theta(z_2))$

$\mathcal{C} := \{ ((z_1, A_1), (z_2, A_2)) \in C_2(S^1 \times S^3) : A_1 = A_2 \}$

$\mathcal{L}_p(F) := \underset{\{+1,-1\}}{\mathrm{sgn}_p(\hat{F}, \mathcal{C})} \cdot \underset{\pi_1(S^1 \times S^3) = \langle t \rangle}{[\gamma_p(F)]}$ where $\gamma_p(F) := F_\theta(\underset{S^1}{\overrightarrow{z_1 z_2}}) \bullet (A \times \overrightarrow{z_2 z_1})$

NOTE: If $(\theta, z_1, z_2) \in \hat{F}^{-1}(\tau\tau)$, then $(\theta, z_2, z_1) \in \hat{F}^{-1}(\tau\tau)$,

and:

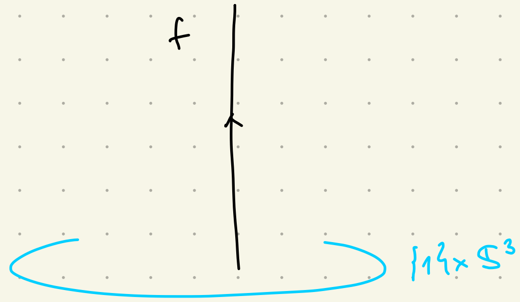
$$\mathcal{L}_{(\theta, z_1, z_2)}(F) = \overline{\mathcal{L}_{(\theta, z_2, z_1)}(F)} \qquad \text{where } \overline{t^k} = t^{-k}$$

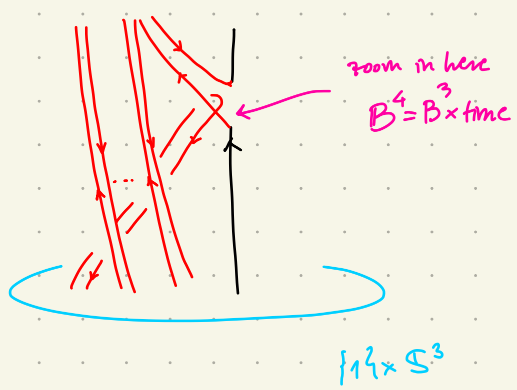
Hence: $W_2(F) \in \Lambda_3^1$ well-defined.

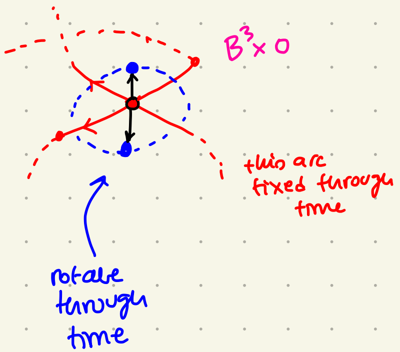
- One checks that it is a homotopy invariant, so $W_2 : \pi_1 \mathrm{Emb}_u(S^1, S^1 \times S^3) \longrightarrow \Lambda_3^1$
- Can construct F s.t. each t^k, $k \geq 2$ realized $\Rightarrow$ W_2 surjective.
- Can prove $W_1 \times W_2 = ev_2$ the map to 2-nd stage of Taylor tower

  Deep theorem of Goodwillie-Klein'15 $\Longrightarrow$ ev_2 is an isomorphism on π_1.

For surjectivity :

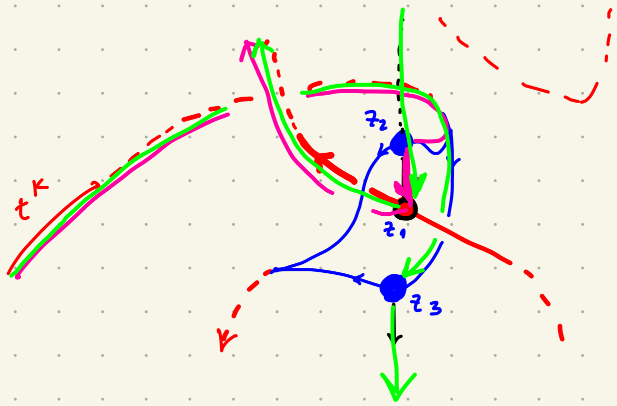






$\Rightarrow$ get one resolution for each blue point

Now for each blue dot have a "resolution":



