KNOT THEORY MEETS HOMOTOPY THEORY

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MPI.

Danica Kosanović
danica@mpim-bonn.mpg.de.

Knots

OBJECTS embeddings : $\mathcal{K} = \mathrm{Emb}_*(S^1, S^3)$

RELATIONS isotopies : paths in $\mathcal{K}$

$\Rightarrow \pi_0 \mathcal{K} =$ comm. monoid of knot types

INVARIANTS $H^0(\mathcal{K}; A) =$ locally constant A-valued maps

QUESTIONS

1) Can we distinguish a given knot from the unknot?
2) How far is it from being the unknot?

Homotopy

maps : $\mathcal{M} = \mathrm{Map}_*(S^1, S^3)$

homotopies : paths in $\mathcal{M}$

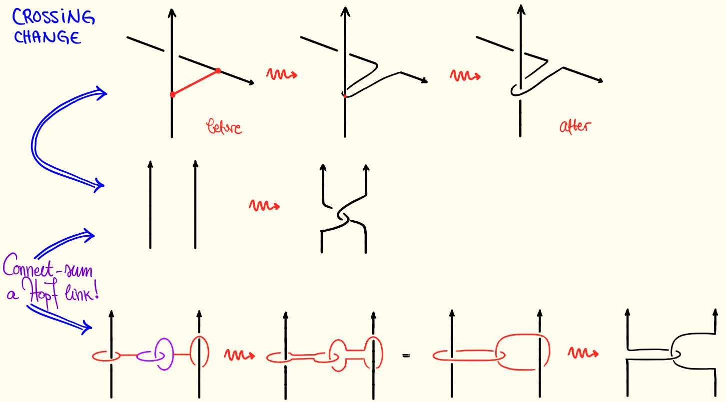
$\Rightarrow \pi_0 \mathcal{M} = [S^1, S^3]_* \cong \pi_1 S^3 = 0.$

End of story? Not quite!

MAYBE HELPS:

What is in $\mathcal{M} \setminus \mathcal{K}$? Look for a filtration?

CROSSING CHANGE



What next? Do multiple crossing changes or ...

IMMERSIONS: $\mathcal{I} := \mathrm{Imm}(S^1, S^3) \subseteq \mathcal{M}$

paths in $\mathcal{I}$: isotopies and $\underbrace{\text{crossing changes}}$.

"

crossing a "wall"

i.e. a codim 1 hyperplane in Σ.

What next?

NOTE: Original approach by Vassiliev:

study the "discriminant" $\Sigma = \mathcal{M} \setminus \mathcal{K}$

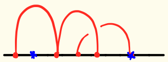
STEP 1. Alexander duality $H^*(\mathcal{K}) \cong H_*(\Sigma)$

STEP 2. Simplicial resolution $\Sigma' \xrightarrow{\sim} \Sigma$

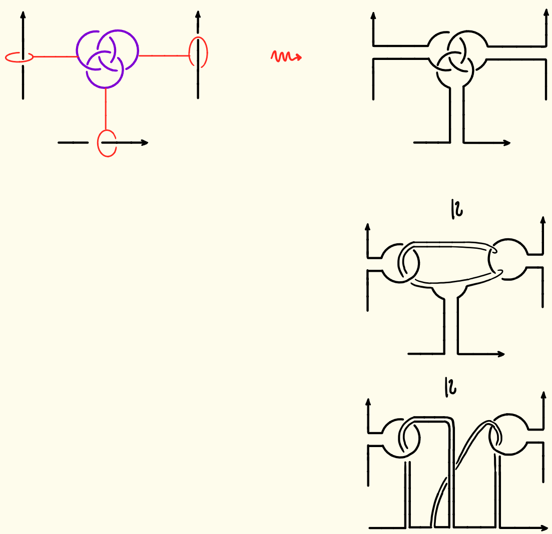
STEP 3. Filtration on Σ' $\rightsquigarrow$ Vassiliev spectral sequence.

roughly (multiple crossing changes & cusps

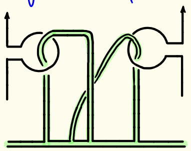
ex. of a corresponding cord diagram:



IDEA: Connect-sum Borromean rings!



NOTE:
Can see something like
a part of a genus 1 surface:



GEOMETRICALLY: these diagrams are too complicated...

HISTORICALLY: Feynman diagrams appeared!

THEOREM [BAR-NATAN '95] $\dfrac{\text{cord diagrams}}{4T,\ 1T} \cong \dfrac{\text{Jacobi diagrams}}{STU}$

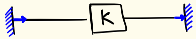
Q: What are Jacobi diagrams geometrically?

SETUP.

