~ A geometric approach to the embedding calculus ~

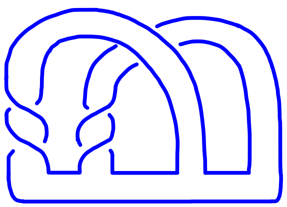
Danica Kosanović

Добродошли!
Willkommen!
Welcome!

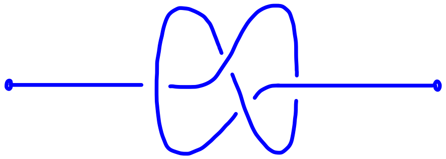
25.9.2020
Bonn
(virtually)

Some objects of interest in GEOMETRIC TOPOLOGY

a knot



a long knot



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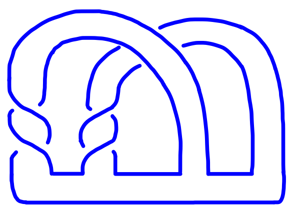
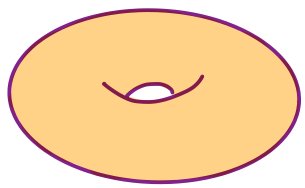


a closed surface



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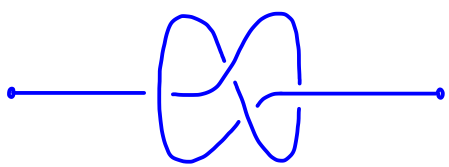
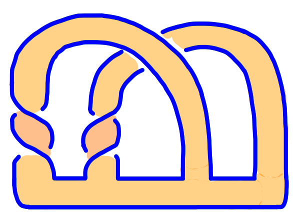


a surface with boundary



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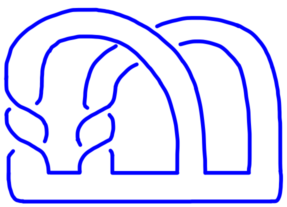

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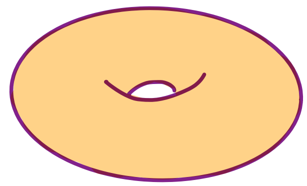
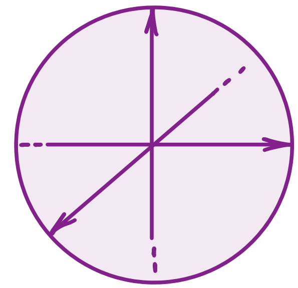


a closed 3-manifold



$\cup \ \infty$

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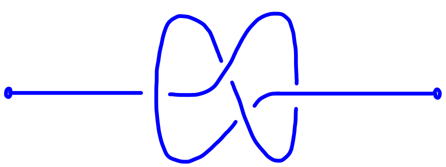

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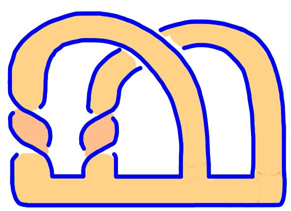

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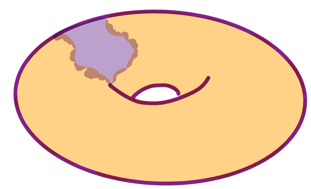

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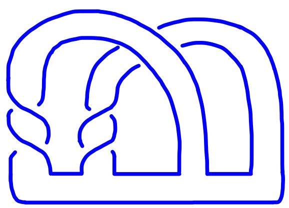

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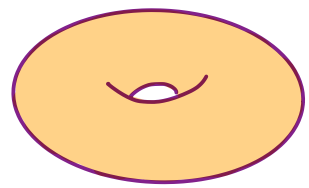

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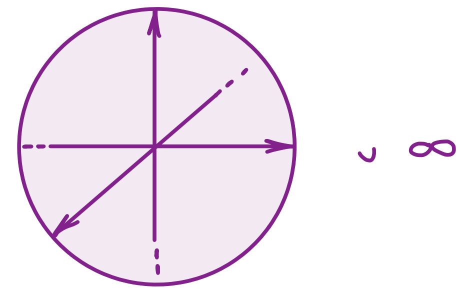

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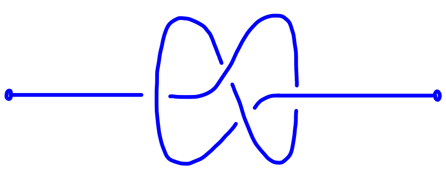

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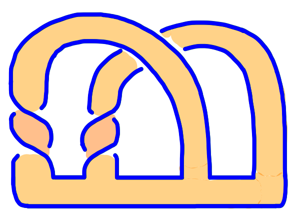