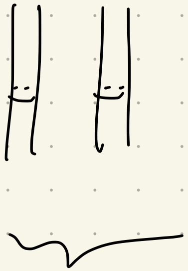
see 2 resolutions in $B^3 \times 0$

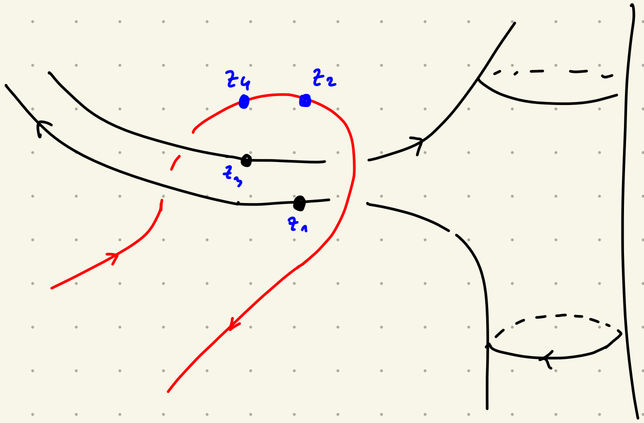
$h_{\theta, z_1, z_2} = t^K$

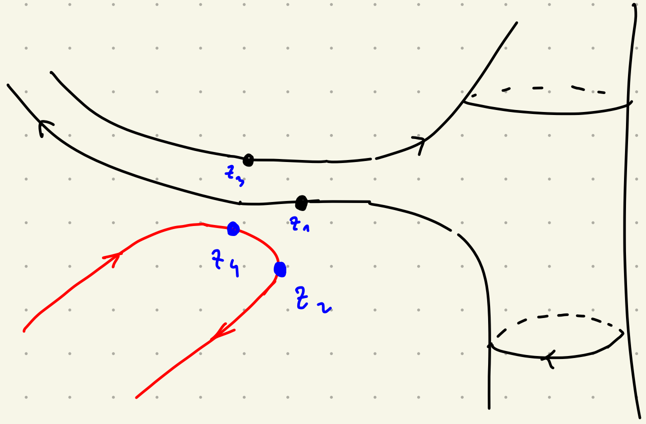
$h_{\theta, z_1, z_3} = -t^{K-1}$

TOTAL: $t^K - t^{K-1}$

Other construction:







(z_1, z_2) $-t^K$

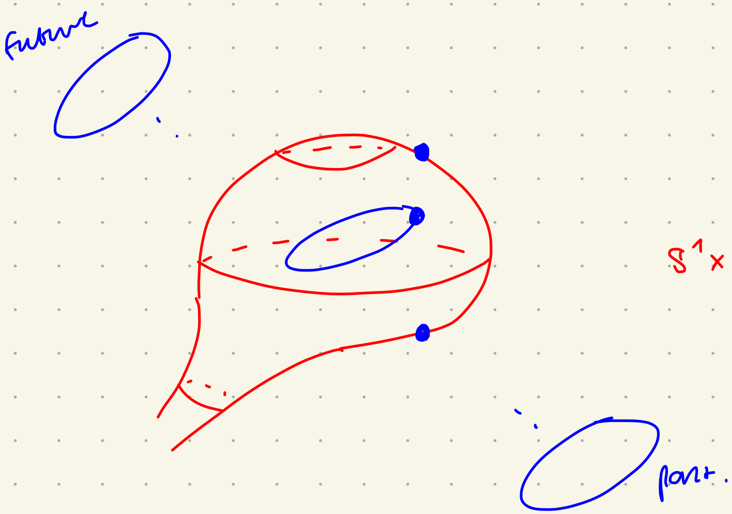
(z_3, z_4) t^{K+1}

(z_1, z_2) $-t^{K-1}$

(z_3, z_4) t^K

TOTAL : $t^{K+1} - t^{K-1}$

another picture:



$S^1 \times D^2$

NB: Does not match what they say in the paper...

t^n