2-knots and knotted families of arcs

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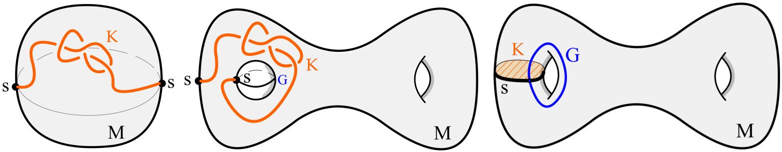
Introduction

- Fix $1 \leq k \leq d$. Let M be a compact smooth d-dimensional manifold and $\mathbf{s}\colon \mathbb{S}^{k-1} \hookrightarrow \partial M$ a smooth embedding. Recall that this means that $\mathbf{s}$ is *injective*, and at any $x \in \mathbb{S}^{k-1}$ the derivative $d\mathbf{s}_x$ is *injective*.
- We consider the space

$$\mathrm{Emb}_\partial(\mathbb{D}^k, M) := \{K\colon \mathbb{D}^k \hookrightarrow M \mid K \text{ is a neat smooth embedding}, K|_{\partial\mathbb{D}^k} = \mathbf{s}\}$$

where neat means transverse to ∂M and $K(\mathbb{D}^k) \cap \partial M = K(\partial\mathbb{D}^k) = \mathbf{s}$.
- For example, for $(k, d) = (1, 3)$ and $(2, 3)$:



- Setting with a dual: If there exists $G\colon \mathbb{S}^{d-k} \hookrightarrow \partial M$, such that G has trivial normal bundle and $G \pitchfork \mathbf{s} = \{pt\}$. Like the second and third examples!

In low dimensional topology...

... one studies codimension two embeddings, where "knotting" occurs.

- For example, (classical) knot theory studies the set of isotopy classes of circles embedded into the 3-space:

$$\pi_0 \operatorname{Emb}_\partial(\mathbb{S}^1, \mathbb{R}^3).$$

- This is in fact in bijection with $\pi_0 \operatorname{Emb}_\partial(\mathbb{D}^1, \mathbb{D}^3)$, the so-called long knots.
- Recently, intensively studied is the set of (long) 2-knots in a 4-manifold M:

$$\pi_0 \operatorname{Emb}_\partial(\mathbb{D}^2, M)$$

  This can be huge – for example, "spinning" a classical knot gives a 2-knot in $\pi_0 \operatorname{Emb}_\partial(\mathbb{S}^2, \mathbb{R}^4) \cong \pi_0 \operatorname{Emb}_\partial(\mathbb{D}^2, \mathbb{D}^4)$.

In this work...

...we compute

$$\pi_0 \,\mathsf{Emb}_\partial(\mathbb{D}^2, M)$$

in the setting with a dual!

$\implies$ We can recover 4d LBT for spheres of Gabai (JAMS '20) and their classification by Schneiderman and Teichner (Duke Math. J. '21), by completely new techniques.

- In fact, our Space Level Light Bulb Theorem expresses $\mathsf{Emb}_\partial(\mathbb{D}^k, M^d)$ for any $1 \leq k \leq d$ in the setting with a dual, as the loop space on another embedding space, of higher codimension.

$\implies$ In particular:

$$\pi_0 \,\mathsf{Emb}_\partial(\mathbb{D}^2, M) \cong \pi_1 \,\mathsf{Emb}_\partial^\varepsilon(\mathbb{D}^1, M \cup_{\nu G} h^3)$$

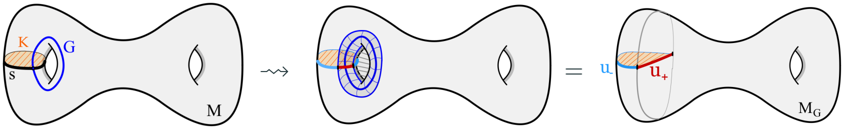
so homotopy groups of embedding spaces show useful in low-dimensional topology!

