A LIGHT BULB THEOREM FOR DISKS

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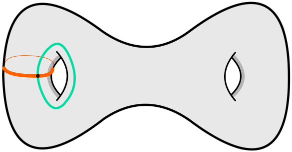
The main trick

Theorem (Space Level LBT)

For $k \leq d \geq 1$ let M be a compact smooth d-manifold with a pair of smoothly embedded spheres $\mathbf{s}\colon \mathbb{S}^{k-1} \hookrightarrow \partial M$ and $\mathbf{G}\colon \mathbb{S}^{d-k} \hookrightarrow \partial M$, such that $\mathbf{G}$ has trivial normal bundle and $\mathbf{G} \pitchfork \mathbf{s} = \{pt\}$.



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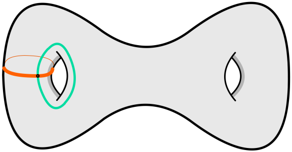

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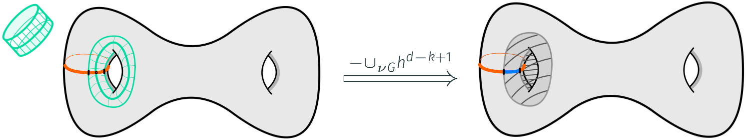
  Neat = transverse to ∂M and $K(X) \cap \partial M = K(\partial X)$.
- For $E = \mathrm{Emb}^\varepsilon_\partial(\mathbb{D}^{k-1}, M \cup_{\nu G} h^{d-k+1})$ the boundary condition is $u_0 := \partial u_+$ and $\Omega E = \mathrm{Map}_*(\mathbb{S}^1, E)$ is the space of loops based at $\mathbf{u}_+ := \mathbf{s} \cap \mathbf{h}^{d-k+1}$.



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- Supscript ε means each embedded disk is equipped with a "push-off"...
- Codimension increased by one! ($\implies$ right hand side is easier)

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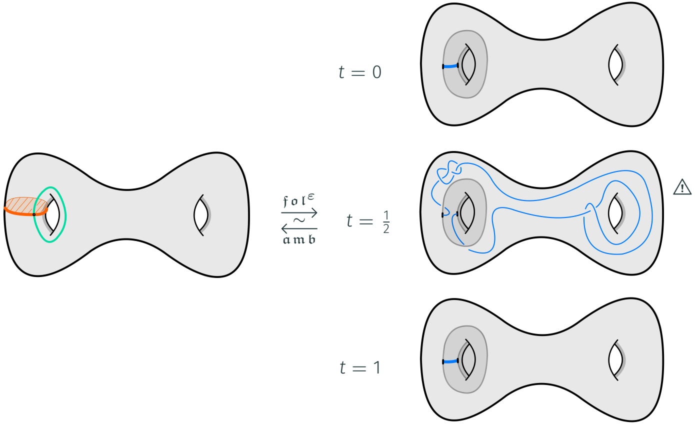

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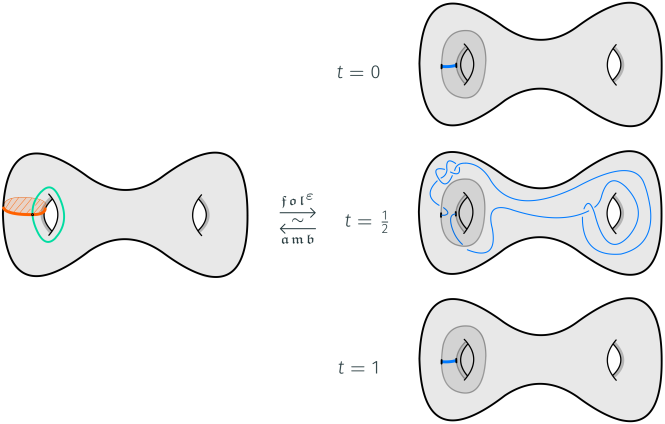

⚠ Only a schematic: for any $t \in [0, 1]$, the time t arc is isotopic to u_+.

Special cases

$k = d$: Recovers a theorem (and proof) of Cerf '68:

$$\mathrm{Diff}^+_\partial(\mathbb{D}^d) = \mathrm{Emb}_\partial(\mathbb{D}^d, \mathbb{D}^d) \simeq \Omega\,\mathrm{Emb}_\partial(\mathbb{D}^{d-1}, \mathbb{D}^d).$$

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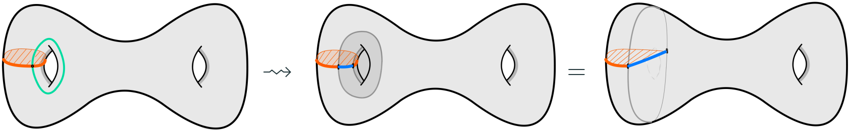
Picture Proof of Space Level LBT



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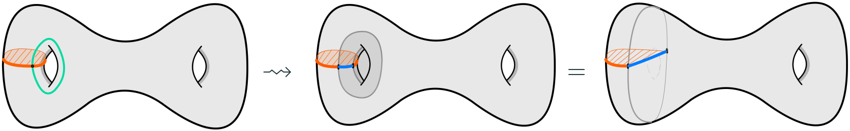