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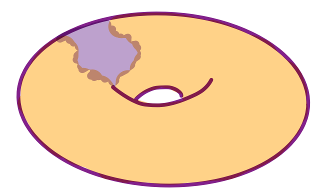
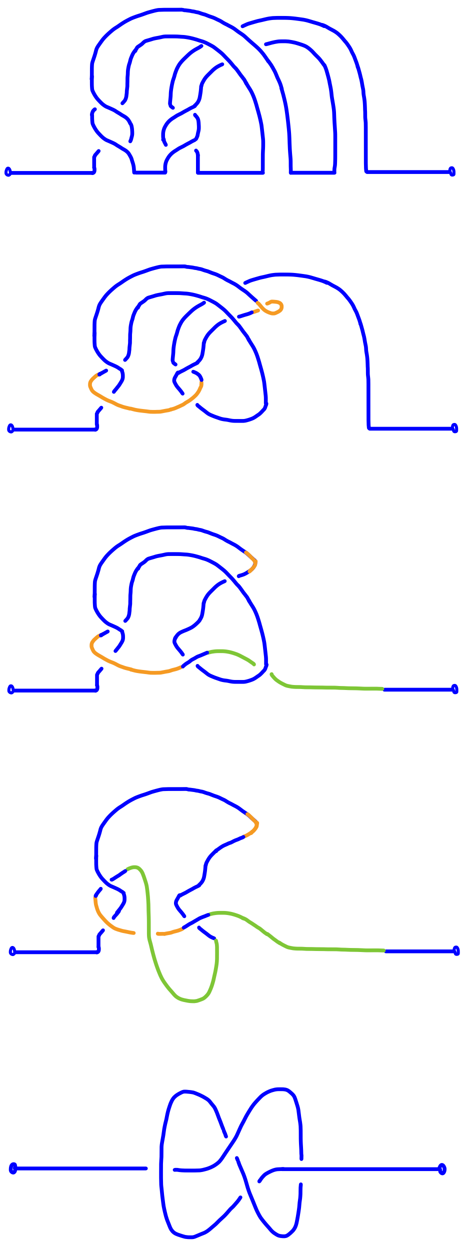


One fundamental question: Describe all long knots up to isotopy.

Another one: Describe all 2-knots in a 4-manifold up to isotopy.

An isotopy:

K



isotopy

K'

$K \simeq K'$

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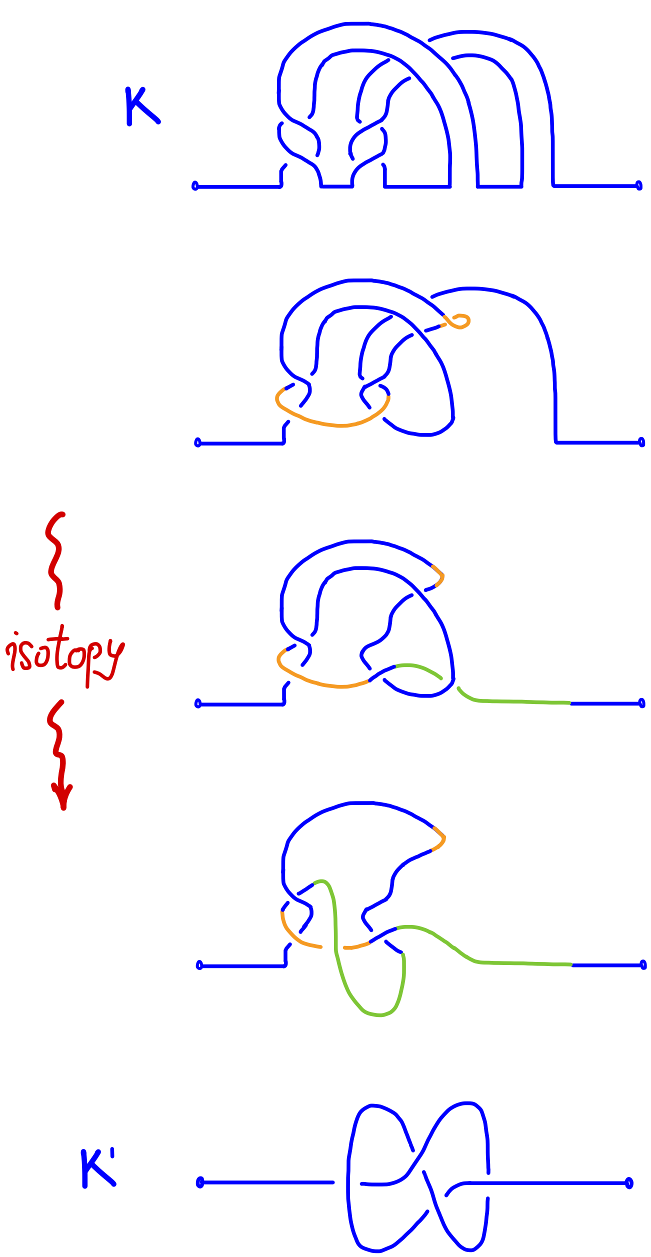