- Using the classical work of Dax from '70s we compute $\pi_k \,\mathsf{Emb}_\partial(\mathbb{D}^2, M)$ for $k \leq d-4$ and any $d \geq 4$. Also: for $d = 4$ get an explicit group structure!

Space Level Light Bulb Theorem

Cerf's trick

Attach a handle h^{d-k+1} to M along the dual $G \implies$
a disk in M with $\partial K = s$ becomes a "half-disk" in $M \cup_{\nu G} h^{d-k+1}$ with $\partial J = u_- \cup u_+$:



Can show this gives a homotopy equivalence $\mathbf{Emb}_\partial(\mathbb{D}^k, M) \simeq \mathbf{Emb}_\partial(\text{ᗡ}^k, M_G)$.

Now consider the fibration sequence (due to Cerf):

$$\mathbf{Emb}_\partial(\text{ᗡ}^k, M_G) \hookrightarrow \mathbf{Emb}_{\mathbb{D}^\varepsilon_-}(\text{ᗡ}^k, M_G) \xrightarrow{K \mapsto K|_{\mathbb{D}^\varepsilon_+}} \mathbf{Emb}^\varepsilon_\partial(\mathbb{D}^{k-1}, M_G)$$

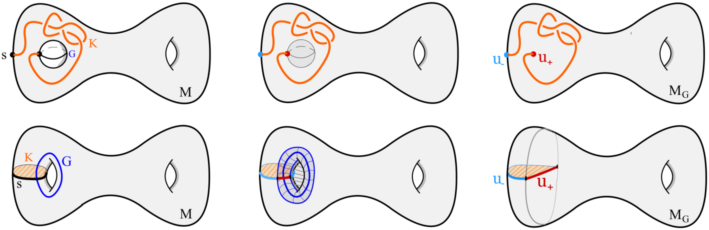
The total space is contractible (shrink the half-disk to its u^ε_--collar), so the connecting map $\mathfrak{amb}_U$ is a homotopy equivalence:

$$\Omega\,\mathbf{Emb}^\varepsilon_\partial(\mathbb{D}^{k-1}, M_G) \underset{\mathfrak{fol}^\varepsilon_U}{\overset{\mathfrak{amb}_U}{\rightleftarrows}} \mathbf{Emb}_\partial(\text{ᗡ}^k, M_G) \quad (\sim)$$

with the inverse $\mathfrak{fol}^\varepsilon_U(K)$ given as the loop of ε-augmented $(k-1)$-disks foliating the sphere $-U \cup K$. □

$k, d = 1, 3$

$k, d = 2, 3$



Theorem [K-Teichner]

In the setting with a dual, there is an explicit pair of homotopy equivalences

$$\mathrm{Emb}_\partial(\mathbb{D}^k, M) \underset{\mathfrak{amb}}{\overset{\mathfrak{fol}^\varepsilon}{\underset{\sim}{\rightleftarrows}}} \mathcal{P}\mathrm{ath}_{\mathrm{u}_-}^{\mathrm{u}_+}\left(\mathrm{Emb}_\partial^\varepsilon(\mathbb{D}^{k-1}, M \cup_{\nu G} h^{d-k+1})\right).$$

Moreover, if $\mathrm{Emb}_\partial(\mathbb{D}^k, M)$ is nonempty, then a choice of a basepoint U yields a homotopy equivalence to the loop space $\Omega_{\mathrm{u}_+}\left(\mathrm{Emb}_\partial^\varepsilon(\mathbb{D}^{k-1}, M \cup_{\nu G} h^{d-k+1})\right)$.