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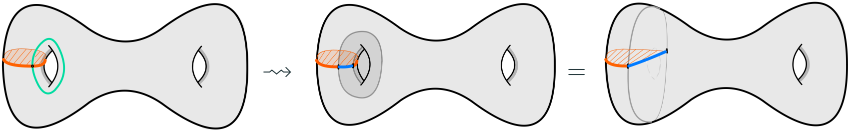

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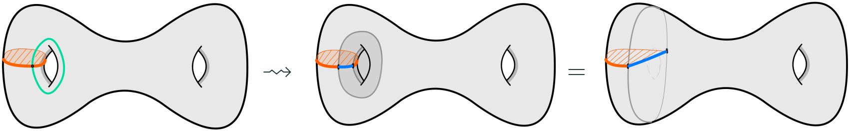

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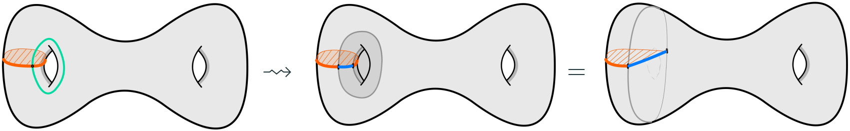

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where: $\mathfrak{amb}_{\mathrm{U}}$ is the connecting map (use the family ambient isotopy theorem to extend loops), $\mathfrak{fol}^\varepsilon_{\mathrm{U}}(K)$ is the loop of ε-augmented $(k-1)$-disks foliating the sphere $-\mathrm{U} \cup K$. □

LBT for 2-disks in 4-manifolds

The 4D setting

Let M be an oriented compact smooth 4-manifold together with

- a knot $s\colon \mathbb{S}^1 \hookrightarrow \partial M$,
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so that s and G intersect transversely and positively in a single point.

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- $m_- = s(-i) \in M$ be the basepoint and denote $\pi = \pi_1(M, m_-)$,
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By Space Level LBT we have $\mathsf{Emb}_\partial[\mathbb{D}^2, M] := \pi_1 \mathsf{Emb}_\partial^\varepsilon(\mathbb{D}^1, M \cup_{\nu G} h^3)$ and we can compute the latter group.

LBT for 2-disks

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$\Longleftrightarrow$ the relative Dax invariant, given by a clever count of double point loops in a homotopy to K, detects the action:

$$\mathrm{Dax}(K + \mathrm{fm}(r)^G, K) = [r].$$

Picture of LBT for 2-disks

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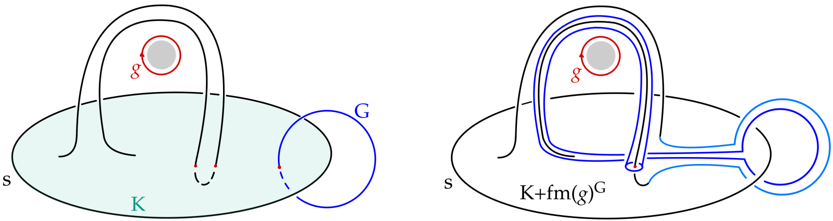

Note: A similar construction by Gabai in "Self-Referential Discs and the Light Bulb Lemma".

Other results

Special case: spheres with a common dual

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Proposition

There is a bijection

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where $s = \partial(\nu_x G)\colon \mathbb{S}^1 \hookrightarrow \partial(N \setminus \nu G)$ is a meridian circle of G at $x \in G$, and its dual is a push-off of G into $\partial(N \setminus \nu G)$.

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Observe: $\partial(N \setminus \nu G) = \partial N \sqcup \partial(\nu G)$ and $\partial(\nu G) \cong \mathbb{S}^1 \times \mathbb{S}^2$. Conversely, if a 4-manifold M has a boundary component $\mathbb{S}^1 \times \mathbb{S}^2$, attaching $\mathbb{D}^2 \times \mathbb{S}^2$ to it takes us to the setup of spheres with a fixed dual.

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The set of spheres homotopic to $[F] \in \mathbf{Emb}^G[\mathbb{S}^2, N] \cong \mathbf{Emb}_\partial[\mathbb{D}^2, M]$ is given by

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- Group structures on sets of isotopy classes, see the next slide.

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for $K_1, K_2 \in \mathsf{Emb}_\partial[\mathbb{D}^2, M]$ and $\tilde{\lambda} = [\lambda(-\mathrm{U} \cup K_1, -\mathrm{U} \cup K_2)] \in \mathbb{Z}[\pi \setminus 1]$.

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Moreover, the sequence of Theorem A becomes *an exact sequence of groups*, with the bijection $-\mathrm{U} \cup \bullet \colon \mathsf{Map}_\partial[\mathbb{D}^2, M] \cong \pi_2 M$ inducing a nonstandard group structure $\star$ on $\pi_2 M$:

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Note:
$\mathsf{Emb}_\partial[\mathbb{D}^2, M]$ is almost never abelian (we have seen $\mathsf{dax}(\pi_3 M) \subset \mathbb{Z}[\pi \setminus 1]^\sigma$ and λ is rarely symmetric, so $\tilde{\lambda}$ not in the image of dax).

Thank you!