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$$\mathbb{K} := \{(\text{long}) \text{ knots}\} \big/ \simeq$$

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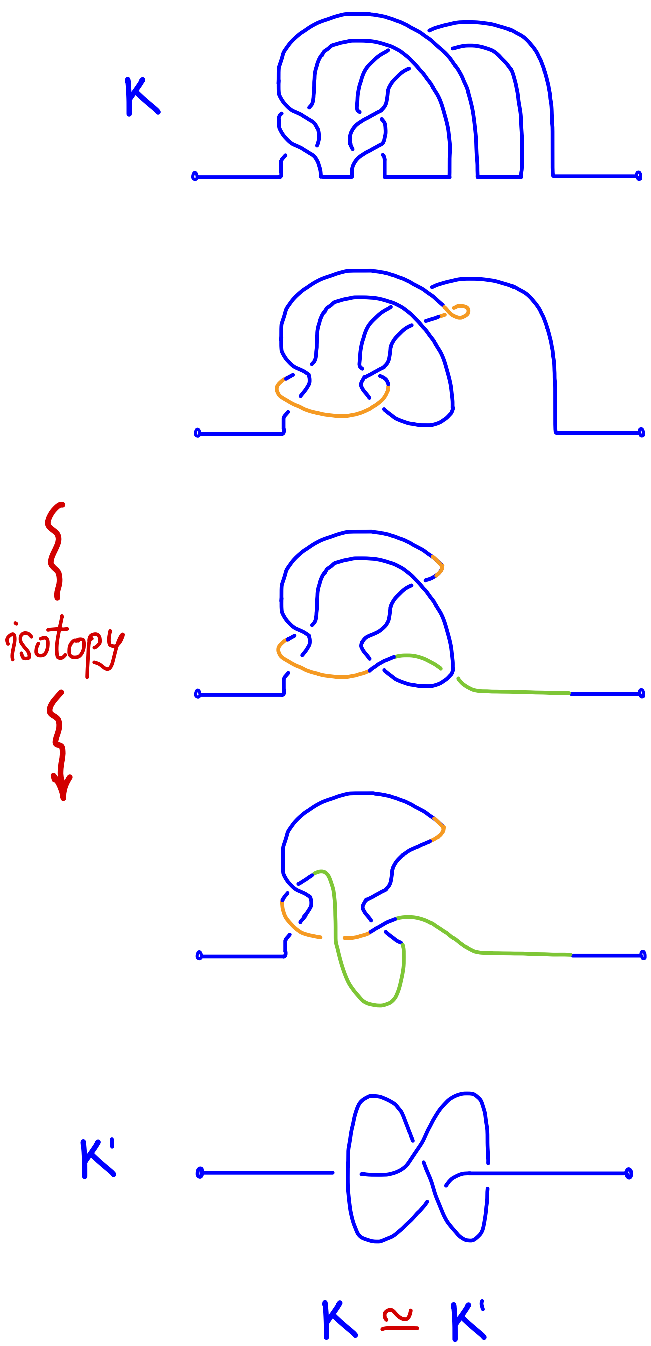

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Is there an additional structure on this set?

What patterns do we see?

What are some interesting subsets?