Applications: the group of 2-disks in a 4-manifold

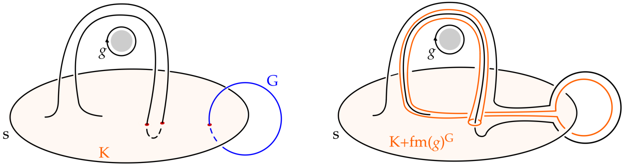
2-knots

Theorem [K-Teichner] Assume for a 4-manifold M and $\mathbf{s}\colon \mathbb{S}^1 \hookrightarrow \partial M$ that there is a dual $G\colon \mathbb{S}^2 \hookrightarrow \partial M$. Then there is an exact sequence of sets

$$\mathbb{Z}[\pi\setminus 1]^\sigma \big/ \mathsf{dax}(\pi_3 M) \underset{\mathsf{Dax}}{\overset{+\,\mathsf{fm}(\bullet)^G}{\rightleftarrows}} \pi_0\,\mathsf{Emb}_\partial(\mathbb{D}^2, M) \xrightarrow{j} \pi_0\,\mathsf{Map}_\partial(\mathbb{D}^2, M) \overset{\mu_2}{\twoheadrightarrow} \mathbb{Z}[\pi\setminus 1]\big/\langle r-\overline{r}\rangle$$

where $\pi := \pi_1 M$ and $\mathbb{Z}[\pi\setminus 1]^\sigma := \left\{ \sum_i \epsilon_i g_i \in \mathbb{Z}[\pi\setminus 1] : \sum_i \epsilon_i g_i = \sum_i \epsilon_i g_i^{-1} \right\}$.

- embeddings homotopic to $K\colon \mathbb{D}^2 \hookrightarrow M$ are obtained from K by the action $+\,\mathsf{fm}(r)^G$: do finger moves along r, and then Norman tricks:



- the relative Dax invariant, given by a clever count of double point loops in a homotopy to K, detects the action: for all $r \in \mathbb{Z}[\pi\setminus 1]^\sigma$ we have

$$\mathsf{Dax}(K + \mathsf{fm}(r)^G, K) = [r].$$

Theorem [K-Teichner]

After choosing an arbitrary basepoint $\mathbb{U} \in \mathsf{Emb}_\partial[\mathbb{D}^2, M]$, the above sequence becomes an exact sequence of groups

$$\mathbb{Z}[\pi \setminus 1]^\sigma \big/ \mathsf{dax}(\pi_3 M) \underset{\mathsf{Dax}}{\overset{+\,\mathsf{fm}(\bullet)^G}{\rightleftarrows}} \pi_0 \,\mathsf{Emb}_\partial(\mathbb{D}^2, M) \xrightarrow{j} \pi_0 \,\mathsf{Map}_\partial(\mathbb{D}^2, M) \overset{\mu_2}{\twoheadrightarrow} \mathbb{Z}[\pi \setminus 1] \big/ \langle r - \bar{r} \rangle$$

with $\mathbb{U}$ as the unit of the almost never abelian group $\pi_0 \,\mathsf{Emb}_\partial(\mathbb{D}^2, M)$, and with a nonstandard group structure on $\pi_2 M \cong \pi_0 \,\mathsf{Map}_\partial(\mathbb{D}^2, M)$.

Moreover, our group structure on $\pi_0 \,\mathsf{Emb}_\partial(\mathbb{D}^2, M)$ is related to the natural one on the mapping class group $\pi_0 \,\mathsf{Diff}_\partial(M)$: Consider the normal subgroup

$$\mathbb{D}(M; \mathsf{s})^0 < \pi_0 \,\mathsf{Emb}_\partial(\mathbb{D}^2, M)$$

of disks K that induce the same framing on $\mathsf{s} = \partial K$ as the chosen undisk $\mathbb{U}$.

Theorem [K-Teichner]

In the above setting, there is a right split short exact sequence of groups

$$\mathbb{D}(M; \mathsf{s})^0 \overset{a_M}{\rightarrowtail} \pi_0 \,\mathsf{Diff}_\partial(M) \overset{\pi_0 i}{\twoheadrightarrow} \pi_0 \,\mathsf{Diff}_\partial(M \cup_G h^3).$$

Vielen Dank!