

FINITE INDEX SOLUTIONS TO THE BERNOULLI PROBLEM IN THREE DIMENSIONS ARE AXIALLY SYMMETRIC

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ABSTRACT. We show that every entire solution to the Bernoulli (or one-phase) free boundary problem with finite Morse index in $\mathbb{R}^3$ is axially symmetric. In fact, we additionally prove that the same result would follow in any dimension $4 \leq n \leq 6$ in which stable entire solutions are shown to be flat.

CONTENTS

1. Introduction	1
1.1. Main results	2
2. Solutions in exterior domains	4
2.1. Improvement of flatness in annuli	5
2.2. Higher-order expansion for entire solutions	10
3. Asymptotic structure of entire solutions with finite index	18
3.1. Preliminaries	18
3.2. Asymptotic structure in the non-compact free boundary case	21
4. Symmetry for entire solutions with finite index	22
4.1. Compact free boundary case	23
4.2. Non-compact free boundary case: symmetry in x_n	23
4.3. Non-compact free boundary case: symmetry in x_1	26
4.4. Proof of Theorems 1.1 and 1.2	30
5. Total mean curvature	30
References	33

1. INTRODUCTION

Given an open set $D \subset \mathbb{R}^n$, we say that $u : D \rightarrow [0, \infty)$ is a classical solution to the Bernoulli (or one-phase) free boundary problem if:

$$\{u > 0\} \text{ is locally a smooth domain in } D \quad \text{and} \quad \begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \cap D, \\ |\nabla u| = 1 & \text{on } \partial\{u > 0\} \cap D. \end{cases} \quad (1.1)$$

Such solutions are critical points of the *Alt-Caffarelli functional*

$$\mathcal{E}(v; A) := \int_A (|\nabla v|^2 + \mathbf{1}_{\{v>0\}}) \, dx$$

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in any open subset A with compact closure in D . This functional was originally introduced by Alt and Caffarelli in [1], and since then, it has become one of the most studied free boundary problems (see also the monographs [6, 49]). In recent years, the one-phase Bernoulli problem has been studied increasingly beyond the classical minimizing framework, [35, 21, 4, 38, 7, 22, 33].

We say that u is *stable in A* if the second variation quadratic form of the Alt–Caffarelli functional is nonnegative. That is,

$$\mathcal{Q}(\varphi, \varphi; A) := \int_{A \cap \{u > 0\}} |\nabla \varphi|^2 dx - \int_{A \cap \text{FB}(u)} H \varphi^2 d\mathcal{H}^{n-1} \geq 0 \quad \forall \varphi \in C_c^\infty(A \cap \overline{\{u > 0\}}), \quad (1.2)$$

where H denotes the mean curvature of the *free boundary* $\text{FB}(u) := \partial\{u > 0\}$ with respect to the outer unit normal $\nu = -\nabla u$. The definition extends to the case where A is noncompact, since we always consider compactly supported variations.

More generally, u is said to have *Morse index $m \in \mathbb{N} \cup \{0\}$ in $A \subset D$* if

$$m = \sup \left\{ \dim E : E \subset C_c^\infty(A \cap \overline{\{u > 0\}}) \text{ is a linear subspace with } \mathcal{Q}(\varphi, \varphi; A) < 0 \ \forall \varphi \in E \setminus \{0\} \right\}.$$

If $D = \mathbb{R}^n$, then the finiteness of index in $\mathbb{R}^n$ implies stability in $\mathbb{R}^n \setminus \overline{B_{R_0}}$ for some $R_0 > 0$; see [4, Proposition 3.4].

The goal of this article is to characterize global classical solutions to the Bernoulli problem with finite Morse index in three dimensions, and more generally in every dimension $3 \leq n \leq 6$ in which stable entire solutions are known to be flat. This had been previously achieved only in the two-dimensional case [4, 47], exploiting the log cut-off trick and a deep, one-to-one correspondence between entire solutions to the Bernoulli problem with finite connectivity in the plane and minimal bigraphs in $\mathbb{R}^3$ —both of which are not available in higher dimensions.

We emphasise that there is a wealth of entire classical solutions to (1.1) without further constraints, as showcased by the examples in [3, 32, 47, 41, 15, 19] and their natural extensions to additional dimensions. They have rich structures in terms of free boundary connectivity, topology and symmetries; in dimension $n \geq 7$, the examples from [15, 19] are even energy-minimizing.

The finite index hypothesis and dimensional restrictions are then essential in our result. They will lead to a decomposition of such solutions—away from a large ball—into a sum of two solutions with connected free boundaries that become progressively flatter at infinity (see Section 3). We will then combine a novel improvement of flatness in annuli technique developed in Section 2 (see also the forthcoming work [23]) with the moving planes method à la Schoen/Serrin [45, 46] in Section 4 and several new arguments particular to the Bernoulli problem, leading to the axial symmetry of all such solutions. See Section 1.1 for more details.

The study of finite index solutions to geometric variational problems, and in particular of entire solutions in Euclidean space, has a long history in the literature. Such solutions provide many of the known examples [12, 16]. Moreover, they are especially relevant to the existence theory in compact geometries, see [42, 28, 11]. For further examples of structure and classification results for solutions of finite index to other geometric variational problems, we refer the reader to [26, 48, 45, 9, 10, 43] for minimal hypersurface theory, [51, 30, 31, 37, 27] for the Allen–Cahn equation, [50, 18] for its elegant free boundary version, and references therein.

1.1. Main results. Our main result in dimension 3 is the following:

Theorem 1.1. *Let $u : \mathbb{R}^3 \rightarrow [0, \infty)$ be an entire classical solution to the Bernoulli problem with finite Morse index which is not one-dimensional. Then, after a suitable scaling, translation, and rotation, we have that:*

- either $u(x) = (1 - |x|^{-1})_+$;

- or $u(x_1, x_2, x_3) = u(r, x_3)$ is axially symmetric, with $r = \sqrt{x_1^2 + x_2^2}$, even in x_3 , and satisfies

$$\partial_{x_3} u > 0 \quad \text{in} \quad \{x_3 > 0\} \cap \{u > 0\}, \quad \partial_r u < 0 \quad \text{in} \quad \{r > 0\} \cap \{u > 0\}.$$

Moreover, $\{u = 0\} = \{(x', x_3) : |x'| \geq 1, |x_3| \leq g(|x'|)\}$ for some $g : [1, \infty) \rightarrow [0, \infty)$ with $g(1) = 0$, $g'(r) > 0$ for $r > 1$, and $\lim_{r \rightarrow \infty} g(r) < \infty$.

In fact, we obtain a sharp conditional result up to dimension 6. Define the *critical dimension*

$$n_* := \min\{n_*^1, n_*^2\}, \quad (1.3)$$

where

$$n_*^1 := \min \left\{ n \in \mathbb{N} : \begin{array}{l} \text{there exists a classical stable solution} \\ \text{in } \mathbb{R}^n \text{ which is not one-dimensional} \end{array} \right\}, \quad (1.4)$$

$$n_*^2 := \min \left\{ n \in \mathbb{N} : \begin{array}{l} \text{there exists a classical stable one-homogeneous} \\ \text{solution in } \mathbb{R}^n \setminus \{0\} \text{ which is not one-dimensional} \end{array} \right\}. \quad (1.5)$$

By [5, 34, 14] and [7, 15, 19] we have $5 \leq n_*^1 \leq 7$ and $4 \leq n_*^2 \leq 7$, hence

$$4 \leq n_* \leq 7.$$

Then, an analogous result holds for $3 \leq n < n_*$:

Theorem 1.2. *Let $3 \leq n < n_*$, with $n_* \geq 4$ given by (1.3). Let $u : \mathbb{R}^n \rightarrow [0, \infty)$ be an entire classical solution with finite Morse index which is not one-dimensional. Then, after a suitable scaling, translation, and rotation, we have that:*

- either $u(x) = \frac{1}{n-2}(1 - |x|^{2-n})_+$;
- or $u(x_1, \dots, x_{n-1}, x_n) = u(r, x_n)$ is axially symmetric, with $r = \sqrt{x_1^2 + \dots + x_{n-1}^2}$, even in x_n , and satisfies

$$\partial_{x_n} u > 0 \quad \text{in} \quad \{x_n > 0\} \cap \{u > 0\}, \quad \partial_r u < 0 \quad \text{in} \quad \{r > 0\} \cap \{u > 0\}.$$

Moreover, $\{u = 0\} = \{(x', x_n) : |x'| \geq 1, |x_n| \leq g(|x'|)\}$ for some $g : [1, \infty) \rightarrow [0, \infty)$ with $g(1) = 0$, $g'(r) > 0$ for $r > 1$, and $\lim_{r \rightarrow \infty} g(r) < \infty$.

Remark 1.3. It is interesting to notice that, in fact, our proof shows that any classical entire solution $u : \mathbb{R}^n \rightarrow [0, \infty)$, decomposing as $u = u_+ + u_-$ away from a large ball B_{R_0} , with $u_{\pm}$ asymptotically flat classical solutions in $\mathbb{R}^n \setminus B_{R_0}$, satisfies the thesis of Theorem 1.2.

Remark 1.4. In fact, we have, for any $\alpha \in (0, 1)$,

$$\left| u - |x_n| + b - \frac{c}{|x|^{n-2}} - \frac{(n-2)bc|x_n|}{|x|^n} \right| \leq \frac{C_\alpha}{|x|^{n-1+\alpha}} \quad \text{in} \quad \{u > 0\},$$

for some $b, c > 0$. In particular, the asymptotic expansion for $g : [1, \infty) \rightarrow [0, \infty)$ is:

$$\begin{aligned} g(r) &= b - cr^{2-n} + O(r^{-n+1-\alpha}), \\ g'(r) &= (n-2)cr^{1-n} + O(r^{-n-\alpha}), \\ g''(r) &= -(n-1)(n-2)cr^{-n} + O(r^{-n-\alpha-1}), \end{aligned}$$

Moreover, we know $g'(r) > 0$ for $r > 1$, and from the sign on the mean curvature of the free boundary we also know $g''(r) \leq -g'(r)(1 + g'(r)^2)/r < 0$ for $r > 1$.

Remark 1.5. In [41], the authors construct an example of an entire axially symmetric solution in all dimensions $n \geq 3$ via gluing methods. From its symmetries and monotonicity relations, it readily follows that it is stable outside of a ball¹, and our method applies (alternatively, one can use

¹We emphasize that this is known to be equivalent to having finite Morse index for a large class of problems and operators, see [17].

Theorem 1.6 below to get their expansion). Furthermore, they show that $\lim_{r \rightarrow \infty} g'(r)r^{n-2} \geq 0$ for their solution, while from the expansions above one deduces that, in fact, $\lim_{r \rightarrow \infty} g'(r)r^{n-1} \in (0, +\infty)$ in their construction.

A main step in the proof, which we believe may be of independent interest, is the study of asymptotically flat solutions in exterior domains. Our approach follows an improvement of flatness-type scheme in which the natural domains are annuli (instead of balls, as is the usual case in improvement of flatness-type iterations). This is a simple yet powerful and flexible tool, and which can be applied to other similar settings, see [23, 27], and compare also with the variational approach of [20].

We say that a (nonzero) classical solution $u : \mathbb{R}^n \setminus B_{R_0} \rightarrow [0, \infty)$ is *asymptotically flat* if for all $R > R_0$ large there exists $e_R \in \mathbb{S}^{n-1}$ such that

$$u = e_R \cdot x + o(R) \quad \text{in } (B_R \setminus B_{R/2}) \cap \{u > 0\}$$

as $R \rightarrow \infty$. The main result in Section 2 is:

Theorem 1.6. *Let $\alpha \in (0, 1)$, $R_0 > 0$, and u be a classical solution to the Bernoulli problem, (1.1), in $\mathbb{R}^n \setminus \overline{B_{R_0}} \subset \mathbb{R}^n$, which is asymptotically flat. Then, after a rotation and for some $(b, c, d) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$ with $d_n = (n-2)bc$, we have*

$$u(x) = x_n - b + \frac{c}{|x|^{n-2}} + \frac{d \cdot x}{|x|^n} + O(|x|^{1-n-\alpha}) \quad \text{in } \{u > 0\} \setminus B_{R_0}, \quad (1.6)$$

as $x \rightarrow \infty$.

Section 3 shows, via a blow-down analysis, that any entire solution $u : \mathbb{R}^n \rightarrow [0, \infty)$ with finite index and $3 \leq n < n_*$ decomposes as $u = u_+ + u_-$ outside of a large ball, where each of the $u_{\pm}$ is an asymptotically flat classical solution. Theorem 1.6 then applies to each of the $u_{\pm}$, yielding a precise asymptotic expansion for u . The resulting asymptotics are then used in a crucial (and subtle) way in Section 4, in combination with the moving planes method à la Schoen/Serrin [45, 46] and global information coming from stationarity, to deduce Theorem 1.2.

Remark 1.7. It is interesting to consider which alternative conditions might also provide such a strong characterization of global solutions. In two dimensions, the natural condition of finite connectivity of the free boundary suffices, by [47]. In higher dimensions, however, even finite topology is far from sufficient anymore, by the examples mentioned above. In fact, one expects a wealth of examples in $\mathbb{R}^3$ with finite topology and no elementary characterization, even with a density condition on the positivity set, as suggested by the homogeneous solutions recently constructed in [33]. In Section 5 we propose a new, related condition: the finiteness of the total mean curvature of the free boundaries, defined in a scale-invariant integral sense (see Definition 5.1). It implies finite connectivity in two dimensions, as shown in [4]. We will show that it ensures the axial symmetry of global solutions in all higher dimensions too, via the methods developed in the article.

2. SOLUTIONS IN EXTERIOR DOMAINS

In this section, we study asymptotically flat solutions in exterior domains. We first prove an annular improvement-of-flatness result and then use it to derive the higher-order expansion at infinity.

Throughout the paper, whenever $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we write

$$B'_r(y') := \{z' \in \mathbb{R}^{n-1} : |z' - y'| < r\}, \quad B'_r := B'_r(0).$$

Moreover, given sets $A_1, A_2, A_3 \subset \mathbb{R}^n$, we say

$$A_1 \subset A_2 \quad \text{in } A_3 \quad \Longleftrightarrow \quad A_1 \cap A_3 \subset A_2 \cap A_3.$$

2.1. Improvement of flatness in annuli. The following quantity records the best trapping of u between two parallel shifted half-plane solutions on a fixed annulus.

Definition 2.1. Let $n \geq 3$. For a given classical solution (1.1) in a domain $D \subset \mathbb{R}^n$ containing the annulus $B_{2r} \setminus \overline{B_{r/2}}$, we define the *shifted annular height* in the direction $e \in \mathbb{S}^{n-1}$, with shift $b \in \mathbb{R}$, and at scale $r > 0$ as:

$$H_b(u, e, r) := \inf \{ h > 0 : (e \cdot x - b - h)_+ \leq u \leq (e \cdot x - b + h)_+ \text{ in } B_{2r} \setminus \overline{B_{r/2}} \}.$$

The *centered annular height* is the particular case $b = 0$, $H_0(u, e, r)$. We also define

$$H(u, r) := \inf_{e \in \mathbb{S}^{n-1}, b \in \mathbb{R}} H_b(u, e, r) \quad \text{and} \quad H_0(u, r) := \inf_{e \in \mathbb{S}^{n-1}} H_0(u, e, r).$$

Notice that we have the scaling

$$H_b(u, e, r) = \rho H_{b/\rho}(u_\rho, e, r/\rho), \quad \text{for } \rho > 0 \text{ and } u_\rho(x) := \frac{u(\rho x)}{\rho}. \quad (2.1)$$

We next state the improvement of flatness argument in annuli. This is one of the main contributions of the work, and which can be extrapolated to other settings:

Theorem 2.2. Let $n \geq 3$, $R \geq 4$, and $\alpha \in (0, 1)$. There are $C = C(n, \alpha)$ and $\eta_0 = \eta_0(n, \alpha) > 0$ such that the following holds:

Let u be a classical solution to the Bernoulli problem, (1.1), in $D = B_R \setminus \overline{B_1} \subset \mathbb{R}^n$, satisfying

$$H_0(u, r) \leq \eta r \quad \text{for all } r \in (2, R/2), \quad (2.2)$$

for some $\eta \leq \eta_0$.