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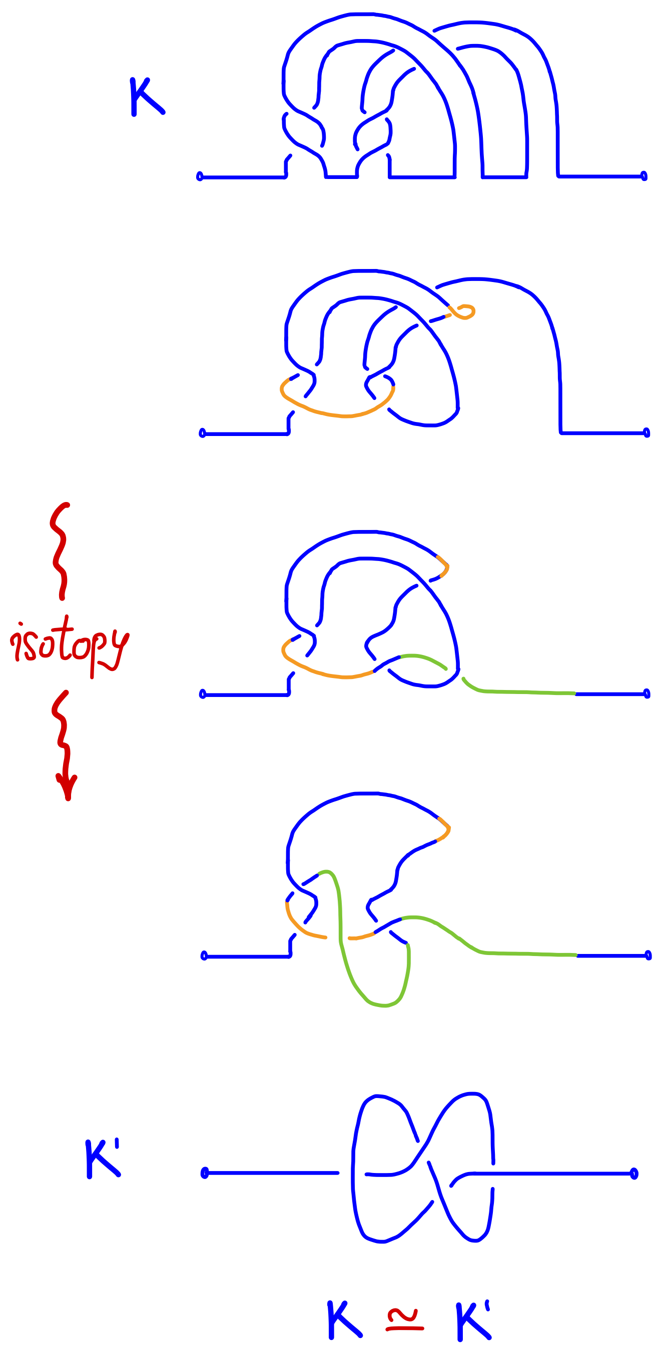

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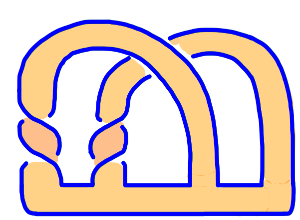
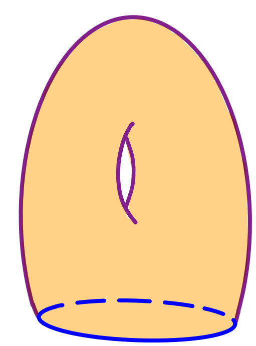
One classical idea:

Assign each (long) knot the smallest genus of a surface bounded by it.



$g(K) = 1$

abstractly:



Another idea:

Instead of surfaces consider
their iterations: **gropes.**

Instead of genus (length),
measure the degree (height).
First appeared in 4-manifold topology...

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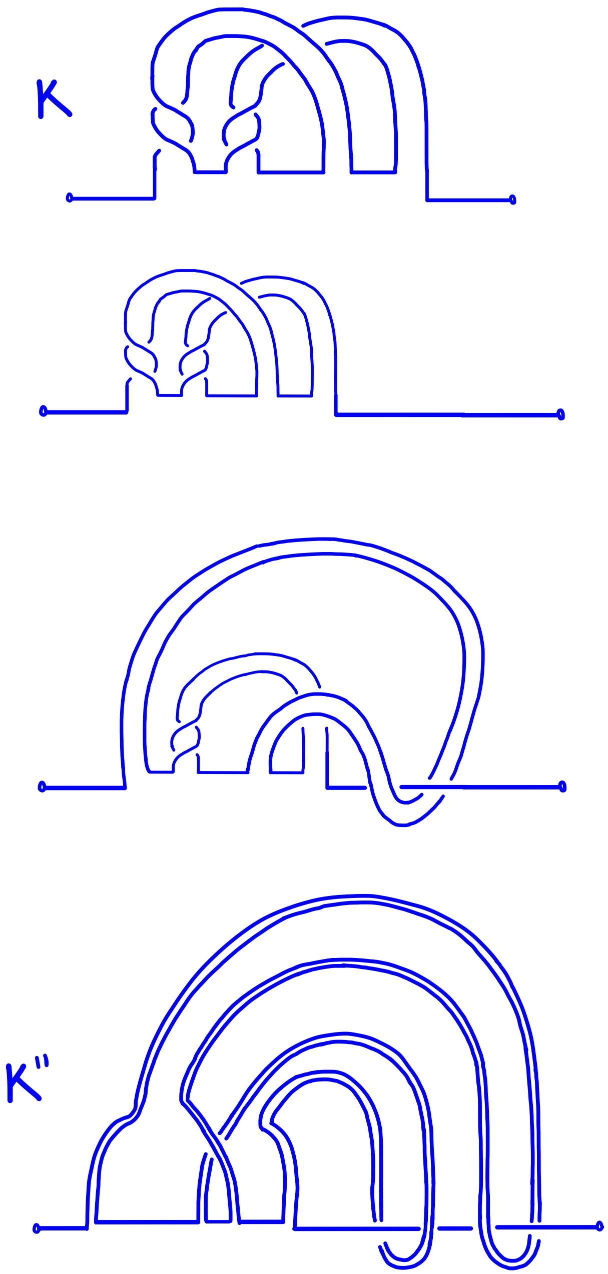