From now on:

$\mathcal{K} := \mathrm{Emb}_\partial(I, I^3)$

$\mathcal{J} := \mathrm{Imm}_\partial(I, I^3)$



Note: $\mathcal{J} \simeq \Omega S^2$

Goodwillie-Weiss embedding calculus ('99):

applied to the functor $\mathrm{Emb}_\partial(-, I^3)\colon \mathcal{O}(I) \longrightarrow \mathrm{Top}$

In some cases $\mathrm{Emb}_\partial(I, I^d) \simeq T_\infty \mathrm{Emb}_\partial(I, I^d)$

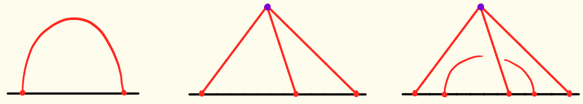
You guess: only for $d \geq 4$!

However:

* **Conjecture.** For every $n \geq 1$
$\pi_0(ev_n)\colon \pi_0\mathcal{K} \longrightarrow \pi_0 T_n\mathcal{K}$
is a universal invariant of type $n-1$.

Def. A *Jacobi forest* is a uni-trivalent graph with $b_1(\Gamma)=0$ whose univalent vertices are attached to a distinguished line and whose trivalent vertices are oriented
i.e. there is a cyclic order on the edges incident to them.

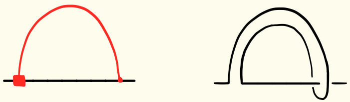
The degree of Γ is half the total number of vertices.



a cord diagram deg 1 — a Jacobi tree deg 2 — a Jacobi forest - NOT separated. deg 3

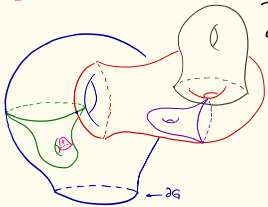
Def. Given a Jacobi tree Γ of degree n and a choice of root Γ_* we construct *an abstract grope* G_{Γ_*} of the shape Γ_* inductively:

no trivalent vertices ⇝ Γ = a single cord ⇝ G_{Γ_*} = a disc

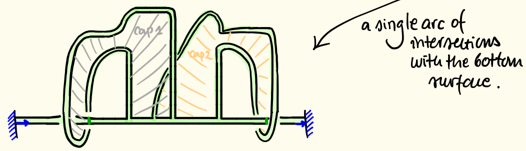


 root edge ⇝ genus 1 surface Σ

subtrees T_1 & T_2 ⇝ attach gropes $G_{T_{1*}}$ and $G_{T_{2*}}$ to simple closed curves representing a symplectic basis for $H_1(\Sigma;\mathbb{Z})$.



IN PRACTICE: Embed $G_{\Gamma_*} \hookrightarrow I^3$ and demand all caps simple !



Def. Two knots K and K' are *directly n-equivalent* if there exists an embedded grope cobordism G_{Γ_*} between them such that $\deg\Gamma = n$.
They are *n-equivalent* if they can be connected by a sequence of direct n-equivalences.

A knot n-equivalent to the unknot is called *n-trivial*.

We obtain the *Gusarov-Habiro filtration*:

$$\pi_0 \mathcal{K} = G_1 \supseteq G_2 \supseteq \dots \supseteq \underset{\underset{\{\text{n-trivial knots}\}}{\shortparallel}}{G_n} \supseteq \dots$$

Remark. If you know about finite type invariants, then: K is n-equivalent to K' iff they have same invariants of type $\leq n-1$.
(Thm by Gusarov, Habiro and Conant-Teichner).

$$
\begin{tikzcd}
 & T_\infty\mathcal{K} := \operatorname{holim} T_n\mathcal{K} \arrow[d, dotted] \\
 & T_{n+1}\mathcal{K} \arrow[d] \\
 & T_n\mathcal{K} \arrow[d, dotted] \\
 & T_2\mathcal{K} \arrow[d] \\
\mathcal{K} = \mathrm{emb}_\partial(I, I\times\mathbb{R}^2) \arrow[r, "ev_1"] \arrow[ur, bend left, "ev_2"] \arrow[uur, bend left, "ev_n"] \arrow[uuur, bend left, "ev_{n+1}"] \arrow[uuuur, bend left, "ev_\infty"] & T_1\mathcal{K} \simeq \mathrm{Imm}_\partial(I, I\times\mathbb{R}^2)
\end{tikzcd}
$$

* **Theorem** [Budney - Conant - Koytcheff - Sinha]
$\pi_0(ev_n)$ is of type $n-1$.

We obtain the same result as the **Corollary.** of:

* **Theorem** [K-Shi-Teichner, work in progress]
Given an abstract grope G of degree n, there is a continuous map:

$$ev : \mathrm{emb}^{sc}(G, I\times\mathbb{R}^2) \longrightarrow P\,T_n\mathcal{K}$$

such that $ev(G)$ is a path from $ev_n(G|_{\partial_0})$ to $ev_n(G|_{\partial_1})$.