Then,

$$H(u, r) \leq C\eta r [r^{2-n-\alpha} + (r/R)^\alpha], \quad \text{for all } r \in (2, R/2). \quad (2.3)$$

The proof of the result relies on the following lemma:

Lemma 2.3. Let $n \geq 3$, $\alpha \in (0, 1)$, and $\varepsilon > 0$. There exists $\delta = \delta(n, \varepsilon, \alpha) > 0$ such that the following holds.

Define $\psi(t) := \max(t^{3-n-\alpha}, t^{1+\alpha})$. Let u be a classical solution to the Bernoulli problem in $D = B_{1/\delta} \setminus B_\delta \subset \mathbb{R}^n$, (1.1), satisfying, for some $e \in \mathbb{S}^{n-1}$, $|b| < \delta$, and $0 < \eta \leq \delta^{2n}$,

$$H_b(u, e, t) \leq \eta \psi(t) \quad \text{for all } 2\delta < t < \frac{1}{2\delta}. \quad (2.4)$$

Then

$$H(u, 1) \leq \varepsilon \eta.$$

Proof. Assume without loss of generality $e = e_n$.

If $\delta > 0$ is small enough, the classical improvement of flatness for the Bernoulli problem, [13], applied at every dyadic scale and rescaled, (2.1), yields

$$\partial\{u > 0\} = \{x_n = b + \eta g(x')\} \quad \text{in } (B_{\frac{1}{4\delta}} \setminus \overline{B_{4\delta}}),$$

where we denote $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and

$$\sup_{B'_{2t} \setminus B'_{t/2}} |g|/t + |\nabla g| + t|D^2 g| \leq C\psi(t)/t$$

for $8\delta < t < \frac{1}{8\delta}$ and some C dimensional, and where B'_r denotes the ball of radius r in $\mathbb{R}^{n-1}$. We have used here that, since $\eta \leq \delta^{n-1+\alpha}$, $\eta\psi(t)/t < C\delta$ for all $t \in (\delta, 1/\delta)$, $\delta > 0$ is small enough depending only on n , and the linear estimates for higher order regularity of the free boundary (see [36, 39] or [7, Lemma 3.7]).

In particular for any $\tilde{e} \in \mathbb{S}^{n-1}$ with $|\tilde{e} - e_n| \leq C\eta\psi(t)/t$ and for all $x \in \partial\{u > 0\} \cap (B_{2t} \setminus \overline{B_{t/2}})$ we have

$$\begin{aligned} |\partial_{\tilde{e}}(u - e_n \cdot x)| &= |\tilde{e} \cdot \nabla u(x) - \tilde{e} \cdot e_n| \leq |1 - \tilde{e} \cdot \nabla u(x)| + |1 - \tilde{e} \cdot e_n| \\ &= \frac{1}{2}|\tilde{e} - \nabla u(x)|^2 + \frac{1}{2}|\tilde{e} - e_n|^2 \leq C(\eta\psi(t)/t)^2, \end{aligned} \quad (2.5)$$

where we have used here that $|1 - e_1 \cdot e_2| = \frac{1}{2}|e_1 - e_2|^2$ for unit vectors $e_1, e_2 \in \mathbb{S}^{n-1}$, the triangle inequality, and the fact that $|\nabla u - e_n| \leq C\eta\psi(t)/t$ on the free boundary in $B_{2t} \setminus \overline{B_{t/2}}$.

Thus, in the coordinates

$$y' = x', \quad y_n = x_n - \eta g(x') - b$$

the function

$$v(y) := \frac{u(x(y)) - x_n(y) + b}{\eta}$$

solves a divergence form equation with Lipschitz coefficients arbitrarily close to the identity in $\{y_n > 0\}$,

$$\operatorname{div}_y (A(y) \nabla_y v) = 0 \quad \text{in } B_{\frac{1}{8\delta}} \setminus B_{8\delta} \cap \{y_n > 0\} \quad \text{where } A(y) := \begin{pmatrix} \operatorname{Id}_{n-1} & -\eta \nabla g(y') \\ -\eta \nabla^\top g(y') & 1 + \eta^2 |\nabla g(y')|^2 \end{pmatrix}.$$

Moreover using (2.5) and by assumption (2.4) we have, for $\delta > 0$ small enough and since $\eta \leq \delta^{2n}$,

$$\begin{cases} |v(y)| \leq C\psi(|y|) & \text{in } B_{\frac{1}{8\delta}} \setminus B_{8\delta} \cap \{y_n > 0\}, \\ |\partial_{y_n} v(y)| \leq C\eta(\psi(|y|)/|y|)^2 \leq \delta & \text{on } B_{\frac{1}{8\delta}} \setminus B_{8\delta} \cap \{y_n = 0\}. \end{cases}$$

Now, up to redefining δ small enough and applying a standard compactness result (see Lemma 2.4 below), we have

$$|v(y) - a \cdot y - d| < \varepsilon/4 \quad \text{in } (B_4 \setminus \overline{B_{1/4}}) \cap \{y_n > 0\},$$

for some universally bounded $a \in \mathbb{R}^n$ and $d \in \mathbb{R}$ with $a_n = 0$. In terms of u we have

$$|u(x) - x_n + b - \eta \bar{a} \cdot x - \eta \bar{d}| \leq \eta \varepsilon/2 \quad \text{in } \{u > 0\} \cap (B_2 \setminus \overline{B_{1/2}})$$

for some universally bounded $\bar{a} \in \mathbb{R}^n$ and $\bar{d}$ with $\bar{a} \cdot e_n = 0$. That is, $||e_n + \eta \bar{a}| - 1| \leq C\eta^2$, and absorbing this term on the right-hand side we obtain the desired result. $\square$

We have used the following standard compactness lemma:

Lemma 2.4. *Let $n \geq 3$ and $\alpha \in (0, 1)$. For any $\varepsilon > 0$ there exists $\delta \in (0, 1/4)$ depending only on n , α , and ε such that the following holds.*

Define $\psi(t) := \max(t^{3-n-\alpha}, t^{1+\alpha})$. Let $v \in C^\infty(\overline{\{x_n > 0\}} \cap (B_{1/\delta} \setminus \overline{B_\delta}))$ satisfy

$$\begin{cases} \operatorname{div}(A(x) \nabla v) = 0, & \text{in } \{x_n > 0\} \cap (B_{1/\delta} \setminus \overline{B_\delta}), \\ |\partial_n v| \leq \delta & \text{on } \{x_n = 0\} \cap (B_{1/\delta} \setminus \overline{B_\delta}), \end{cases}$$

$$\sup_{x \in (B_{1/\delta} \setminus \overline{B_\delta}) \cap \{x_n \geq 0\}} \|A(x) - \operatorname{Id}\|_2 \leq \delta, \quad \text{and} \quad |v(x)| \leq \psi(|x|) \quad \text{for } x \in (B_{1/\delta} \setminus \overline{B_\delta}) \cap \{x_n \geq 0\}.$$

Then, there are $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

$$|v(x) - a \cdot x - b| \leq \varepsilon \quad \text{in } (B_4 \setminus \overline{B_{1/4}}) \cap \{x_n \geq 0\},$$

where $a_n = 0$ and $|a| + |b| \leq C(n)$.

Proof. The proof is standard by compactness. Assuming by contradiction that the statement does not hold, there is $\varepsilon_o > 0$ and a sequence v_k satisfying the assumptions with $\delta = 1/k$ for $k \rightarrow \infty$ such that the conclusion fails with $\varepsilon = \varepsilon_o$.

By standard regularity estimates and Arzelà-Ascoli (see, e.g., [40]), v_k converges locally uniformly to v_∞ , which satisfies:

$$\begin{cases} \Delta v_\infty = 0, & \text{in } \{x_n > 0\}, \\ \partial_n v_\infty = 0 & \text{on } \{x_n = 0\} \setminus \{0\}, \end{cases}$$

where

$$|v_\infty(x)| \leq \psi(|x|) \quad \text{in } \{x_n > 0\}.$$

By taking the even extension across $\{x_n = 0\}$, denoted $\bar{v}_\infty$, we have that $\Delta \bar{v}_\infty = 0$ in $\mathbb{R}^n \setminus \{0\}$. The bound by $\psi(|x|)$ implies a removable singularity at the origin, and thus, by Liouville's theorem ($\bar{v}_\infty$ being subquadratic) we have $\bar{v}_\infty$ is affine and even in x_n . This contradicts the negation of the thesis. $\square$

We can now give the proof of the improvement of flatness in exterior domains for the Bernoulli problem:

Proof of Theorem 2.2. We argue by contradiction. Let $C_\star$ be a positive constant, to be chosen large enough (depending only on n and α). Put $\phi(t) := t(t^{2-n-\alpha} + (t/R)^\alpha)$. Define

$$Q_\star := \sup_{r \in [2, R/2]} \frac{H(u, r)}{\phi(r)},$$

and assume for contradiction that

$$Q_\star \geq C_\star \eta.$$

By continuity, there exist $r_\star \in [2, R/2]$ and $b_\star \in \mathbb{R}$, $e_\star \in \mathbb{S}^{n-1}$ such that

$$H(u, r_\star) = Q_\star \phi(r_\star) \quad \text{and} \quad H(u, r_\star) = H_{b_\star}(u, e_\star, r_\star).$$

More generally, for every $r \in [2, R/2]$ there exist $b_r \in \mathbb{R}$ and $e_r \in \mathbb{S}^{n-1}$ such that

$$H(u, r) \leq Q_\star \phi(r) = H(u, r_\star) \frac{\phi(r)}{\phi(r_\star)} \quad \text{and} \quad H(u, r) = H_{b_r}(u, e_r, r) \leq \eta r, \quad (2.6)$$

where we have additionally used assumption (2.2) in the last inequality.

We proceed in several steps.

Step 1. Rescaling.

Consider $\bar{u} := \frac{u(r_\star x)}{r_\star}$. Denoting

$$t = r/r_\star \in (2/r_\star, R/(2r_\star)), \quad \bar{b}_t = b_r/r_\star, \quad \text{and} \quad \bar{e}_t = e_r,$$

(2.6) becomes (recall (2.1))

$$H_{\bar{b}_t}(\bar{u}, \bar{e}_t, t) \leq \frac{H(u, r_\star)}{r_\star} \frac{\phi(tr_\star)}{\phi(r_\star)} \quad \text{and} \quad H_{\bar{b}_t}(\bar{u}, \bar{e}_t, t) \leq \eta t. \quad (2.7)$$

Noticing that

$$\frac{\phi(tr_\star)}{\phi(r_\star)} \leq \max(t^{3-n-\alpha}, t^{1+\alpha}) =: \psi(t), \quad (2.8)$$

this gives

$$H_{\bar{b}_t}(\bar{u}, \bar{e}_t, t) \leq \eta_\star \psi(t), \quad \text{with} \quad \eta_\star := \frac{H(u, r_\star)}{r_\star} \leq \eta \leq \eta_0. \quad (2.9)$$

We want to apply Lemma 2.3 to $\bar{u}$. Note that (2.2) implies

$$|b_r| < CH(u, r), \quad \text{thus} \quad |\bar{b}_t| < C \frac{H(u, tr_\star)}{r_\star} \leq C \eta t. \quad (2.10)$$

Step 2. Coefficient comparison.

By definition of $H_{\bar{b}_t}(\bar{u}, \bar{e}_t, t)$, for every t as above we have:

$$\sup_{B_{2t} \setminus B_{t/2}} |\bar{u} - (\bar{e}_t \cdot x + \bar{b}_t)_+| \leq H_{\bar{b}_t}(\bar{u}, \bar{e}_t, t).$$

Hence, using $|\bar{b}_t| \leq t/100$ (for η small enough) and the triangle inequality we obtain

$$\begin{aligned} t|\bar{e}_{2t} - \bar{e}_t| + |\bar{b}_{2t} - \bar{b}_t| &\leq C \sup_{B_{2t} \setminus B_t} |(\bar{e}_{2t} \cdot x + \bar{b}_{2t})_+ - (\bar{e}_t \cdot x + \bar{b}_t)_+| \\ &\leq C \sup_{B_{4t} \setminus B_t} |\bar{u} - (\bar{e}_{2t} \cdot x + \bar{b}_{2t})_+| + C \sup_{B_{2t} \setminus B_{t/2}} |\bar{u} - (\bar{e}_t \cdot x + \bar{b}_t)_+| \\ &\leq C(H_{\bar{b}_t}(\bar{u}, \bar{e}_t, t) + H_{\bar{b}_{2t}}(\bar{u}, \bar{e}_{2t}, 2t)). \end{aligned}$$

By (2.9) we then have

$$t|\bar{e}_{2t} - \bar{e}_t| + |\bar{b}_{2t} - \bar{b}_t| \leq C\eta_*(\psi(2t) + \psi(t)),$$

which summing the geometric series yields (recall $e_* = \bar{e}_1$), for $2/r_* < t < R/(2r_*)$,

$$t|\bar{e}_t - e_*| + |\bar{b}_t - \bar{b}_1| \leq C\eta_*\psi(t).$$

Combined with (2.9), this implies (recall Definition 2.1) that

$$H_{\bar{b}_1}(\bar{u}, \bar{e}_1, t) \leq \bar{\eta}_*\psi(t) \quad \text{for all } t \in (2/r_*, R/(2r_*)), \quad \text{with } \bar{\eta}_* := C\eta_* \leq C\eta_0.$$

As in the rest of the proof, here $C = C(n, \alpha)$. Moreover, $|\bar{b}_1| \leq C\eta \leq C\eta_0$ by (2.10).

Step 3. Estimate for r_* .

By (2.2) we know that $H(u, r)/r \leq \eta$ for all $r \in (2, R/2)$, thus

$$Q_* \leq \frac{\eta}{r_*^{2-n-\alpha} + (r_*/R)^\alpha} \leq \eta \max \left\{ \frac{1}{r_*^{2-n-\alpha}}, \frac{1}{(r_*/R)^\alpha} \right\}.$$

Applying this and $Q_* \geq C_*\eta$ twice, we find that

$$C_*^{\frac{1}{n+\alpha-2}} \leq (Q_*/\eta)^{\frac{1}{n+\alpha-2}} \leq r_* \leq R(\eta/Q_*)^{\frac{1}{\alpha}} \leq RC_*^{-\frac{1}{\alpha}}, \quad (2.11)$$

so that $r_* \in [C_*^{\frac{1}{n+\alpha-2}}, RC_*^{-\frac{1}{\alpha}}]$. Since our arguments above were valid for all $t \in (2/r_*, R/(2r_*))$, given $\delta > 0$ this includes all $t \in (\delta, \frac{1}{\delta})$ up to making C_* large enough.

Step 4. Conclusion.

Let $\varepsilon > 0$, to be fixed. Combining all of the above, by choosing C_* large and η_0 small (depending only on n , α , and ε), the setting of Lemma 2.3 is satisfied. This gives

$$H(\bar{u}, 1) \leq \varepsilon \bar{\eta}_* = C\varepsilon \frac{H(u, r_*)}{r_*},$$

or (since $\bar{u}(x) = \frac{u(r_*x)}{r_*}$, and thus $H(\bar{u}, 1) = \frac{H(u, r_*)}{r_*}$), equivalently,

$$H(u, r_*) \leq C\varepsilon H(u, r_*).$$

Once again, we emphasize that $C = C(n, \alpha)$. Fixing finally ε so that $C\varepsilon = \frac{1}{2}$, which in turn fixes the choice of C_* and η_0 in terms of n , and α , we arrive at a contradiction. $\square$

As a consequence, we immediately obtain a first-order expansion at infinity of global solutions with flat free boundaries in exterior domains:

Corollary 2.5. *Let $n \geq 3$ and $\alpha \in (0, 1)$. There are $C = C(n, \alpha)$ and $\eta_0 = \eta_0(n, \alpha) > 0$ such that the following holds:*

Let u be a classical solution to the Bernoulli problem, (1.1), in $\mathbb{R}^n \setminus \overline{B_1} \subset \mathbb{R}^n$, satisfying

$$H_0(u, r) \leq \eta r \quad \text{for all } r \geq 2, \quad (2.12)$$

for some $\eta \leq \eta_0$.

Then, there is $|b| \leq 1/16$ such that, up to a rotation,

$$|u - x_n + b| \leq \frac{C}{|x|^{n-3+\alpha}} \quad \text{in } \{u > 0\} \setminus B_1. \quad (2.13)$$

Moreover,

$$\{u > 0\} \setminus (B'_2 \times [-2, 2]) = \{|x'| > 2, x_n > g(x')\} \cup \{|x'| \leq 2, x_n > 2\}$$

for some smooth $g : \mathbb{R}^{n-1} \setminus B'_2 \rightarrow [-1, 1]$.