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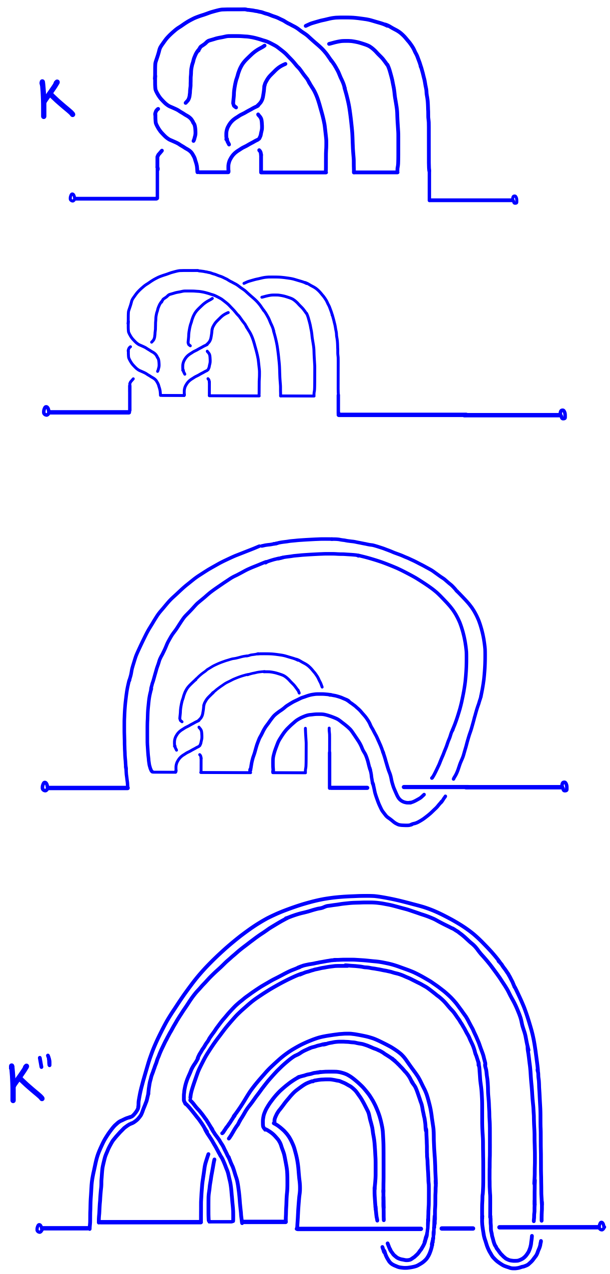

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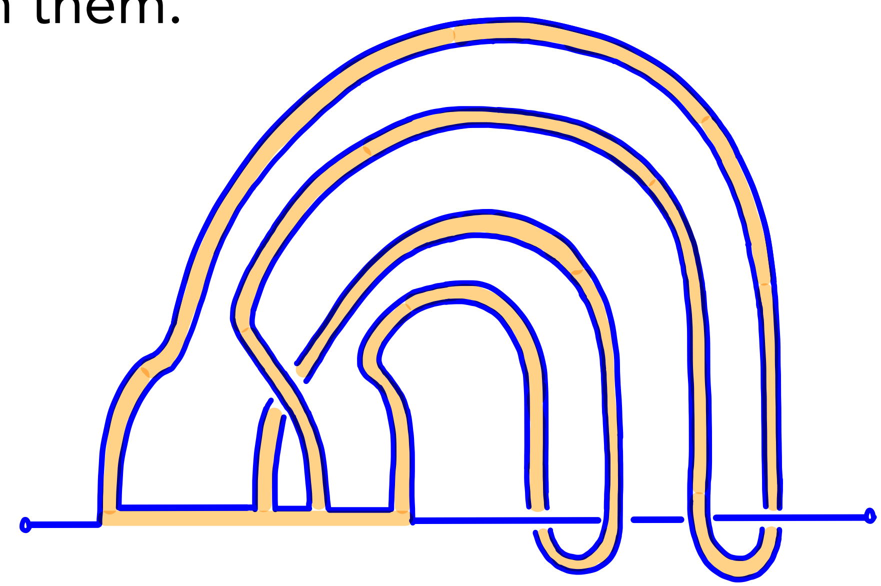
Two knots are **n-equivalent** if there exists a capped grope cobordism of degree n between them.



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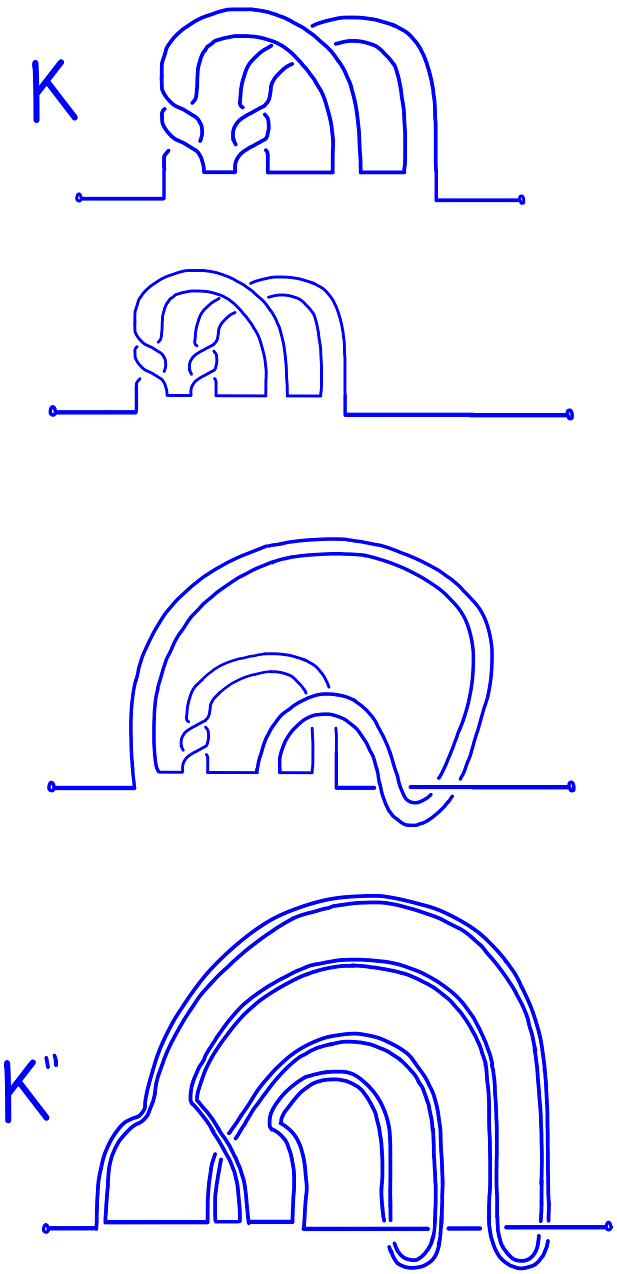

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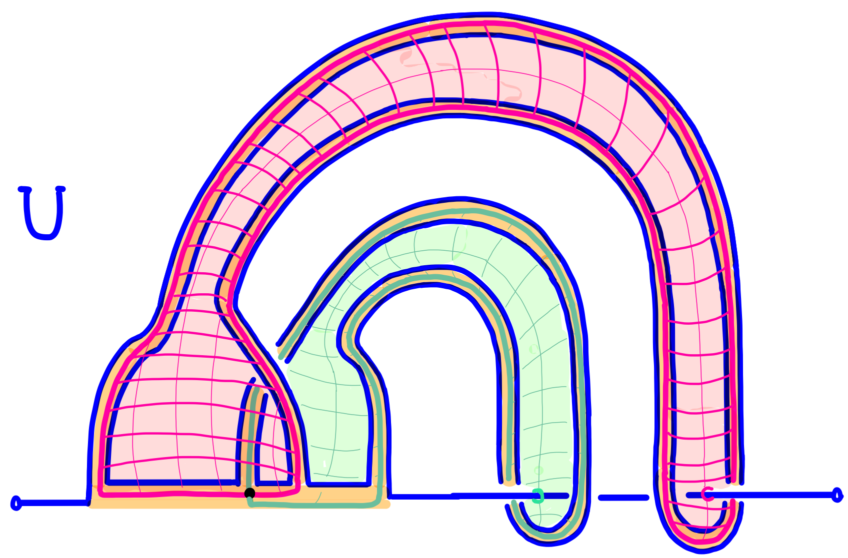