Proof. By Theorem 2.2 applied with $R \nearrow \infty$, we know

$$H(u, r) \leq C\eta r^\gamma \quad \text{for all } r \geq 2,$$

where we have denoted $\gamma = 3 - n - \alpha < 0$. By definition, there are $e_k \in \mathbb{S}^{n-1}$ and $b_k \in \mathbb{R}$ such that, for any $k \in \mathbb{N}$,

$$(e_k \cdot x - b_k - h_k)_+ \leq u \leq (e_k \cdot x - b_k + h_k)_+ \quad \text{in } B_{2^{k+1}} \setminus B_{2^k},$$

where $h_k := C\eta 2^{k\gamma}$. Moreover, by the same reasoning with $k+1$ we deduce that, in the annulus $B_{2^{k+1}} \setminus B_{2^k} \subset B_{2^{k+1}} \setminus B_{2^{k-1}}$ we also have

$$(e_{k+1} \cdot x - b_{k+1} - h_{k+1})_+ \leq u \leq (e_{k+1} \cdot x - b_{k+1} + h_{k+1})_+ \quad \text{in } B_{2^{k+1}} \setminus B_{2^k}.$$

For η sufficiently small, this means that the free boundary is trapped inside two different thin strips with directions e_k and e_{k+1} , and shifts b_k and b_{k+1} . In particular, by the triangle inequality, we must have

$$2^k |e_{k+1} - e_k| + |b_{k+1} - b_k| \leq C(h_k + h_{k+1}) \leq C\eta 2^{k\gamma}.$$

From here, using again the triangle inequality, we deduce

$$\sum_{i \geq k} |e_{i+1} - e_i| \leq C\eta \sum_{i \geq k} 2^{i(\gamma-1)} \leq C\eta 2^{k(\gamma-1)}, \quad \sum_{i \geq k} |b_{i+1} - b_i| \leq C\eta \sum_{i \geq k} 2^{i\gamma} \leq C\eta 2^{k\gamma},$$

where now the constants also depend on α . In particular, both sequences $\{e_k\}_{k \in \mathbb{N}}$ and $\{b_k\}_{k \in \mathbb{N}}$ are Cauchy, and there exist some $e_\infty \in \mathbb{S}^{n-1}$ and $b_\infty \in (-1/16, 1/16)$ —making η smaller if necessary—such that

$$|e_\infty - e_k| \leq C\eta 2^{k(\gamma-1)} \quad \text{and} \quad |b_\infty - b_k| \leq C\eta 2^{k\gamma}.$$

Up to a rotation bringing e_∞ to e_n , by the triangle inequality this implies

$$(x_n - b_\infty - h_k)_+ \leq u \leq (x_n - b_\infty + h_k)_+ \quad \text{in } B_{2^{k+1}} \setminus B_{2^k}, \quad \text{for } h_k = C\eta 2^{k\gamma}.$$

Combining this with the classical improvement of flatness for the Bernoulli problem [13], the desired result follows. $\square$

Remark 2.6. In Corollary 2.5, we obtain a shift b because the flatness (2.12) is compatible with sufficiently small vertical translations. In fact,

$$|b| \leq C\eta,$$

for some C universal. That is, b can be made as small as needed by taking η small.

2.2. Higher-order expansion for entire solutions. The goal of this subsection is to improve Corollary 2.5 to a higher-order expansion:

Theorem 2.7. *Let $n \geq 3$ and $\alpha \in (0, 1)$. In the setting of Corollary 2.5, there are $C = C(n, \alpha)$ and $\eta_0 = \eta_0(n, \alpha) > 0$ as in the corollary such that, for some v of the form*

$$v(x) = x_n - b + \frac{c}{|x|^{n-2}} + \frac{d \cdot x}{|x|^n}, \quad \text{for } (b, c, d) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n, \quad \text{with } d_n = (n-2)bc, \quad (2.14)$$

we have

$$|u(x) - v(x)| \leq \frac{C}{|x|^{n-1+\alpha}} \quad \text{in } \{u > 0\} \setminus B_2. \quad (2.15)$$

Moreover, we also have

$$|\nabla u(x) - \nabla v(x)| \leq \frac{C}{|x|^{n+\alpha}} \quad \text{in } \{u > 0\} \setminus B_2. \quad (2.16)$$

We now turn to the proof of Theorem 2.7. The starting point is Corollary 2.5, which gives b_0 with $|b_0| \leq 1/16$ such that

$$|u - x_n + b_0| \leq \frac{C}{|x|^{n-3+\alpha}} \quad \text{in } \{u > 0\} \setminus B_1. \quad (2.17)$$

We next introduce the finite-dimensional family of harmonic profiles used to improve this approximation.

Definition 2.8 (Family of Ansätze). Let $n \geq 3$. We define the class of *approximating Ansätze*, denoted $\mathcal{A}(\kappa)$, as the class of all functions of the form

$$\hat{v}(x) = x_n + \hat{b} + \frac{\hat{c}}{|x|^{n-2}} + \frac{\hat{d}' \cdot x'}{|x|^n}, \quad (2.18)$$

where $(\hat{b}, \hat{c}, \hat{d}') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1}$, such that

$$\max(|\hat{b}|, |\hat{c}|, |\hat{d}'|) \leq \kappa. \quad (2.19)$$

For a given $\hat{v}$ of the form (2.18), we denote $\hat{v} = \hat{v}[\hat{b}, \hat{c}, \hat{d}']$.

We notice that any $\hat{v} \in \mathcal{A}(\kappa)$ is harmonic in $\mathbb{R}^n \setminus \{0\}$. Besides being harmonic, the key property that Ansätze satisfy is:

Lemma 2.9. *Let $n \geq 3$ and $\kappa > 0$. For any $\hat{v} \in \mathcal{A}(\kappa)$ we have*

$$||\nabla \hat{v}|^2 - 1| \leq C\kappa^2 \quad \text{in } \{|\hat{v}| < \kappa\} \cap (B_4 \setminus B_{1/4}),$$

where C depends only on n .

Proof. We first notice that

$$|x_n| \leq C(|\hat{b}| + |\hat{c}| + |\hat{d}'| + t) \quad \text{in } \{|\hat{v}| < t\} \cap (B_4 \setminus B_{1/4}). \quad (2.20)$$

By an explicit computation

$$\nabla \hat{v} = e_n + \hat{c}(2-n)\frac{x}{|x|^n} + \frac{(\hat{d}', 0)}{|x|^n} - n\frac{(\hat{d}' \cdot x')x}{|x|^{n+2}}$$

Hence

$$\nabla \hat{v} \cdot \nabla \hat{v} = 1 + 2\hat{c}(2-n)\frac{e_n \cdot x}{|x|^n} - 2n\frac{(\hat{d}' \cdot x')(e_n \cdot x)}{|x|^{n+2}} + O(\kappa^2).$$

But then, using (2.20), in $\{|\hat{v}| < \kappa\} \cap (B_4 \setminus B_{1/4})$ we can replace $e_n \cdot x = x_n$ by $-\hat{b}$ up to introducing new error terms of size $O(\kappa)$. By (2.19), the lemma follows. $\square$

The second lemma we need is as follows, where from now on we denote

$$u_r(x) := \frac{1}{r}u(rx + b_0 e_n), \quad \text{where } b_0 \text{ is given by (2.17).} \quad (2.21)$$

Alternatively, the reader may choose to think that after a translation, $b_0 = 0$, and (2.21) is the usual Bernoulli rescaling. Such a rescaling will be undone in the last part of the last step of the proof of Theorem 2.7, in (2.49).

Lemma 2.10. *Let u be as in Theorem 2.7. Let $\theta \in (0, 1)$ and $\hat{v} \in \mathcal{A}(1/8)$. Suppose that, for some $r \geq 4$ and $0 < \hat{h} \leq 1/8$, we have (recall (2.21))*

$$|u_r - \hat{v}| \leq \hat{h} \quad \text{in} \quad \{u_r > 0\} \cap (B_4 \setminus B_{1/4}) \neq \emptyset. \quad (2.22)$$

Then,

$$|\nabla(u_r - \hat{v})| \leq C\hat{h}^\theta \quad \text{in} \quad \{u_r > 0\} \cap (B_2 \setminus B_{1/2}),$$

where C depends only on n and θ .

Proof. We assume $r = 4$ and write u instead of u_4 (as a small abuse of notation), otherwise we argue by scaling. By higher order regularity estimates for the Bernoulli problem (see, e.g., [7, Proposition 3.7]) we have that assuming small flatness, $|D^k u| \leq C$ in $\{u > 0\} \setminus B_{1/4}$, and $\{u > 0\}$ is a C^k domain in $\mathbb{R}^n \setminus B_{1/4}$, for any $k \in \mathbb{N}$. From the definition of $\hat{v}$, we also have $|D^k \hat{v}| \leq C$ in $\mathbb{R}^n \setminus B_{1/4}$. The constant C might depend on k . Hence, $u - \hat{v}$ is a small harmonic function in a smooth domain with bounded higher order estimates. By standard interpolation estimates (see, e.g., [25, Section A.3]) we get

$$\begin{aligned} \|u - \hat{v}\|_{C^{0,1}(\{u>0\} \cap B_2 \setminus B_{1/2})} &\leq C \|u - \hat{v}\|_{L^\infty(\{u>0\} \cap B_3 \setminus B_{1/3})}^{1-1/k} \|u - \hat{v}\|_{C^{k-1,1}(\{u>0\} \cap B_3 \setminus B_{1/3})}^{1/k} \\ &\leq C\hat{h}^{1-1/k} C_k^{1/k}. \end{aligned}$$

Choosing k large depending on θ we get the desired result. $\square$

The following result controls the flux of the difference $u_r - \hat{v}$ along the free boundary.

Lemma 2.11 (Flux control). *Let u be as in Theorem 2.7. Let $\theta \in (0, 1)$, $\kappa \in (0, 1/8)$, and $\hat{v} \in \mathcal{A}(\kappa)$. Let $r \geq 4$ and assume that (2.22) holds for some $0 < \hat{h} \leq 1/8$.*

Then, recalling (2.21), we have

$$|\nabla(u_r - \hat{v}) \cdot \nabla u_r| \leq C(\hat{h}^{2\theta} + \kappa^2) \quad \text{on} \quad \partial\{u_r > 0\} \cap (B_2 \setminus B_{1/2}),$$

for some C depending only on n and θ .

Proof. As in Lemma 2.10, the result is scaling invariant, so we can prove it for $r = 4$. We write u instead of u_4 .

Notice first that (2.22) implies

$$\partial\{u > 0\} \subset \{|\hat{v}| < \hat{h}\} \quad \text{inside } B_2 \setminus B_{1/2}.$$

Since $|\nabla u|^2 = 1$ on $\partial\{u > 0\} \cap B_4 \setminus B_{1/4}$ we have

$$2(\nabla u \cdot \nabla(u - \hat{v})) = 2 - 2\nabla u \cdot \nabla \hat{v} = |\nabla u - \nabla \hat{v}|^2 + 1 - |\nabla \hat{v}|^2$$

on $\partial\{u > 0\}$. Then we apply Lemmas 2.9 and 2.10. $\square$

Lemma 2.12 (Compactness). *Let $n \geq 3$ and $\alpha \in (0, 1)$. For any $\varepsilon > 0$ there exists $\delta > 0$ depending only on n , α , and ε , such that the following holds.*

Let $g : B'_{1/\delta} \rightarrow \mathbb{R}$ with $\|g\|_{C^3(B'_{1/\delta})} \leq \delta$ and put

$$\Omega := \{x_n > g(x')\} \cap (B_{1/\delta} \setminus B_\delta).$$

Let $w : \Omega \rightarrow \mathbb{R}$ be such that

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ |\partial_\nu w| \leq \delta & \text{on } \{x_n = g(x')\} \cap \Omega. \end{cases}$$

Assume, moreover, that $w : \Omega \rightarrow \mathbb{R}$ satisfies

$$|w| \leq |x|^\alpha + |x|^{1-n-\alpha} \quad \text{in } \Omega.$$

Then, there exist $b_*, c_* \in \mathbb{R}$ and $d'_* \in \mathbb{R}^{n-1}$ with $|b_*| + |c_*| + |d'_*| \leq 50$ such that

$$\left| w - b_* - c_* |x|^{2-n} - (d'_* \cdot x') |x|^{-n} \right| \leq \varepsilon \quad \text{in } (B_4 \setminus B_{1/4}) \cap \{x_n > g(x')\}.$$

Proof. The proof follows by compactness. Assume by contradiction that the statement does not hold for some sequence w_k and $\delta = 1/k$.

By standard $W^{1,p}$ regularity estimates for harmonic functions with L^p Neumann boundary data in smooth domains (see, e.g., [29]), w_k converges locally uniformly (by the Sobolev embedding theorem) in $\{x_n \geq 0\} \setminus \{0\}$ to some w_∞ in $\{x_n \geq 0\}$. By weak $W^{1,2}$ convergence, w_∞ is harmonic in $\{x_n > 0\}$ with zero Neumann condition on $\{x_n = 0\} \setminus \{0\}$. If we denote by $\bar{w}_\infty$ the even extension of w_∞ about $\{x_n = 0\}$ (i.e., $\bar{w}_\infty(x', x_n) = w_\infty(x', |x_n|)$) we have

$$\begin{cases} \Delta \bar{w}_\infty = 0 & \text{in } \mathbb{R}^n \setminus \{0\} \\ |\bar{w}_\infty| \leq |x|^\alpha + |x|^{1-n-\alpha} & \text{in } \mathbb{R}^n, \\ \bar{w}_\infty \text{ is even in } x_n. \end{cases}$$

In particular, one can take $\bar{w}_\infty \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\Delta \bar{w}_\infty$ is a distribution in $\mathbb{R}^n$ supported at $\{0\}$. From the growth condition at zero, it has at most order 1, so we must have

$$\Delta \bar{w}_\infty = a\delta_0 + \sum_{i=1}^n b_i \partial_i \delta_0$$

in the distributional sense. Together with Liouville's theorem for harmonic functions with sublinear growth (see, e.g., [24, Proposition 1.19]) we deduce

$$\bar{w}_\infty = \bar{c} + \bar{a}|x|^{2-n} + \sum_{i=1}^n \bar{b}_i x_i |x|^{-n}.$$

This is a contradiction for k large enough. The bounds on $\bar{a}, \bar{b}$, and $\bar{c}$, follow from the growth assumption, and $\bar{b}_n = 0$ by evenness. $\square$

Before we can give the proof of Theorem 2.7, we need to introduce a fine-grained version of $H(u, r)$ using approximating Ansätze as in (2.18)-(2.19) (with bounded coefficients).

Definition 2.13. Let $n \geq 3$. Let u be a classical solution, (1.1), in a domain $\mathbb{R}^n \setminus B_1$ satisfying (2.17). Let $r > 4$. We define the *fine annular height* at scale r as

$$\hat{H}(u, r) := \min_{\hat{v} \in \mathcal{A}(1/8)} \|u_r - \hat{v}\|_{L^\infty(\{u_r > 0\} \cap (B_2 \setminus \overline{B_{1/2}}))},$$

where we recall that $\mathcal{A}(\kappa)$ is given by Definition 2.8, and u_r is given by (2.21).

Proof of Theorem 2.7. We take as starting point (2.17) rescaled: for all $r \geq 4$

$$|u_r - x_n| \leq \frac{\bar{C}}{r^{n-2+\alpha}} \quad \text{in } \{u_r > 0\} \cap (B_2 \setminus B_{1/2}). \quad (2.23)$$

Set

$$\hat{h}_r := \hat{H}(u, r), \quad r \geq 4.$$

By (2.23) (recall that from Corollary 2.5 we may assume $|b_0| \leq 1/16$), we have

$$r\hat{h}_r \leq \bar{C}r^{3-n-\alpha} \quad \text{for all } r \geq 4. \quad (2.24)$$

We claim that there exists $C_\star \geq 1$ (large) such that

$$r\hat{h}_r \leq C_\star r^{1-n-\alpha} \quad \text{for all } r \geq 4. \quad (2.25)$$

To prove this, define (for $C_\star$ to be fixed)

$$\varepsilon_\star := \inf \left\{ \varepsilon > 0 : r\hat{h}_r \leq C_\star r^{1-n-\alpha} + \varepsilon \quad \text{for all } r \geq 4 \right\}.$$

We argue by contradiction and assume that $\varepsilon_\star > 0$. Then, there exists $r_\star \geq 4$ such that

$$r_\star \hat{h}_{r_\star} \geq C_\star r_\star^{1-n-\alpha} + \frac{\varepsilon_\star}{2}. \quad (2.26)$$

Combining (2.26) with (2.24), we get $C_\star r_\star^{1-n-\alpha} \leq \bar{C} r_\star^{3-n-\alpha}$ and hence

$$r_\star \geq \left(\frac{C_\star}{\bar{C}} \right)^{1/2}. \quad (2.27)$$

That is, by taking $C_\star$ large, we can make $r_\star$ large.