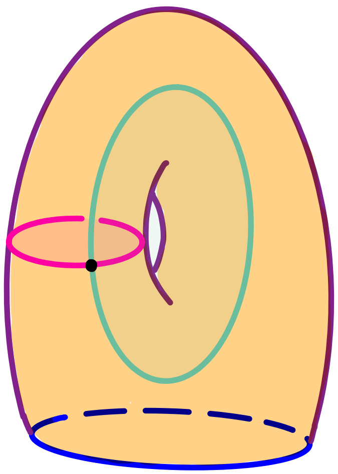

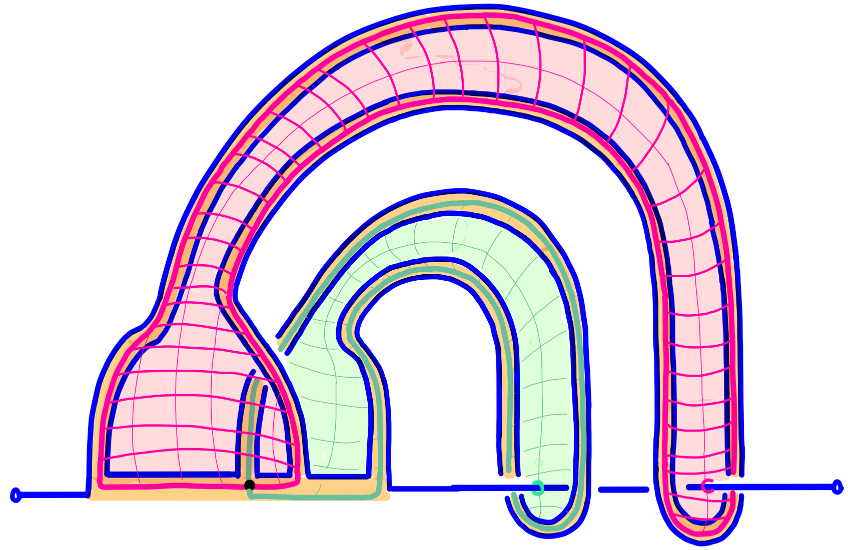

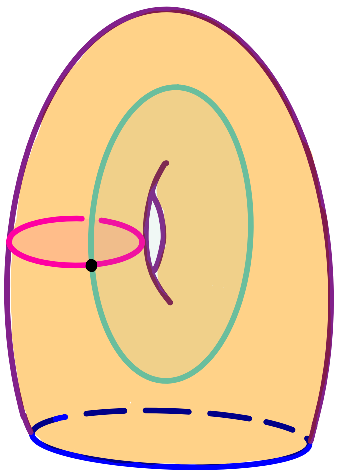

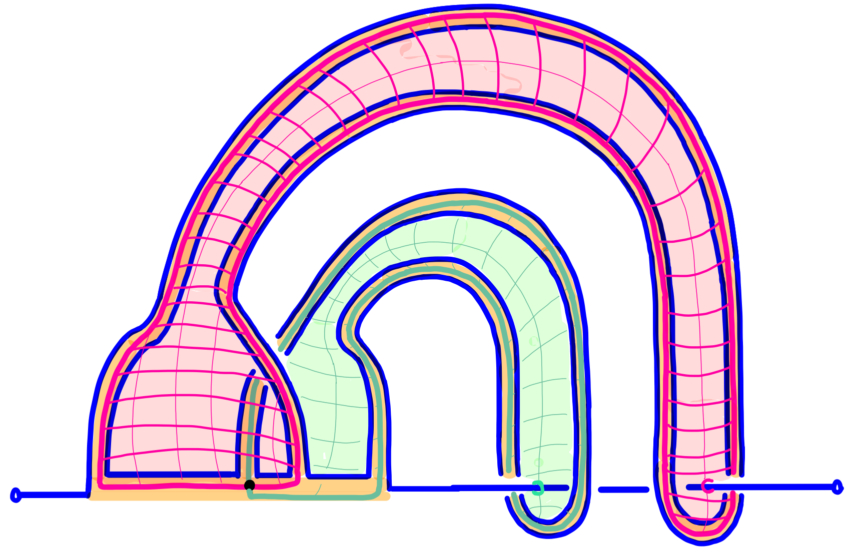

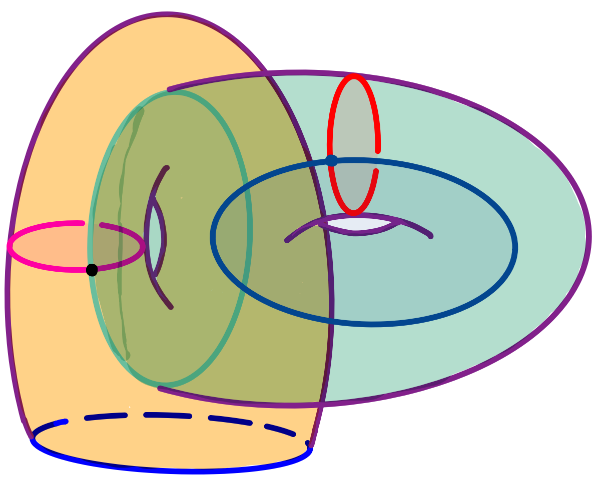

Does not exist for this knot.
That is, K is **not** 3-equivalent
to the unknot U.