Let $\delta > 0$ be a small constant to be fixed, that will depend only on n and α . By taking $C_\star$ large enough (depending on $\delta > 0$), we may assume

$$r_\star \geq 4/\delta.$$

For any $r > 4$, let us choose coefficients

$$(\hat{b}_r, \hat{c}_r, \hat{d}'_r) \in [-1/8, 1/8] \times [-1/8, 1/8] \times \overline{B'_{1/8}}$$

such that (recall Definition 2.8)

$$\hat{h}_r = \|u_r - \hat{v}[\hat{b}_r, \hat{c}_r, \hat{d}'_r]\|_{L^\infty(\{u_r > 0\} \cap (B_2 \setminus \overline{B_{1/2}}))}. \quad (2.28)$$

We now divide the proof into multiple steps.

Step 1. Control of approximating Ansatz at scale $r_\star$. We claim that

$$|\hat{b}_{r_\star}| + |\hat{c}_{r_\star}| + |\hat{d}'_{r_\star}| \leq C(\bar{C}) r_\star^{-\beta} (\hat{h}_{r_\star})^{1/2}, \quad \text{where} \quad \beta := \frac{n-2-\alpha}{2} > 0. \quad (2.29)$$

Indeed, on the one hand by (2.28), for any $r \geq 4$,

$$\left| u_r - x_n - \hat{b}_r - \frac{\hat{c}_r}{|x|^{n-2}} - \frac{\hat{d}'_r \cdot x'}{|x|^n} \right| \leq \hat{h}_r \quad \text{in } \Omega_r,$$

where

$$\Omega_r := \{u_r > 0\} \cap (B_2 \setminus B_{1/2}) \supset \{x_n \geq \tfrac{1}{8}\} \cap (B_2 \setminus B_{1/2}).$$

Using that $u_{r/2}(x) = 2u_r(x/2)$, and that

$$\Omega_{r/2} \cap 2\Omega_r \supset \{x_n \geq \tfrac{1}{4}\} \cap (B_2 \setminus B_1) =: \tilde{\Omega},$$

the triangle inequality gives

$$\left| \hat{b}_{r/2} + \frac{\hat{c}_{r/2}}{|x|^{n-2}} + \frac{\hat{d}'_{r/2} \cdot x'}{|x|^n} - 2\hat{b}_r - 2^{n-1} \frac{\hat{c}_r}{|x|^{n-2}} - 2^n \frac{\hat{d}'_r \cdot x'}{|x|^n} \right| \leq \hat{h}_{r/2} + 2\hat{h}_r \quad \text{in } \tilde{\Omega}. \quad (2.30)$$

Recall that, by definition of $\varepsilon_\star$ and (2.24),

$$\hat{h}_r \leq C_\star r^{-n-\alpha} + \varepsilon_\star/r \quad \text{and} \quad \hat{h}_r \leq \bar{C} r^{2-n-\alpha}. \quad (2.31)$$

In what follows we absorb (harmless) powers of $\bar{C}$ into $C(\bar{C})$. Setting

$$b_r := r\hat{b}_r, \quad c_r := r^{n-1}\hat{c}_r, \quad d'_r := r^n\hat{d}'_r, \quad (2.32)$$

and using (2.30) at finitely many points, and (2.31), we obtain

$$\begin{aligned} |c_{r/2} - c_r| &\leq C(\overline{C}) r^{n-1} \left((C_\star r^{-n-\alpha})^{1/2} + (\varepsilon_\star/r)^{1/2} \right) r^{(2-n-\alpha)/2} \\ &\leq C(\overline{C}) \left((C_\star)^{1/2} r^{-\alpha} + (\varepsilon_\star/r)^{1/2} r^{(n-\alpha)/2} \right), \\ |d'_{r/2} - d'_r| &\leq C(\overline{C}) r^n \left((C_\star r^{-n-\alpha}) + (\varepsilon_\star/r) \right) \\ &\leq C(\overline{C}) \left(C_\star r^{-\alpha} + (\varepsilon_\star/r) r^n \right). \end{aligned}$$

Let now $k_\star \in \mathbb{N}$ be such that $2^{-k_\star} r_\star \in (8, 16)$ and put

$$r_0 = 2^{-k_\star} r_\star.$$

Summing the corresponding geometric series (using that $n \geq 3$) we have

$$\sum_{j=0}^{k_\star-1} (2^{-j} r_\star)^{-\alpha} \leq C, \quad \sum_{j=0}^{k_\star-1} (2^{-j} r_\star)^{(n-1-\alpha)/2} \leq C r_\star^{(n-1-\alpha)/2}, \quad \sum_{j=0}^{k_\star-1} (2^{-j} r_\star)^{n-1} \leq C r_\star^{n-1},$$

and we thus obtain

$$\begin{aligned} |c_{r_0} - c_{r_\star}| &\leq C \overline{C} \left((C_\star)^{1/2} + (\varepsilon_\star/r_\star)^{1/2} r_\star^{(n-\alpha)/2} \right), \\ |d'_{r_0} - d'_{r_\star}| &\leq C \left(C_\star + (\varepsilon_\star/r_\star) r_\star^n \right). \end{aligned}$$

On the other hand, by (2.23),

$$|c_{r_0}| + |d'_{r_0}| \leq C(|\hat{c}_{r_0}| + |\hat{d}'_{r_0}|) \leq C(\overline{C}),$$

because $r_0 \in (8, 16)$. Hence, by the triangle inequality (and since $C_\star \geq 1$),

$$\begin{aligned} |c_{r_\star}| &\leq C(\overline{C}) \left((C_\star)^{1/2} + (\varepsilon_\star/r_\star)^{1/2} r_\star^{(n-\alpha)/2} \right), \\ |d'_{r_\star}| &\leq C(\overline{C}) \left(C_\star + (\varepsilon_\star/r_\star) r_\star^n \right). \end{aligned}$$

Recalling now (2.26),

$$C_\star \leq \hat{h}_{r_\star} r_\star^{n+\alpha} \quad \text{and} \quad \varepsilon_\star/r_\star \leq 2\hat{h}_{r_\star},$$

we infer (recall also (2.32))

$$\begin{aligned} |\hat{c}_{r_\star}| &= r_\star^{1-n} |c_{r_\star}| \\ &\leq C(\overline{C}) r_\star^{1-n} \left((C_\star)^{1/2} + (\varepsilon_\star/r_\star)^{1/2} r_\star^{(n-\alpha)/2} \right) \\ &\leq C(\overline{C}) \left((\hat{h}_{r_\star})^{1/2} r_\star^{1-n+(n+\alpha)/2} + (\hat{h}_{r_\star})^{1/2} r_\star^{1-n+(n-\alpha)/2} \right). \end{aligned}$$

Since

$$1 - n + \frac{n + \alpha}{2} = -\beta, \quad 1 - n + \frac{n - \alpha}{2} = \frac{2 - n - \alpha}{2} \leq -\beta,$$

it follows that

$$|\hat{c}_{r_\star}| \leq C(\overline{C}) (\hat{h}_{r_\star})^{1/2} r_\star^{-\beta}. \tag{2.33}$$

Similarly,

$$|\hat{d}'_{r_\star}| = r_\star^{-n} |d'_{r_\star}| \leq C(\overline{C}) \left(C_\star r_\star^{-n} + \varepsilon_\star/r_\star \right) \leq C(\overline{C}) \left(\hat{h}_{r_\star} r_\star^\alpha + \hat{h}_{r_\star} \right) \leq C(\overline{C}) \hat{h}_{r_\star} r_\star^\alpha,$$

since $r_\star \geq 8$. Finally, using (2.24) we obtain

$$\hat{h}_{r_\star} r_\star^\alpha = (\hat{h}_{r_\star})^{1/2} \left((\hat{h}_{r_\star})^{1/2} r_\star^\alpha \right) \leq C(\hat{h}_{r_\star})^{1/2} r_\star^{(2-n-\alpha)/2+\alpha} = C(\hat{h}_{r_\star})^{1/2} r_\star^{-\beta}.$$

Therefore,

$$|\hat{d}'_{r_\star}| \leq C(\overline{C}) (\hat{h}_{r_\star})^{1/2} r_\star^{-\beta}.$$

Combining the last two bounds proves the bound for $|\hat{c}_{r_\star}|$ and $|\hat{d}'_{r_\star}|$ in (2.29). It remains only to prove the bound for $|\hat{b}_{r_\star}|$.

We proceed as follows: by (2.23) we have, arguing as before,

$$\lim_{r \rightarrow \infty} b_r = 0 \quad \text{and} \quad |b_r| \leq C\overline{C} r^{3-n-\alpha}. \quad (2.34)$$

On the other hand, (2.31) and (2.26) imply:

$$r\hat{h}_r \leq 2r_\star\hat{h}_{r_\star} \quad \text{for all } r \geq r_\star$$

But then, with a triangle inequality argument similar to that above

$$|b_{2r} - b_r| \leq Cr(\hat{h}_{2r} + \hat{h}_r) \leq Cr_\star\hat{h}_{r_\star} \quad \text{for all } r \geq r_\star.$$

Hence,

$$|b_{r_\star} - b_R| \leq Cr_\star\hat{h}_{r_\star} \log(R/r_\star).$$

Now, let us take $R = r_\star^p$ with $p > 1$ large. By (2.34) and triangle inequality we find:

$$|b_{r_\star}| \leq |b_{r_\star} - b_R| + |b_R| \leq Cr_\star(\hat{h}_{r_\star} \log(r_\star) + r_\star^{(3-n-\alpha)p-1})$$

Let us fix $p \geq \frac{n+\alpha+1}{n-3+\alpha}$ so that $r_\star^{(3-n-\alpha)p-1} \leq C\hat{h}_{r_\star}$ (recall again (2.26)). Then, we obtain

$$|b_{r_\star}| \leq Cr_\star\hat{h}_{r_\star} \log(r_\star) \quad \text{and thus} \quad |\hat{b}_{r_\star}| \leq C\hat{h}_{r_\star} \log(r_\star).$$

But by (2.24) we have $\hat{h}_{r_\star} \leq \overline{C}r_\star^{2-n-\alpha}$ and thus

$$|\hat{b}_{r_\star}| \leq C\hat{h}_{r_\star}^{1/2} r_\star^{-(n-2+\alpha)/2} \log(r_\star) \leq Cr_\star^{-\beta} \hat{h}_{r_\star}^{1/2},$$

as we wanted.

Step 2. Normalization at the bad scale. Let us write

$$\hat{v}^r := \hat{v}[\hat{b}_r, \hat{c}_r, \hat{d}'_r] \quad \text{and} \quad \hat{v}_\star := \hat{v}^{r_\star}. \quad (2.35)$$

Since $r_\star$ is large and (2.17) holds, the free boundary in the annulus

$$B_{r_\star/\delta} \setminus B_{\delta r_\star}$$

is a C^3 graph in the x_n -direction with arbitrarily small C^3 -norm after scaling. More precisely, after rescaling by $r_\star$, we may write

$$\begin{aligned} \Omega_\star &:= r_\star^{-1} \left(\{u(\cdot + b_0 e_n) > 0\} \cap (B_{r_\star/\delta} \setminus B_{\delta r_\star}) \right) \\ &= \{u_{r_\star} > 0\} \cap (B_{1/\delta} \setminus B_\delta) \\ &= \{x_n > g_\star(x')\} \cap (B_{1/\delta} \setminus B_\delta), \end{aligned}$$

with

$$\|g_\star\|_{C^3(B'_{1/\delta})} \leq \delta, \quad (2.36)$$

provided $C_\star$ is chosen large enough (again, depending on δ).

Define

$$w(x) := \frac{u_{r_\star} - \hat{v}_\star}{\hat{h}_{r_\star}} \quad \text{in } \Omega_\star.$$

Since both $u_{r_\star}$ and $\hat{v}_\star$ are harmonic in $\{u_{r_\star} > 0\} \setminus \{0\}$, we have

$$\Delta w = 0 \quad \text{in } \Omega_\star.$$

Also, by (2.28),

$$|w| \leq 1 \quad \text{in } \Omega_\star \cap (B_2 \setminus B_{1/2}). \quad (2.37)$$

Step 3. Small Neumann datum. We apply the flux-control lemma, Lemma 2.11, to $u_{r_\star}$ and $v_\star$, using $\theta = \frac{3}{4}$. After scaling, we obtain, denoting $\rho = r/r_\star$, and recalling (2.29),

$$|\partial_\nu w| \leq \frac{C}{\hat{h}_{r_\star}} \left[(\hat{h}_r)^{2\theta} + C(\bar{C}) r_\star^{-2\beta} \hat{h}_{r_\star} \right] \quad \text{on } \partial\Omega_\star \cap (B_{2\rho} \setminus B_{\rho/2}).$$

We remark that the constants above depend on δ as well.

Since $\hat{h}_r \leq C_\delta \hat{h}_{r_\star}$ for all $r \in (\delta r_\star, r_\star/\delta)$ (by definition of $r_\star$ and $\hat{h}_{r_\star}$; see (2.42) for a precise estimate in terms of ρ) we have, for some C that might depend on δ (and $\bar{C}$),

$$|\partial_\nu w| \leq C (\hat{h}_{r_\star})^{2\theta-1} + C r_\star^{-2\beta} \quad \text{on } \partial\Omega_\star \cap (B_{1/\delta} \setminus B_\delta). \quad (2.38)$$

Since $\hat{h}_{r_\star} \leq \bar{C} r_\star^{2-n-\alpha}$ and we have (for $\theta = \frac{3}{4}$)

$$\hat{h}_{r_\star}^{2\theta-1} \rightarrow 0 \quad \text{as } r_\star \rightarrow \infty,$$

by taking $C_\star$ large enough we can ensure

$$|\partial_\nu w| \leq \delta \quad \text{on } \partial\Omega_\star \cap (B_{1/\delta} \setminus B_\delta). \quad (2.39)$$

Step 4. Growth control of w on all intermediate scales. We claim that

$$|w(x)| \leq C \left(|x|^\alpha + |x|^{1-n-\alpha} \right) \quad \text{in } \Omega_\star, \quad (2.40)$$

for some $C = C(n, \alpha)$. Indeed, by the definition of $\varepsilon_\star$,

$$r \hat{h}_r \leq C_\star r^{1-n-\alpha} + \varepsilon_\star \quad \text{for all } r \geq 4, \quad (2.41)$$

while (2.26) also holds. Thus, for $r = tr_\star$,

$$\frac{r \hat{h}_r}{r_\star \hat{h}_{r_\star}} \leq 2 \left(t^{1-n-\alpha} + 1 \right) \quad \text{for } t \in (\delta, 1/\delta). \quad (2.42)$$

Let now $s \in (\delta, 1/\delta)$ be fixed, and denote $\rho := sr_\star$. By triangle inequality, we have (recall the notation (2.35))

$$\|u_{r_\star} - \hat{v}_\star\|_{L^\infty(\partial B_s \cap \{u_{r_\star} > 0\})} \leq s \|u_\rho - \hat{v}^\rho\|_{L^\infty(\partial B_1 \cap \{u_\rho > 0\})} + s \|\hat{v}^\rho - \hat{v}_\star(s \cdot)/s\|_{L^\infty(\partial B_1 \cap \{u_\rho > 0\})}. \quad (2.43)$$

On the one hand, we have

$$s \|u_\rho - \hat{v}^\rho\|_{L^\infty(\partial B_1 \cap \{u_\rho > 0\})} \leq s \hat{h}_\rho \leq 2 \hat{h}_{r_\star} (s^{1-n-\alpha} + 1),$$

thanks to (2.42). On the other hand, recalling (2.32),

$$s \|\hat{v}^\rho - \hat{v}_\star(s \cdot)/s\|_{L^\infty(\partial B_1 \cap \{u_\rho > 0\})} \leq C_n \left(r_\star^{-1} |b_\rho - b_{r_\star}| + r_\star^{-1} \rho^{2-n} |c_\rho - c_{r_\star}| + r_\star^{-1} \rho^{1-n} |d'_\rho - d'_{r_\star}| \right). \quad (2.44)$$

As in Step 1, whenever $r, 2r \in [\delta r_\star, \delta^{-1} r_\star]$, we have

$$|b_{2r} - b_r| + r^{2-n} |c_{2r} - c_r| + r^{1-n} |d'_{2r} - d'_r| \leq C_{n,\alpha} r (\hat{h}_{2r} + \hat{h}_r). \quad (2.45)$$

Summing dyadically, and using (2.42), we deduce

$$\begin{aligned} |b_\rho - b_{r_\star}| &\leq C_{n,\alpha} r_\star \hat{h}_{r_\star} (s^{1-n-\alpha} + \log s), \\ |c_\rho - c_{r_\star}| &\leq C_{n,\alpha} r_\star^{n-1} \hat{h}_{r_\star} (s^{-1-\alpha} + s^{n-2}), \\ |d'_\rho - d'_{r_\star}| &\leq C_{n,\alpha} r_\star^n \hat{h}_{r_\star} (s^{-\alpha} + s^{n-1}). \end{aligned}$$

In (2.44) we thus get

$$s \|\hat{v}^\rho - \hat{v}_\star(s \cdot)/s\|_{L^\infty(\partial B_1 \cap \{u_\rho > 0\})} \leq C_{n,\alpha} \hat{h}_{r_\star} (s^{1-n-\alpha} + |\log s|).$$

Since $|\log s| \leq C(s^{1-n-\alpha} + s^\alpha)$, putting it all together in (2.43) we get (2.40).

Step 5. Compactness improvement. By (2.36), (2.39), and (2.40), we may apply the perturbative compactness lemma, Lemma 2.12. Up to dividing by a constant depending only on n and α for the application of Lemma 2.12, we may assume that C in (2.40) is 1. We obtain coefficients

$$b^\sharp, c^\sharp \in \mathbb{R}, \quad d^{\sharp'} \in \mathbb{R}^{n-1},$$

bounded by a universal constant (depending on n and α , due to the constant in (2.40)), such that

$$\left| w(x) - b^\sharp - c^\sharp |x|^{2-n} - (d^{\sharp'} \cdot x') |x|^{-n} \right| \leq \varepsilon \quad \text{in } \Omega_\star \cap (B_2 \setminus B_{1/2}), \quad (2.46)$$

where $\varepsilon > 0$ will be fixed small.

We thus obtain a new Ansatz at scale $r_\star$ given by:

$$\tilde{v}(x) = \hat{v}_\star(x) + \hat{h}_{r_\star} b^\sharp + \hat{h}_{r_\star} c^\sharp |x|^{2-n} + \hat{h}_{r_\star} (d^{\sharp'} \cdot x') |x|^{-n}.$$

From (2.46), we get

$$|u_{r_\star} - \tilde{v}| \leq \hat{h}_{r_\star} \varepsilon \quad \text{in } \{u_{r_\star} > 0\} \cap (B_2 \setminus B_{1/2}).$$

We can now fix $\varepsilon = \frac{1}{2}$ (which fixes δ from Lemma 2.12, depending only on n and α) and $r_\star$ large enough so that (also using that $\tilde{v} \in \mathcal{A}(1/8)$ because the coefficients are *small*)

$$\hat{h}_{r_\star} = \hat{H}(u, r_\star) \leq \|u_{r_\star} - \tilde{v}\|_{L^\infty(\{u_{r_\star} > 0\} \cap (B_2 \setminus B_{1/2}))} \leq \frac{1}{2} \hat{h}_{r_\star};$$

a contradiction. This proves $\varepsilon_\star = 0$, and therefore

$$\hat{h}_r \leq C_\star r^{-n-\alpha} \quad \text{for all } r \geq 4. \quad (2.47)$$

Step 6. Passage to the asymptotic coefficients. From (2.47) and using (2.28)-(2.32) we know, for all $r \geq 4$ (up to a constant)

$$|u(x + b_0 e_n) - w^r(x)| \leq r \hat{h}_r \leq C_\star r^{1-n-\alpha} \quad \text{in } \{u(\cdot + b_0 e_n) > 0\} \cap B_{2r} \setminus B_{r/2}$$

where

$$w^r(x) := x_n + b_r + \frac{c_r}{|x|^{n-2}} + \frac{d'_r \cdot x'}{|x|^n}.$$

Notice, also, that thanks to Lemma 2.10, we know

$$|\nabla u(x + b_0 e_n) - \nabla w^r(x)| \leq \hat{h}_r^\theta \leq C r^{\theta(1-n-\alpha)} \quad \text{in } \{u(\cdot + b_0 e_n) > 0\} \cap B_{2r} \setminus B_{r/2}. \quad (2.48)$$

Denoting r_k the dyadic scale $r_k := 2^k$, and

$$(b_k, c_k, d'_k) := (b_{r_k}, c_{r_k}, d'_{r_k}),$$

by (2.34)-(2.45)-(2.47) and summing a geometric series, we know $(b_k, c_k, d'_k) \rightarrow (0, c_\infty, d'_\infty)$ with

$$|b_k| \leq C r_k^{3-n-\alpha}, \quad |c_k - c_\infty| \leq C r_k^{-1-\alpha}, \quad |d'_k - d'_\infty| \leq C r_k^{-\alpha}.$$

Thus, denoting w^{r_∞} the corresponding function with coefficients $(0, c_\infty, d'_\infty)$, we have

$$\begin{aligned} |u(x + b_0 e_n) - w^{r_\infty}(x)| &\leq |u(x + b_0 e_n) - w^{r_k}(x)| + |w^{r_k}(x) - w^{r_\infty}(x)| \\ &\leq C r_k^{1-n-\alpha}. \end{aligned}$$

in $\{u(\cdot + b_0 e_n) > 0\} \setminus B_{r_k/2}$. This yields the following bound, using that $|b_0|$ is bounded,

$$\left| u(x) - x_n + b_0 - \frac{c_\infty}{|x - b_0 e_n|^{n-2}} - \frac{d'_\infty \cdot x'}{|x - b_0 e_n|^n} \right| \leq \frac{C}{|x|^{n-1+\alpha}} \quad \text{in } \{u > 0\} \setminus B_1. \quad (2.49)$$

Taking a Taylor expansion we obtain (2.14). Notice, in particular, the appearance of a new term $d_n x_n |x|^{-n}$, with $d_n = (n-2)b_0 c_\infty$. The bound (2.16) for the gradients follows in the same way, using (2.48), up to choosing a larger $\alpha \in (0, 1)$ and θ very close to 1. $\square$

In particular, we obtain the:

Proof of Theorem 1.6. It follows from Theorem 2.7. $\square$

3. ASYMPTOTIC STRUCTURE OF ENTIRE SOLUTIONS WITH FINITE INDEX

We focus now on entire solutions $u : \mathbb{R}^n \rightarrow [0, \infty)$ with finite Morse index.

3.1. Preliminaries. Our goal is to show:

Proposition 3.1. *Let $3 \leq n < n_*$, and let $u : \mathbb{R}^n \rightarrow [0, \infty)$ be a classical solution to the Bernoulli problem with finite Morse index and which is not one-dimensional.*

Then, either $\text{FB}(u)$ is bounded, or there exists some $R_0 = R_0(u) > 0$ such that we can decompose $u = u_+ + u_-$ in $D := \mathbb{R}^n \setminus B_{R_0}$, where $\text{supp } u_+ \cap \text{supp } u_- = \emptyset$, and each of the $u_\pm$ is a classical solution to the Bernoulli problem in D with

$$H_0(u_\pm, r) \leq o(r) \quad \text{as } r \rightarrow \infty. \quad (3.1)$$

We first collect some important properties for local and global solutions.

The following gradient bound is well-known (see, e.g., [35, Proposition A.5]):

Lemma 3.2. *Let $u : \mathbb{R}^n \rightarrow [0, \infty)$ be an entire solution to the Bernoulli problem. Then, $|\nabla u| \leq 1$.*

Let us recall Weiss's monotonicity formula.

Lemma 3.3 ([52, Theorem 3.1]). *Let $u : B_1 \rightarrow [0, \infty)$ be an inner stationary solution to the Bernoulli problem, and let*

$$W(r) := \frac{1}{r^n} \int_{B_r} (|\nabla u|^2 + \mathbf{1}_{\{u>0\}}) - \frac{1}{r^{n+1}} \int_{\partial B_r} u^2 d\mathcal{H}^{n-1} \quad (3.2)$$

for $r \in (0, 1)$. Then $\frac{d}{dr} W(r) \geq 0$; moreover, $\frac{d}{dr} W(r) = 0$ precisely if u is one-homogeneous in B_r .

We have outer stability for solutions with finite index.

Lemma 3.4 ([4, Proposition 3.4]). *Let u be a classical solution to the Bernoulli problem in $\mathbb{R}^n$ with finite Morse index. Then, there is some $R_0 = R_0(u) > 0$ such that u is stable in $\mathbb{R}^n \setminus \overline{B_{R_0}}$.*

We also record a nondegeneracy property.

Lemma 3.5. *Let u be a classical solution to the Bernoulli problem in $B_1 \subset \mathbb{R}^n$ with $0 \in \text{FB}(u)$ and $|D^2 u| \leq C_\circ$ in $\{u > 0\} \cap B_1$. Then, there is $C = C(n, C_\circ)$ such that, up to a rotation, $u \geq x_n - Cr^2$ in B_r for all $r \in (0, 1/2)$.*

Proof. Let us assume $\nabla u(0) = e_n$. It now follows by observing that $v(x) := u(x) - x_n$ satisfies $v(0) = |\nabla v(0)| = 0$ and $|D^2 v| \leq C_\circ$ in $\{u > 0\} \cap B_1$. Notice, also, that the Hessian bound implies the same curvature bound on the free boundary by the constant Neumann condition $|\nabla u| = 1$ on $\text{FB}(u)$. $\square$

We will need the next lemma on solutions that are close to a vee.

Lemma 3.6 ([7, Lemma 3.15]). *Let $n \geq 2$. Given $C_1 \geq 1$ there exists $\varepsilon_1 > 0$, depending only on n and C_1 , such that the following holds.*

Let u be a global classical solution to the Bernoulli problem in $\mathbb{R}^n$. Suppose that $|D^2u| \leq C_1 \varrho^{-1}$ in $B_{2\varrho} \cap \{u > 0\}$ and

$$|u - V_{0,e_n}| \leq \varepsilon \varrho \leq \varepsilon_1 \varrho \quad \text{in } B_{2\varrho}, \quad (3.3)$$

where e_n is the n -th vector in the canonical basis. Then

$$\varrho^2 \|D^2u\|_{L^\infty(\{u>0\} \cap B_\varrho)} \leq C\varepsilon \varrho$$

for some C depending only on n . Moreover,

$$\{u > 0\} = \{x_n > \tilde{g}^{(+)}(x_1, \dots, x_{n-1})\} \cup \{x_n < \tilde{g}^{(-)}(x_1, \dots, x_{n-1})\} \quad \text{in } B_\varrho,$$

where $\tilde{g}^{(\pm)} : B'_\varrho \rightarrow \mathbb{R}$, $\tilde{g}^{(-)} < \tilde{g}^{(+)}$, and

$$\|\tilde{g}^{(\pm)}\|_{L^\infty(B'_\varrho)} + \varrho^2 \|D^2\tilde{g}^{(\pm)}\|_{L^\infty(B'_\varrho)} \leq C\varepsilon \varrho$$

for some C depending only on n .

The following important result shows that entire solutions are close to a wedge solution at all large scales where $\text{FB}(u)$ is present (assuming u is not one-dimensional).

Lemma 3.7 (Compactness toward a two-plane configuration). *For every $\varepsilon \in (0, 1)$ there exists $\delta = \delta(n, \varepsilon) > 0$ such that the following holds.*

Let $n < n_$, and let u be a classical solution in $\mathbb{R}^n$, stable in $\mathbb{R}^n \setminus B_\delta$. Assume that*

$$W(u, 2) - W(u, 1) \leq \delta$$

and

$$\text{FB}(u) \cap (B_2 \setminus \overline{B_1}) \neq \emptyset.$$

Then, there exists a unit vector $e \in \mathbb{S}^{n-1}$ such that either

$$\|u - (e \cdot x)_+\|_{L^\infty(B_{16})} \leq \varepsilon \quad \text{or} \quad \|u - |e \cdot x|\|_{L^\infty(B_{16})} \leq \varepsilon. \quad (3.4)$$

Moreover, in the second case, letting $\mathcal{R}_e : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a rotation sending e to e_n , we have:

$$\mathcal{R}_e(\{u > 0\}) = \{x_n > \tilde{g}^{(+)}(x')\} \cup \{x_n < \tilde{g}^{(-)}(x')\} \quad \text{in } B_8 \setminus \overline{B_{1/2}},$$

where $\tilde{g}^{(\pm)} : B'_8 \rightarrow \mathbb{R}$, $\tilde{g}^{(-)} < \tilde{g}^{(+)}$, and there is $C = C(n) > 0$ such that

$$\|\tilde{g}^{(\pm)}\|_{L^\infty(B'_8)} + C^{-1} \|D^2\tilde{g}^{(\pm)}\|_{L^\infty(B'_8)} \leq \varepsilon \leq 1/4.$$

In particular,

$$\text{FB}(u) \cap \partial B_r \neq \emptyset \quad \text{for every } r \in (1, 4).$$

Proof. From the assumption that entire stable solutions are flat, it follows (arguing² as in [7]) that

$$|D^2u| \leq \frac{C}{r} \quad \text{in } \{0 < \text{dist}(x, \{u = 0\}) \leq r/3\} \cap (B_{3r} \setminus \overline{B_{r/3}}), \quad (3.5)$$

for every $r \geq 6\delta$.

By Lemmas 3.2 and 3.5, together with a Harnack chain argument and the fact that $\text{FB}(u) \cap (B_2 \setminus \overline{B_1}) \neq \emptyset$, we also have:

$$u \leq C_M \quad \text{in } B_M, \quad (3.6)$$

and

$$u \geq c_M > 0 \quad \text{in } \{\text{dist}(x, \{u = 0\}) \geq M^{-1}\} \cap (B_M \setminus \overline{B_{M^{-1}}}). \quad (3.7)$$

²The results for curvature estimates are stated for $n = 3$ in [7], but the exact same proof works for $n < n_*$ more generally.

Together with (3.5), this shows that

$$|D^2u| \leq C_M \quad \text{in} \quad (B_M \setminus \overline{B_{M^{-1}}}). \quad (3.8)$$

Let us show (3.4) by contradiction. Assume that there exist sequences $\delta_i = \frac{1}{i}$, u_i , with $W(u_i, 2) - W(u_i, 1) \leq \delta_i \rightarrow 0$, contradicting the thesis. By our estimates above, up to a subsequence $u_i \rightarrow u_\infty$ in $C_{loc}^0(\mathbb{R}^n \setminus \{0\})$, where u_∞ is an inner stable and stationary solution. Moreover, $u_\infty \not\equiv 0$, e.g., by passing the nondegeneracy condition (3.7) to the limit. In fact, thanks to (3.7) we see that the sets $\{u_k > 0\}$, $\text{FB}(u_k)$, and $\{u_k = 0\}$ converge—locally in the Hausdorff distance sense—to $\{u_\infty > 0\}$, $\text{FB}(u_\infty)$, and $\{u_\infty = 0\}$.

These properties show that the Weiss energies of the u_i locally converge to that of u_∞ ; by Weiss's monotonicity formula, u_∞ is 1-homogeneous. Two cases arise:

- If two parallel sheets of $\text{FB}(u_i)$ collapse together (tangentially, by the curvature estimates) at some $y \in \text{FB}(u_\infty) \setminus \{0\}$, then $\frac{1}{r}u_\infty(rx + y) \xrightarrow{r \rightarrow 0} |x \cdot e|$ for some $e \in \mathbb{S}^{n-1}$. An argument using Weiss's monotonicity formula again, cf. [7, Proof of Lemma 4.4], shows then that u_∞ is also conical around y , thus $u_\infty \equiv |(x - y) \cdot e|$, contradicting the negation of (3.4).
- Otherwise, u_∞ is a classical solution outside of the origin. By (1.3) then u_∞ is a half-space solution, contradicting the negation of (3.4) again.