This refines the relation of isotopy $\cong$ to a sequence of relations $\sim_n$ for n=1,2,3...

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Theorem [Conant-Teichner 2004]

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Theorem [Gusarov 2000, Habiro 2000, Conant-Teichner 2004]

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Question Are there any torsion elements in $\mathbb{K}/\sim_n$?

Conjecture [Budney-Conant-Scannell-Sinha 2005]

For each n≥1 there is
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$$\pi_0 ev_n : \mathbb{K}/_{\sim_n} \longrightarrow \pi_0 P_n$$

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Corollaries of Theorem E.

BCSS Conjecture is true:

1) over $\mathbb{Q}$.
2) over $\mathbb{Z}_p$ in a range (for $n \leq p+2$).
3) for $n \leq 7$.

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4) As a consequence we obtain Goodwillie-Klein [2015] estimates for the connectivity of ev_n in some missing cases:

$$ev_n : \mathrm{Emb}_\partial(L, M) \longrightarrow P_n(L, M)$$

is $\big(3 - \dim M + (n+1)(\dim M - \dim L - 2)\big)$ - connected

also for $L = I$ and $\dim M = 3$.

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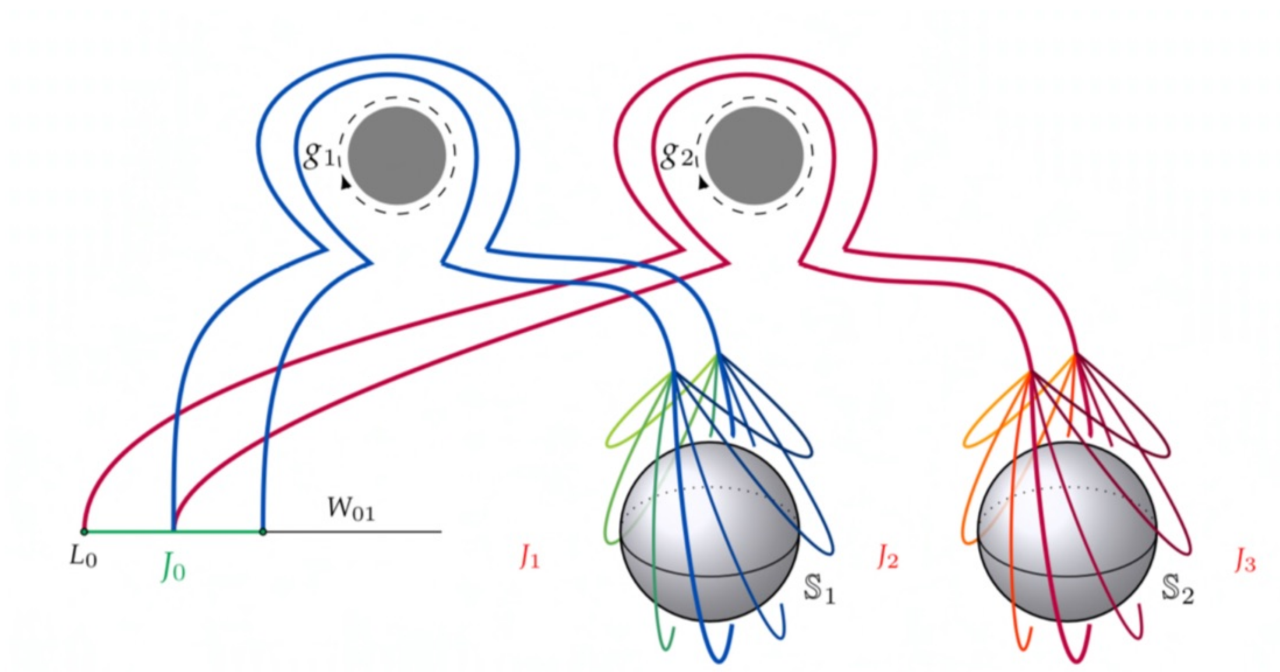

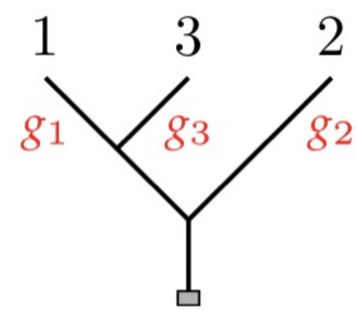

$g_i \in \pi_1 M$

Thank you for your attention!

MPIM Bonn 2019

Srni 2018