Assume now that we are in the second case in (3.4). Perform a rotation so that $e = e_n$. Up to making δ smaller, we can then apply Lemma 3.6 around every $x \in (B_8 \setminus \overline{B_{1/2}}) \cap \text{FB}(u)$, which gives the graphical description of $\{u > 0\}$. Finally, from this and $|\tilde{g}^\pm| \leq 1/4$ we find that $\text{FB}(u) \cap \partial B_r \neq \emptyset$ for every $r \in (1, 4)$. $\square$

We can now give:

Proof of Proposition 3.1. Assume $\text{FB}(u)$ to be unbounded. After a dilation, we can assume u to be stable outside of B_1 by Lemma 3.4. Moreover, from $|\nabla u| \leq 1$ (see Lemma 3.2) we have $W(R) \leq |B_1|$, and in particular $W(R)$ is bounded; by monotonicity (Lemma 3.3), given $\delta > 0$ there exists then R_0 such that $W(R) - W(r) \leq \delta$ for every $R_0 \leq r \leq R$.

We can then apply Lemma 3.7—appropriately rescaled—to u for every $r \geq R_0$ such that $\text{FB}(u) \cap \partial B_r \neq \emptyset$, with associated $\varepsilon_r \rightarrow 0$ as $r \rightarrow \infty$. Up to making R_0 larger, if $r \geq R_0$ the first case in the Lemma never holds: if, arguing by contradiction, there were $r_k \rightarrow \infty$ with

$$\|u - (e_{r_k} \cdot x)_+\|_{L^\infty(B_{16r_k})} \leq \varepsilon_{r_k} \xrightarrow{r_k \rightarrow \infty} 0,$$

the improvement of flatness result in [13] would imply that $u \equiv (x \cdot e)_+$ for some $e \in \mathbb{S}^{n-1}$, and u would be one-dimensional.

Let now $r \geq R_0$ with $\text{FB}(u) \cap \partial B_r \neq \emptyset$, which we can find since we are assuming $\text{FB}(u)$ to be unbounded. By the previous argument, the second case in Lemma 3.7 applies, i.e.

$$\|u - |e_r \cdot x|\|_{L^\infty(B_{16r})} \leq \varepsilon_r \xrightarrow{r \rightarrow \infty} 0.$$

Moreover, $\text{FB}(u) \cap \partial B_s \neq \emptyset$ for every $s \in (r, 4r)$. But then, in particular we can apply Lemma 3.7 again with $2r$ in place of r ; iterating this, we conclude that in fact $\text{FB}(u) \cap \partial B_r \neq \emptyset$ for every $r \geq R_0$. Moreover, putting $A_r := B_{8r} \setminus B_r$, Lemma 3.7 gives a decomposition $\{u > 0\} \cap A_r = U_r^+ \cup U_r^-$, with associated $e_r \in \mathbb{S}^{n-1}$ such that

$$\mathcal{R}_{e_r}(U_r^+) = \{x_n > \tilde{g}_r^{(+)}(x')\} \cap A_r, \quad \mathcal{R}_{e_r}(U_r^-) = \{x_n < \tilde{g}_r^{(-)}(x')\} \cap A_r,$$

with $\tilde{g}_r^{(\pm)} : B'_{8r} \rightarrow \mathbb{R}$, $\tilde{g}_r^{(-)} < \tilde{g}_r^{(+)}$, and

$$\|\tilde{g}_r^{(\pm)}\|_{L^\infty(B'_{8r})} + C^{-1}\|D^2\tilde{g}_r^{(\pm)}\|_{L^\infty(B'_{8r})} \leq r\varepsilon_r \leq r/4.$$

By considering the overlap among the A_r , we see that—up to changing the sign of some of the e_r , and making ε_r small enough by taking r large enough—we can ensure that $|e_{\lambda r} - e_r| < 1/8$ for

all $\lambda \in (1/128, 128)$, thus also $U_{\lambda r}^+ \cap U_r^- = \emptyset$. Defining $U^+ = \cup_{r \geq R_0} U_r^+$ and $U^- = \cup_{r \geq R_0} U_r^-$ it follows that $U^+ \cap U^- = \emptyset$, and we put $u_+ = u \mathbf{1}_{U^+}$ and $u_- = u \mathbf{1}_{U^-}$. Finally, letting $e_r^\pm = \pm e_r$, since $|u - |e_r \cdot x|| \leq o(r)$ in $B_{2r} \setminus B_{r/2}$ it follows that $|u_\pm - (e_r^\pm \cdot x)_+| \leq o(r)$ as well—for instance, by a compactness-plus-contradiction argument. This shows that $H_0(u_\pm, r) \leq o(r)$ and concludes the proof. $\square$

3.2. Asymptotic structure in the non-compact free boundary case. Combining our results up to now, we obtain a detailed expansion for u in the non-compact case:

Corollary 3.8. *Let $3 \leq n < n_*$ and $\alpha \in (0, 1)$. Let $u : \mathbb{R}^n \rightarrow [0, \infty)$ be a classical solution to the Bernoulli problem with finite Morse index, non-compact free boundary, and which is not one-dimensional.*

After a rescaling, rotation, and translation, we have the decomposition $u = u_+ + u_-$ in $D := \mathbb{R}^n \setminus (B'_1 \times [-1, 1])$, where each of the $u_\pm$ is a classical solution to the Bernoulli problem in D . Letting $\Omega_\pm = \{u_\pm > 0\} \subset D$, we have $\Omega_+ \cap \Omega_- = \emptyset$ and

$$\Omega_\pm = \{|x'| > 1, \pm x_n > g^{(\pm)}(x')\} \cup \{|x'| \leq 1, \pm x_n > 1\},$$

for some smooth $g^{(\pm)} : \mathbb{R}^{n-1} \setminus B'_1 \rightarrow [-1, 1]$.

Moreover, the $u_\pm$ satisfy

$$|u_\pm - v_\pm| + |x| |\nabla(u_\pm - v_\pm)| \leq |x|^{1-n-\alpha} \quad \text{in } \Omega_\pm,$$

where

$$v_\pm = \pm x_n - b + \frac{c_\pm}{|x|^{n-2}} + \frac{d_\pm \cdot x}{|x|^n},$$

with $b \geq 0$, $c_\pm \in \mathbb{R}$, $d_\pm \in \mathbb{R}^n$, and satisfying $(d_\pm)_n = \pm(n-2)bc_\pm$ and $|b| + |c_\pm| + |d_\pm| \leq 1/2$.

Proof. Decompose $u = u_+ + u_-$ via Proposition 3.1. By (3.1), we can apply Corollary 2.5 and Theorem 2.7—appropriately rescaled—to each of u_+ and u_- , separately. We obtain Ansätze $v_\pm = e_\pm \cdot x - b_\pm + \frac{c_\pm}{|x|^{n-2}} + \frac{d_\pm \cdot x}{|x|^n}$, where $e_\pm \in \mathbb{S}^{n-1}$, such that

$$|u_\pm - v_\pm| + |x| |\nabla(u_\pm - v_\pm)| \leq C|x|^{1-n-\alpha} \quad \text{in } \Omega_\pm,$$

and an associated graphicality condition for the $\Omega_\pm$ in the directions $e_\pm$. Now, since $\text{supp } u_+ \cap \text{supp } u_- = \emptyset$, it follows that necessarily $e_- = -e_+$; hence, after a rotation we have $e_\pm = \pm e_n$. From this, we see that $(d_\pm)_n = \pm(n-2)bc_\pm$ (by the relation between the coefficients from Theorem 2.7) and that $\Omega_\pm$ satisfy the graphicality condition in x_n from the statement. Similarly, necessarily $b_+ \geq -b_-$, and considering the translated solution $u(x', x_n + \frac{b_+ - b_-}{2})$ instead we can impose that $b_+ = b_- =: b \geq 0$ as well. Notice that by translating we are also altering the coefficients, e.g., we need to expand $c_\pm |x + \frac{b_+ - b_-}{2} e_n|^{2-n}$ around $c_\pm |x|^{2-n}$, but the relationship between them remains (one sees this either by direct computation, or by just reapplying Theorem 2.7). Up to rescaling again so that the bounds on the $b, c_\pm, d_\pm$ and the error terms hold, we obtain the result as stated. $\square$

We find then a natural lower barrier for our solution:

Lemma 3.9. *In the setting of Corollary 3.8, it holds $u \geq |x_n| - b$.*

Proof. It is enough to show that $u \geq x_n - b$ in $\mathbb{R}^n$; indeed, arguing symmetrically, $u \geq -x_n - b$ as well. Let

$$h_* := \inf\{h > 0 : u \geq x_n - h \quad \text{in } \mathbb{R}^n\}.$$

By Corollary 3.8 we have

$$u \geq x_n - b - |x|^{2-n} \quad \text{in } \mathbb{R}^n \setminus B_1, \tag{3.9}$$

thus $h_* < \infty$. Moreover, it is a minimum by continuity, thus it suffices to show that $h_* \leq b$. Suppose that $h_* > b$ instead; then

$$u \geq x_n - b - |x|^{2-n} > x_n - h_* + \varepsilon \quad \text{in } \mathbb{R}^n \setminus B_R$$

for $\varepsilon = (h_* - b)/2$ and some $R > h_*$ large enough (so that $h_* > b + 2R^{2-n}$). That is, we have two ordered harmonic functions $u \geq v := x_n - h_*$ in $\{x_n > h_*\} \cap B_R$, with $u > v + \varepsilon$ on ∂B_R . By definition of h_* (and the fact that u is 1-Lipschitz in $\mathbb{R}^n$) and harmonicity, we must then have $u = v = 0$ at some point on $\{x_n = h_*\} \cap B_R$. But this is a contradiction with Hopf's lemma, by the Bernoulli condition on u and the fact that $\partial_n v = 1$. $\square$

The coefficients further satisfy:

Lemma 3.10. *In the setting of Corollary 3.8, we have $b, c_{\pm}, \pm(d_{\pm})_n > 0$.*

In particular, up to a rescaling we can additionally assume that $g^{(\pm)} > 0$ in Corollary 3.8, thus $\Omega_{\pm} \subset \{\pm x_n > 0\}$.

Proof. The fact that $b > 0$ follows from Lemma 3.9: otherwise $u \geq |x_n|$, which would contradict the assumption that u be a classical solution with unbounded free boundary. Together with the asymptotics for $g^{(\pm)}$ in Corollary 3.8, it follows that $g^{(\pm)}(x') > 0$ in $\mathbb{R}^{n-1} \setminus B_M$ for some $M > 0$, thus in $\mathbb{R}^{n-1} \setminus B_1$ up to a rescaling.

It suffices then to show that $c_+, (d_+)_n > 0$, since the corresponding properties for $c_-, (d_-)_n$ follow then by symmetry.

First, from Corollary 3.8 and $u \geq x_n - b$ we have

$$-c_+ |x|^{2-n} \leq (u - x_n + b) - c_+ |x|^{2-n} \leq C |x|^{1-n} \quad \text{in } \{x_n > b\}.$$

Letting $|x| \rightarrow \infty$, this shows that $-c_+ \leq 0$, i.e. $c_+ \geq 0$.

If $c_+ > 0$, since $(n-2)bc_+ = (d_+)_n$ we see that $(d_+)_n > 0$ as well, concluding the proof. Otherwise, we would have $c_+ = (d_+)_n = 0$; let us show that this leads to a contradiction. First, by Corollary 3.8 we find

$$0 < w \leq |x|^{1-n} \left(C |x|^{-\alpha} + \frac{d'_+ \cdot x'}{|x|} \right) \quad \text{in } \{x_n > b\},$$

where $w := u - x_n + b > 0$ is harmonic in $\{x_n > b\}$. Considering $x = (-\lambda d'_+, 2b)$ and sending $\lambda \rightarrow \infty$ shows that $d'_+ = 0$, thus

$$0 \leq w \leq C |x|^{1-n-\alpha} \quad \text{in } \{x_n > b\}.$$

If $w > 0$, we obtain a contradiction with the maximum decay of order $|x|^{1-n}$ —given by the Poisson kernel of the half-space—for positive harmonic functions. Otherwise $w(x) = 0$ at some point, thus $u \equiv x_n - b$ in $\{x_n \geq b\}$ by the (strong) maximum principle. Then, $\bar{u} = u \mathbf{1}_{\{x_n > -b\}}$ is a classical entire solution to the Bernoulli problem. By improvement of flatness, the component of $\{\bar{u} > 0\}$ containing $\{u_+ > 0\}$ must be flat, and there cannot be other components, as they would be bounded. $\square$

4. SYMMETRY FOR ENTIRE SOLUTIONS WITH FINITE INDEX

We now use the asymptotic information obtained above to prove the symmetry of finite-index entire solutions. The proof separates the compact and non-compact free boundary cases.

4.1. Compact free boundary case. The compact case reduces to the classical exterior over-determined problem:

Proposition 4.1. *Let u be a classical solution to the Bernoulli problem in $\mathbb{R}^n$, $n \geq 3$, with compact free boundary. Then, either u is constant, or after a translation and rescaling we have $u = \frac{1}{n-2}(1 - |x|^{2-n})_+$.*

Proof. We can assume that $\text{FB}(u)$ is nonempty: otherwise u is a nonnegative harmonic function in $\mathbb{R}^n$ and thus constant. Since $\text{FB}(u)$ is compact, then $\{u > 0\}$ consists of a (single) unbounded component. Let R_0 be such that $\text{FB}(u) \subset B_{R_0}$, so that u is a positive harmonic function in $\mathbb{R}^n \setminus B_{R_0}$, satisfying $|u| \leq C(1 + |x|)$ since $|\nabla u| \leq 1$. Separation of variables³ gives then $a, d \in \mathbb{R}^n$ and $b, c \in \mathbb{R}$ such that

$$u = a \cdot x + b + \frac{c}{|x|^{n-2}} + \frac{d \cdot x}{|x|^n} + O(|x|^{-n}) \quad \text{in } \mathbb{R}^n \setminus B_{R_0}.$$

Since $u \geq 0$, necessarily $a = 0$. Combined with standard harmonic estimates, $v := b - u$ satisfies $|v| + |\nabla v| \rightarrow 0$ as $|x| \rightarrow \infty$, and $v \geq 0$ by the maximum principle.

Then, the main result in [44]—whose proof is based on the moving planes method—shows that v (hence, u) is radially symmetric, and $\{v = b\}$ (hence, $\{u = 0\}$) is a ball.

Let R be the radius of this ball; after a rescaling and translation, we can assume that $R = 1$ and it is centered at the origin. Integrating radially the equation $\Delta u(r) = 0$ —which is now an ODE, since $u = u(r)$ is radial—for $r \geq 1$ and imposing the initial conditions $u(1) = 0$, $u'(1) = 1$ yields the desired form for u . $\square$

4.2. Non-compact free boundary case: symmetry in x_n . Let us now assume that $3 \leq n < n_*$, and that $u : \mathbb{R}^n \rightarrow [0, \infty)$ is a classical solution to the Bernoulli problem with finite Morse index, non-compact free boundary, and which is not one-dimensional.

We can (and do) assume then that we are in the setting given by Proposition 3.1 and Corollary 3.8 (recall also Lemmas 3.9 and 3.10). In particular

$$\Omega_{\pm} := \{u_{\pm} > 0\} \subset D := \mathbb{R}^n \setminus (B'_1 \times [-1, 1]), \quad (4.1)$$

and the $\Omega_{\pm}$ are given by

$$\Omega_+ = \{|x'| > 1, x_n > g^{(+)}(x')\} \cup \{|x'| \leq 1, x_n > 1\}, \quad (4.2)$$

$$\Omega_- = \{|x'| > 1, x_n < -g^{(-)}(x')\} \cup \{|x'| \leq 1, x_n < -1\}, \quad (4.3)$$

with (by Lemma 3.10)

$$g^{(+)} > 0 \quad \text{and} \quad g^{(-)} > 0. \quad (4.4)$$

Remark 4.2. In fact, the simpler asymptotics in Corollary 2.5 would suffice in this section.

This section will show:

Proposition 4.3. *In the setting of Corollary 3.8, u is even in x_n . Moreover, $\partial_{x_n} u > 0$ in $\{x_n > 0\} \cap \{\overline{u > 0}\}$.*

We start with some asymptotic properties.

Lemma 4.4 (Gradient asymptotics). *There is $M > 0$ such that*

$$\partial_{x_n} u_+ > 3/4 \quad \text{in } \Omega_+ \cap \{|x| \geq M\} \quad \text{and} \quad \partial_{x_n} u_- < -3/4 \quad \text{in } \Omega_- \cap \{|x| \geq M\}.$$

Proof. The result follows from (4.2)–(4.4) and the gradient comparison in Corollary 3.8. $\square$

³Alternatively one can use a Green's function representation, or a (quite simpler) version of the arguments in Section 2.

In the following, we define for $x = (x', x_n)$ the symmetry:

$$\begin{aligned} \sigma_t(x) &:= (x', 2t - x_n) \quad \text{and} \quad x_t^* := \sigma_t(x), \\ u_t^* &= u \circ \sigma_t \quad \text{i.e.,} \quad u_t^*(x', x_n) := u(x', 2t - x_n). \end{aligned}$$

Lemma 4.5 (Vertical initialization and reflection asymptotics). *In the setting of Corollary 3.8, we have:*

(a) *There exists $t_0 > 8$ (large) such that for all $t > t_0$ we have:*

$$u \geq u_t^* \quad \text{in} \quad \{x_n \geq t\}$$

(b) *Given $\delta > 0$, there is $R_\delta \geq 0$ such that if $t \geq \delta$,*

$$u \geq u_t^* \quad \text{in} \quad \{x_n \geq t\} \setminus B_{R_\delta}.$$

Proof. We first show (a). Let $x \in \{x_n \geq t\}$ with $t \geq t_0 > 8$; by Corollary 3.8 we have $u(x) > 0$. If $u_t^*(x) = 0$ there is nothing to prove, so we can also assume $u_t^*(x) > 0$, that is, $\sigma_t(x) \in \{u > 0\}$. We now have three options, where we denote by $M \geq 1$ the constant from Lemma 4.4:

- If $\sigma_t(x) \in \{u > 0\} \cap B_M$, since u is 1-Lipschitz (and $\partial B_2 \cap \{u = 0\} \neq \emptyset$) we know $u_t^*(x) \leq 4M$. On the other hand, $u(x) \geq |x_n| - b \geq t_0 - b$, so that taking t_0 large we get $u(x) \geq u_t^*(x)$.
- If $\sigma_t(x) \in \Omega_+ \setminus B_M$, then $0 < (\sigma_t(x))_n < x_n$, $|\sigma_t(x)| \geq M$, and (by (4.2)–(4.4)) $u > 0$ on the segment between $\sigma_t(x)$ and x . Lemma 4.4 gives that $u(x) \geq u_t^*(x)$ as well.
- Finally, if $\sigma_t(x) \in \Omega_- \setminus B_M$, since $x \in \Omega_+ \setminus B_M$ Corollary 3.8 gives (up to maybe making M universally larger)

$$u(x) = u_+(x) \geq x_n - b - 1 \quad \text{and} \quad u_t^*(x) = u_-(\sigma_t(x)) \leq x_n - 2t - b + 1.$$

From $t_0 > 1$ (so that $-2t + 1 \leq -2t_0 + 1 \leq -1$) we conclude that $u(x) \geq u_t^*(x)$ in this case as well.

Let us now show (b). If $t > t_0$, where $t_0 > 8$ was found in (a), we already conclude. Hence, it suffices to show that given $\delta > 0$, there exists $R_\delta \geq t_0$ such that, for $t \in (\delta, R_\delta)$, we have

$$u \geq u_t^* \quad \text{in} \quad \{x_n \geq t\} \setminus (B'_{R_\delta} \times (0, 3R_\delta)).$$

Let $x \in \{x_n \geq t\} \setminus (B'_{R_\delta} \times (0, 3R_\delta))$. If $u_t^*(x) = 0$ there is nothing to prove, so we can assume that $u_t^*(x) > 0$. From $R_\delta > 1$ one sees that $\sigma_t(x) \in \mathbb{R}^n \setminus (B'_1 \times [-1, 1])$, thus $\sigma_t(x) \in \Omega_+ \cup \Omega_-$.

Let us assume first that $u(x) > 0$ as well; we will reduce to this case immediately after. By (4.2)–(4.4) we have $x \in \Omega_+$.

If also $\sigma_t(x) \in \Omega_+$, Lemma 4.4 shows that $u(x) > u(\sigma_t(x)) = u_t^*(x)$, just as in case (a). If $\sigma_t(x) \in \Omega_-$ instead, by the asymptotics in Corollary 3.8 we can ensure, up to making R_δ large enough, that

$$u(x) = u_+(x) \geq x_n - b - \delta/3 \quad \text{and} \quad u_t^*(x) = u_-(\sigma_t(x)) \leq x_n - 2t - b + \delta/3.$$

Since $t \geq \delta$ this shows that $u(x) > u_t^*(x)$ also in this case.

It only remains to argue that $u(x) > 0$ indeed above. If $\sigma_t(x) \in \Omega_+$, this is automatic by Lemma 4.4. Assume that $\sigma_t(x) \in \Omega_-$ and $u(x) = 0$ instead; increasing x_n until we reach $\text{FB}(u)$, and using the case from before, we then find a contradiction. More precisely: let $s := \max\{\lambda \in [x_n, \infty) : u(x', \lambda) = 0\}$ and $y = (x', s) \in \mathbb{R}^n$, so that $y \in \text{graph } g^{(+)} \subset \text{FB}(u)$ (recall (4.2)–(4.4)). Then $u(y + he_n) > 0 \forall h > 0$; moreover, by Lemma 4.4 applied with u_- we have $u(\sigma_t(y + he_n)) \geq u(\sigma_t(x)) > 0$ as well. We can therefore apply the previous case with $y + he_n$ in place of x , finding

$$u(y + he_n) \geq u(\sigma_t(y + he_n)) \geq u(\sigma_t(x)) > 0 \quad \forall h > 0.$$

Sending $h \rightarrow 0$ then $u(y) \geq u(\sigma_t(x)) > 0$, a contradiction with $y \in \text{FB}(u)$. $\square$

In the proof of Proposition 4.3, we will need the following result from [46], which we reproduce here for the reader's convenience:

Lemma 4.6 ([46, Lemma 1]). *Let U^* be a domain with C^2 boundary and let H be a plane containing the normal to ∂U^* at some point $x \in \partial U^*$. Let U then denote the portion of U^* lying on some particular side of H .*

Suppose that v is of class C^2 in the closure of U and satisfies

$$\Delta v \leq 0 \quad \text{in } U,$$

while also $v \geq 0$ in U and $v = 0$ at x . Let s be any direction at x which enters U non-tangentially. Then either

$$\frac{\partial v}{\partial s}(x) > 0 \quad \text{or} \quad \frac{\partial^2 v}{\partial s^2}(x) > 0,$$

unless $v \equiv 0$.

We can now proceed with:

Proof of Proposition 4.3. Let $T \subset [0, \infty)$ be defined as

$$T := \{t \in [0, \infty) : u \geq u_t^* \text{ in } \{x_n \geq t\} \text{ and } \partial_{x_n} u > 0 \text{ in } \{x_n > t\} \cap \overline{\{u > 0\}}\}.$$

We want to show that $0 \in T$: This gives $u \geq u_0^*$ and $\partial_{x_n} u > 0$ in $\{x_n > 0\} \cap \overline{\{u > 0\}}$, and arguing symmetrically we deduce $u = u_0^*$, i.e., u is even in x_n , as we wanted.

We start by observing that, if $t \in T$ and $s > t$, then $s \in T$ as well. To show it, it is enough to verify that $u \geq u_s^*$ in $\{x_n \geq s\}$ —we will repeatedly use that u is monotone nondecreasing in the x_n direction in $\{x_n > t\}$ (since there it is monotone when positive, and it is nonnegative):

- If $s \leq x_n \leq 2s - t$, so that $t \leq 2s - x_n \leq s$, we have

$$u_s^*(x', x_n) = u(x', 2s - x_n) \leq u(x', s) \leq u(x', x_n).$$

- If $x_n \geq 2s - t$, so that $x_n - 2(s - t) \geq t$, we have

$$u_s^*(x', x_n) = u(x', 2s - x_n) = u_t^*(x', x_n - 2(s - t)) \leq u(x', x_n - 2(s - t)) \leq u(x', x_n),$$

by monotonicity and the fact that $u \geq u_t^*$ in $\{x_n \geq t\}$.

Hence, T is an interval (a half-line). Notice also that, by definition and continuity of u , T is closed. Moreover, T is nonempty by (a) in Lemma 4.5 and Lemma 4.4. We will show that T is relatively open in $[0, \infty)$, thus concluding that $0 \in T$.

Let $t > 0$, with $t \in T$. By definition, $\partial_{x_n} u \geq 0$ in $\{x_n = t\} \cap \overline{\{u > 0\}}$: indeed, the only nontrivial case to consider is if $x \in \text{FB}(u) \cap \{x_n = t\}$. The fact that $u \geq u_t^*$ in $\{x_n \geq t\}$ applied around x then forces $\partial_{x_n} u(x) = \nu_{\text{FB}(u)}(x) \cdot e_n \geq 0$.

Step 1. We show that $\partial_{x_n} u > 0$ also on $\{x_n = t\} \cap \overline{\{u > 0\}}$.

Assume there exists some $x \in \{x_n = t\} \cap \overline{\{u > 0\}}$ with $\partial_{x_n} u(x) = 0$ instead. Consider $v := u - u_t^*$, so that $v \geq 0$ in $\{x_n \geq t\}$, $v = 0$ on $\{x_n = t\}$, $\partial_{x_n} v(x) = 0$, and $v > 0$ in $\{u_t^* > 0\} \cap \{x_n > t\}$ (i.e., where v is harmonic, since $u \geq u_t^*$ in $\{x_n > t\}$). There are two cases:

- If $u(x) > 0$, Hopf's lemma (applied to v in a small half-ball $D := B_r(x) \cap \{x_n \geq t\} \subset \{u > 0\} \cap \{u_t^* > 0\}$) gives $\partial_{x_n} v(x) > 0$, which contradicts the fact that $\partial_{x_n} v(x) = 0$.
- If $u(x) = 0$ instead, we have $x \in \text{FB}(u) \cap \{x_n = t\}$. Observe that $\nabla u(x) = \nabla u_t^*(x)$ (i.e., $\nabla v(x) = 0$) since $\partial_{x_n} u(x) = 0$. This shows that $\text{FB}(u)$ and $\text{FB}(u_t^*)$ are tangent at x . Put $U^* = \{u_t^* > 0\}$ and $H = \{x_n = t\}$, so that $x \in \partial U^* \cap H$ and $\nu_{\partial U^*}(x) \cdot e_n = 0$.

The same computation⁴ as in [46] shows that $D^2 v$ vanishes at x as well. But then,

⁴This follows by symmetry and because, denoting $\nu(x) = e_1$, the constant Neumann condition and harmonicity of u impose $\partial_{x_1, x_i}^2 u(x) = 0$ for all $i \neq 1$.

the second-order Hopf boundary lemma, Lemma 4.6—applied to v and the domain $U = U^* \cap \{x_n > t\}$ —gives a contradiction.

Step 2. Conclusion.

We split our analysis into several subdomains:

- By Lemma 4.4, $\partial_{x_n} u > 0$ in $\{u > 0\} \cap \{x_n \geq t/3\} \cap \{|x| \geq M\}$.
- By the previous substep, we also know that $\partial_{x_n} u > 0$ in $\{x_n \geq t\} \cap \overline{\{u > 0\}}$. By continuity, there is $\varepsilon \in (0, 2t/3]$ such that $\partial_{x_n} u > 0$ in $\{x_n \geq t - \varepsilon\} \cap \overline{\{u > 0\}} \cap \{|x| \leq M\}$ as well. Together with the first bullet, we have $\partial_{x_n} u > 0$ in $\{x_n \geq t - \varepsilon\} \cap \overline{\{u > 0\}}$. This ensures that $u \geq u_s^*$ in $\{s \leq x_n \leq s + \varepsilon/2\}$ for every $s \in [t - \varepsilon/4, t]$. Note that $s + \varepsilon/2 \geq t + \varepsilon/4$ for this range of s .
- By (b) in Lemma 4.5 we also have $u \geq u_s^*$ in $\{x_n \geq s\} \cap \{|x| \geq M_t\}$, for some $M_t \geq M$.
- Finally, there is $\tilde{\varepsilon} \in (0, \varepsilon)$ such that $u \geq u_s^*$ in $\{x_n \geq s + \varepsilon/2\} \cap \{|x| \leq M_t\}$ for all $s \in [t - \tilde{\varepsilon}/4, t]$:

By assumption, $u \geq u_t^*$ in $\{x_n \geq t\}$. Then, there is $\delta_\varepsilon > 0$ such that $u > u_t^* + \delta_\varepsilon$ in $\overline{\{u_t^* > 0\}} \cap \{x_n \geq t + \varepsilon/4\} \cap \{|x| \leq M_t\}$: if equality were achieved at some x with $u_t^*(x) > 0$, this would contradict the interior maximum principle, and likewise if $x \in \text{FB}(u_t^*)$ (and thus $x \in \text{FB}(u)$ as well) by Hopf's lemma.

Hence, $u > u_s^* + \delta_\varepsilon/2$ in $\overline{\{u_s^* > 0\}} \cap \{x_n \geq t + \varepsilon/4\} \cap \{|x| \leq M_t\}$ for any $s \in [t - \tilde{\varepsilon}/4, t]$ as well, for some $0 < \tilde{\varepsilon} \leq \varepsilon$ small, by continuity and compactness. Since $s + \varepsilon/2 \geq t + \varepsilon/4$ for this range of s , this confirms the claim opening this bullet.

Putting all of the above together, we have shown that there is some $\tilde{\varepsilon} > 0$ such that: For every $s \in [t - \tilde{\varepsilon}/4, t]$, we have $\partial_{x_n} u > 0$ in $\{x_n \geq s\} \cap \overline{\{u > 0\}}$ and $u \geq u_s^*$ in $\{x_n \geq s\}$. Hence $[t - \tilde{\varepsilon}/4, t] \subset T$ as well. Since $t \in T$ with $t > 0$ was arbitrary, this shows that T is relatively open in $[0, \infty)$ and concludes the proof. $\square$

4.3. Non-compact free boundary case: symmetry in x_1 . We assume the same setting as in the previous section.

Lemma 4.7. *In the setting of Corollary 3.8, u is even in x_n and up to a translation in the $x' = (x_1, \dots, x_{n-1})$ coordinates and a further rescaling⁵ we have*

$$|u - \tilde{v}| + |x| |\nabla(u - \tilde{v})| \leq |x|^{1-n-\alpha} \quad \text{in } \{u > 0\} \cap \{x_n > 0\} \setminus B_1,$$

with

$$\tilde{v}(x) := x_n - b + \frac{c}{|x - be_n|^{n-2}},$$

for some $b, c > 0$, $|b| + |c| \leq 1/2$.

Proof. The evenness in x_n follows from Proposition 4.3. Take $b, c = c_+$ and $d = d_+$ from Corollary 3.8, with $c > 0$ and $d_n = (n-2)bc$ by Lemma 3.10 and Corollary 3.8. Set

$$x'_0 := \frac{d'}{(n-2)c}$$

and replace $u(x', x_n)$ by $u(x' + x'_0, x_n)$. Then, using

$$|x + (x'_0, 0)|^{2-n} = |x|^{2-n} - (n-2) \frac{x'_0 \cdot x'}{|x|^n} + O(|x|^{-n}),$$

⁵That is, considering $u(x' + x'_0, x_n)$ for some appropriate $x'_0 \in \mathbb{R}^{n-1}$ in place of u .

together with the analogous differentiated estimate, the resulting horizontal dipole term cancels that of v_+ in Corollary 3.8: Outside of a large ball,

$$|u - (x_n - b + c|x|^{2-n} + d_n x_n |x|^{-n})| \leq C|x|^{1-n-\alpha}.$$

Notice that, since $d_n = (n-2)bc$,

$$c|x - be_n|^{2-n} = c|x|^{2-n} + d_n x_n |x|^{-n} + O(|x|^{-n}).$$

Therefore, also incorporating the differentiated quantities,

$$|u - \tilde{v}| + |x| |\nabla(u - \tilde{v})| \leq C|x|^{1-n-\alpha},$$

in $\{u > 0\} \cap \{x_n > 0\}$ outside of a large ball. Finally, after a harmless rescaling—already allowed in Corollary 3.8—we may assume that the estimate holds in $\mathbb{R}^n \setminus B_1$ with constant 1, and that $|b| + |c| \leq 1/2$. $\square$

We put ourselves in the setting given by this lemma for the remainder of the section. We will now show:

Proposition 4.8. *In the setting of Lemma 4.7, u is even in x_1 . Moreover, $\partial_{x_1} u < 0$ in $\{x_1 > 0\} \cap \overline{\{u > 0\}}$.*

As in the previous section, we start with some precise asymptotic properties.

Lemma 4.9 (Gradient asymptotics). *For any $\delta > 0$, there is $M = M(\delta, n) > 0$ such that:*

$$\partial_{x_1} u(x) < 0 \quad \text{in} \quad \{x_1 \geq \delta/2\} \cap \{|x| \geq M\} \cap \{u > 0\}.$$

Proof. It follows from Lemma 4.7 using

$$\partial_{x_1} \tilde{v}(x) = \partial_{x_1} \left(\frac{c}{|x - be_n|^{n-2}} \right) = -(n-2)c \frac{x_1}{|x - be_n|^n} < 0$$

in $\{x_1 > 0\}$. $\square$

We write $x = (x_1, x'')$, with $x'' = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$. In this splitting we denote by

$$B_r''(y'') := \{z'' \in \mathbb{R}^{n-1} : |z'' - y''| < r\}, \quad B_r'' := B_r''(0),$$

the ball in the x'' -variables. We define

$$\begin{aligned} \sigma_t^\circ(x) &:= (2t - x_1, x'') \quad \text{and} \quad x_t^\circ := \sigma_t^\circ(x), \\ u_t^\circ &= u \circ \sigma_t^\circ \quad \text{i.e.,} \quad u_t^\circ(x_1, x'') := u(2t - x_1, x''). \end{aligned}$$

Lemma 4.10 (Horizontal initialization and reflection asymptotics). *In the setting of Lemma 4.7, we have:*

(a) *There exists $t_0 > 8$ (large) such that for all $t > t_0$ we have:*

$$u \leq u_t^\circ \quad \text{in} \quad \{x_1 \geq t\}$$

(b) *Given $\delta > 0$, there is $R_\delta \geq 0$ such that, if $t \geq \delta$,*

$$u \leq u_t^\circ \quad \text{in} \quad \{x_1 \geq t\} \setminus B_{R_\delta}.$$

Proof. Given $x \in \mathbb{R}^n$ and $s > 0$, we write

$$x_s := x - se_1.$$

Recall that

$$\tilde{v}(x + be_n) := x_n + \frac{c}{|x|^{n-2}}.$$

On the one hand, by the mean value theorem, given $x \in \{x_1 \geq t\}$, $s \in (0, 2x_1)$, for any given T_1 and T_2 with

$$|x_1 - s| \leq T_1 < T_2 \leq x_1$$

there exists $\xi \in (T_1, T_2)$ such that

$$\begin{aligned} \tilde{v}(x_s + be_n) - \tilde{v}(x + be_n) &= c((x_1 - s)^2 + |x''|^2)^{1-n/2} - c((x_1)^2 + |x''|^2)^{1-n/2} \\ &\geq c(T_1^2 + |x''|^2)^{1-n/2} - c((T_2)^2 + |x''|^2)^{1-n/2} \\ &= c(n/2 - 1)2\xi(\xi^2 + |x''|^2)^{-n/2}(T_2 - T_1). \end{aligned}$$

Hence, after a vertical translation (using $b < 1$):

$$\tilde{v}(x_s) - \tilde{v}(x) \geq \bar{c} \frac{T_1(T_2 - T_1)}{(T_2 + |x''| + b)^n} \quad \text{in } \{x_n > 0\} \setminus B_4. \quad (4.5)$$

On the other hand, denoting $[x - se_1, x]$ the segment joining the points $x - se_1$ and x for $s > 0$, the following implication holds:

$$[x_s, x] \subset \{u > 0\} \cap \{x_n > 0\} \setminus B_1 \implies |(u - \tilde{v})(x_s) - (u - \tilde{v})(x)| < |x''|^{-n-\alpha}s. \quad (4.6)$$

Indeed, it follows from the mean value theorem applied to $(u - \tilde{v})$, using the gradient bound in Corollary 3.8.

Step 1. We begin by showing (a).

Let M_1 be given by Lemma 4.9, applied with $\delta = 1$. Assume that $t > t_0$, with $t_0 > M_1 + 1$ large to be chosen. We want to show that $u_t^\circ(x) \geq u(x)$ for all $x \in \{x_1 \geq t\}$. If $x \in \{u = 0\}$ there is nothing to show, so we may assume—taking also into account the symmetry in the x_n direction—that

$$x \in \{u > 0\} \cap \{x_1 \geq t\} \cap \{x_n \geq 0\} = \Omega_+ \cap \{x_1 \geq t\}.$$

We now distinguish four cases:

- If $t < x_1 < 2t - 1$ and $x_t^\circ \in \mathbb{R}^n \setminus B_{t_0}$ we conclude $u(x_t^\circ) - u(x) > 0$ by monotonicity, i.e. Lemma 4.9.
- If $2t - 1 < x_1 \leq |x''|$ (so, in particular, $|x''| > 2t_0 - 1 > M_1$), we claim that $u(x_s) \geq u(x) > 0$ for all $s \in (0, 2(x_1 - t))$. First, if $s \in (0, x_1 - 1)$, then $x_s \in \{y : y_1 > 1\} \setminus B_{M_1}$; by monotonicity (i.e. Lemma 4.9) we find that $u(x_s) \geq u(x) > 0$ again.

But then, from this point on, we can actually continue up to reaching the point x_t° (that is, $s = 2(x_1 - t)$) while keeping this relation. Indeed, (4.5) for $T_1 = \max(x_1/2, |x_1 - s|)$ and $T_2 = x_1$ (using $x_1 - s \leq 1$ and hence $x_1 - |x_1 - s| \geq t$ then $T_2 - T_1 \geq t/2$) tells us:

$$\tilde{v}(x_s) - \tilde{v}(x) \geq \bar{c} \frac{\max(x_1/2, |x_1 - s|)t}{|x|^n} > 0.$$

Then (4.6) gives:

$$u(x_s) - u(x) > \tilde{v}(x_s) - \tilde{v}(x) - |x''|^{-\alpha-n}s \geq \frac{\bar{c}x_1t/4}{|x|^n} - 2x_1|x''|^{-\alpha-n}.$$

For $t \geq t_0$ (large enough), since $|x|$ and $|x''|$ are comparable by the assumption $x_1 \leq |x''|$, the right hand-side above is positive, and we can continue until proving $u(x_t^\circ) - u(x) > 0$.

- If $x_1 \geq \max(2t - 1, |x''|)$ and $x_t^\circ \notin B_{t_0}$: Assume first that additionally $x_t^\circ \in \{u > 0\}$; we will later reduce to this case. Then, by the triangle inequality and using (4.5) with

$$s = 2(x_1 - t), |x_1 - s| = |2t - x_1|, T_1 = \max(|2t - x_1|, |x''| - 1), T_2 = T_1 + 1 \leq x_1:$$

$$\begin{aligned} u(x_t^\circ) &\geq \tilde{v}(x_t^\circ) - \frac{1}{|x_t^\circ|^{n+\alpha-1}} \\ &\geq \tilde{v}(x) + c_1 \frac{\max(|2t - x_1|, |x''| - 1)}{(\max(|2t - x_1| + 1, |x''|) + |x''| + b)^n} - |x_t^\circ|^{1-n-\alpha} \\ &\geq \tilde{v}(x) + c_1 \frac{|x_t^\circ|}{(|2t - x_1| + 1 + |x''| + b)^n} - |x_t^\circ|^{1-n-\alpha} \\ &\geq u(x) + c_2 |x_t^\circ|^{1-n} - |x_t^\circ|^{1-n-\alpha} - |x|^{1-n-\alpha} \\ &\geq u(x) + |x_t^\circ|^{1-n} (c_2 - 2|x_t^\circ|^{-\alpha}) \\ &> u(x) + \frac{c_2}{2} |x_t^\circ|^{1-n} \end{aligned}$$

whenever t_0 is large enough, where we have also used $|x_t^\circ| \leq |x|$. In particular, $u(x_t^\circ) > u(x)$.

It remains to remove the additional assumption that $x_t^\circ \in \{u > 0\}$. Assume, for contradiction, that $x_t^\circ \in \{u = 0\}$ instead; sliding in the e_n direction we find $s > 0$ with $\sigma_t^\circ(x + se_n) \in \text{FB}(u)$ and $(x + se_n) \in \{u > 0\}$. Then, for every $h > 0$ we have $\sigma_t^\circ(x + (s+h)e_n) \in \{u > 0\}$ and $(x + (s+h)e_n) \in \{u > 0\}$, thus we fall into one of the previous cases. In either case, this shows that $u(\sigma_t^\circ(x + (s+h)e_n)) > u(x + (s+h)e_n)$, thus (sending $h \rightarrow 0$) we find $u(\sigma_t^\circ(x + se_n)) \geq u(x + se_n)$. On the other hand $u(x + se_n) > 0$ and $\sigma_t^\circ(x + se_n) \in \text{FB}(u)$, reaching a contradiction.

- Finally, if $x_t^\circ \in B_{M_1}$ then $x \in B_{M_1}(2te_1)$. In this case, using Lemma 3.9, the interior maximum principle, and Hopf's lemma, there is $\eta > 0$ such that

$$u \geq (|x_n| - b + \eta)_+ \quad \text{in } B_{M_1} \iff u_t^\circ \geq (|x_n| - b + \eta)_+ \quad \text{in } B_{M_1}(2te_1).$$

On the other hand, by the expansion in Lemma 4.7 we know that

$$u(x + 2te_1) \rightarrow (|x_n| - b)_+ \quad \text{uniformly in } B_{M_1}, \quad \text{as } t \rightarrow \infty.$$

That is, for t large enough, we have $u < |x_n| - b + \eta$ in $B_{M_1}(2te_1)$, which shows $u_t^\circ \geq u$ in $B_{M_1}(2te_1)$.

Step 2. Let us now show (b). We may assume that $t < t_0$ since otherwise we use case (a). We need to show that, given $\delta \in (0, t_0)$, there is $R_\delta > t_0$ such that, for any $t \in (\delta, t_0)$,

$$u \leq u_t^\circ \quad \text{in } \{x_1 \geq t\} \setminus \mathcal{C}_{R_\delta}, \quad \text{where } \mathcal{C}_{R_\delta} := (t, 3R_\delta) \times B_{R_\delta}''$$

As before, we may assume $x \in \{u > 0\} \cap \{x_1 \geq t\} \cap \{x_n > 0\} \setminus \mathcal{C}_{R_\delta}$. To prove that $u(x_t^\circ) > u(x)$ we argue as follows by separating between two regimes:

- If $x_1 \leq |x''|$, we will show that $u(x_s) > u(x) > 0$ for all $s \in (0, 2(x_1 - t))$. If $s \in (0, x_1 - \delta/2)$ then $x_s \in \{x_1 > \delta/2\}$ and we argue by monotonicity (provided we choose $R_\delta > M_{\delta/2}$). For $s \in (x_1 - \delta/2, 2(x_1 - t))$, instead, we use (4.5) and (4.6).

Arguing as in the second bullet in part (a), we have

$$u(x_s) - u(x) > \tilde{v}(x_s) - \tilde{v}(x) - |x''|^{-\alpha-n}s \geq c \frac{x_1(x_1 - |x_1 - s|) - 2x_1|x''|^{-\alpha}}{|x''|^n}.$$

Since $x_1 - |x_1 - s| > \delta/2$ and $|x''| \geq R_\delta$, for R_δ large enough we get the positive sign on the right-hand side, for all s up to $2(x_1 - t)$.

- If $x_1 > |x''|$, since $t < R_\delta$ and $x \in \mathbb{R}^n \setminus \mathcal{C}_{R_\delta}$, we have $x_t^\circ \in \mathbb{R}^n \setminus B_{R_\delta}$. Thus, if $u(x_t^\circ) > 0$, using (4.5) with $T_1 = \max(|2t - x_1|, |x''| - \delta)$, $T_2 = T_1 + \delta \leq x_1$ as in the third bullet of part (a) we get

$$\begin{aligned} u(x_t^\circ) &\geq \tilde{v}(x) + c_1 \delta \frac{\max(|2t - x_1|, |x''| - \delta)}{(\max(|2t - x_1| + \delta, |x''|) + |x''| + b)^n} - |x_t^\circ|^{1-n-\alpha} \\ &\geq \tilde{v}(x) + c_1 \delta \frac{|x_t^\circ|}{(|2t - x_1| + \delta + |x''| + b)^n} - |x_t^\circ|^{1-n-\alpha} \\ &\geq u(x) + |x_t^\circ|^{1-n}(c_2 - 2|x_t^\circ|^{-\alpha}) > u(x), \end{aligned}$$

since $|x_t^\circ| > R_\delta$, by taking R_δ large enough. The case $u(x_t^\circ) = 0$ can be argued as before. $\square$

Proof of Proposition 4.8. Let $T \subset [0, \infty)$ be defined as

$$T := \{t \in [0, \infty) : u \leq u_t^\circ \text{ in } \{x_1 \geq t\} \text{ and } \partial_{x_1} u < 0 \text{ in } \{x_1 > t\} \cap \overline{\{u > 0\}}\}.$$

We want to show that $0 \in T$: This gives $u \leq u_0^\circ$ and $\partial_{x_1} u < 0$ in $\{x_1 > 0\} \cap \overline{\{u > 0\}}$, and arguing symmetrically we deduce $u = u_0^\circ$, i.e., u is even in x_1 , as we wanted.

The proof proceeds now exactly as that of Proposition 4.3, using Lemmas 4.9 and 4.10 in place of Lemmas 4.4 and 4.5. $\square$

4.4. Proof of Theorems 1.1 and 1.2. We can now prove the main theorems, by putting together the results in the article.

Proof of Theorem 1.2. By Proposition 3.1, either $\text{FB}(u)$ is bounded or there is some $R_0 = R_0(u)$ such that $\text{FB}(u) \cap \partial B_R \neq \emptyset$ for every $R \geq R_0$. In the first case, Proposition 4.1 gives the result. In the second one, we can put ourselves in the setting of Proposition 3.1 and Corollary 3.8. Then, Proposition 4.3 shows that u is even in x_n and $\partial_{x_n} u > 0$ in $\{x_n > 0\} \cap \{u > 0\}$. By Lemma 4.7, performing a translation in the $x' = (x_1, \dots, x_{n-1})$ coordinates, and up to a further rescaling, it additionally satisfies the asymptotics indicated there.

To see that u is radially symmetric in $x_1, \dots, x_{n-1}$, i.e. that $u(x^1) = u(x^2)$ if $x_n^1 = x_n^2$, $|(x^1)'| = |(x^2)'|$ and $x^1 \neq x^2$, let $e = \frac{x^2 - x^1}{|x^2 - x^1|} \in \text{span}(e_1, \dots, e_{n-1}) \cap \mathbb{S}^{n-1}$. The reflection with respect to $H_e = \{e \cdot y = 0\}$ sends x^1 to x^2 , thus it suffices to see that u is even with respect to H_e . Let $\mathcal{R} : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ denote a rotation in the x' coordinates which sends e_1 to e . The new function $\tilde{u}(x) = u(\mathcal{R}x', x_n)$ is still a solution to the Bernoulli problem, even in x_n , satisfying the asymptotics in Lemma 4.7. Then, Proposition 4.8 shows that $\tilde{u}$ is even with respect to $H_1 = \{x_1 = 0\}$, which means that u is even with respect to H_e . It moreover shows that $\partial_{x_1} \tilde{u} < 0$ in $\{x_1 > 0\} \cap \{u > 0\}$, from which the negativity condition on $\partial_r u$ in Theorem 1.2 is seen to follow. Finally, combining the symmetries and monotonicity relations for u with its asymptotic expansion in Lemma 4.7, the graphical description for $\{u = 0\}$ —including the explicit asymptotics in Remark 1.4—follows. $\square$

The three-dimensional case follows unconditionally:

Proof of Theorem 1.1. The result follows from Theorem 1.2 since $n = 3$ is admissible there. $\square$

5. TOTAL MEAN CURVATURE

Definition 5.1. Let $\Sigma^{n-1} \subset \mathbb{R}^n$ be an embedded hypersurface. We define the *total mean curvature* of Σ in an open subset $U \subset \mathbb{R}^n$ as

$$\mathbb{H}(\Sigma, U) := \int_{\Sigma \cap U} |H_\Sigma|^{n-1} d\mathcal{H}^{n-1}.$$

We say that Σ has *finite total mean curvature* if $\mathbb{H}(\Sigma, \mathbb{R}^n) < \infty$.

Remark 5.2. Mean curvature plays a special role in the theory of the Bernoulli problem: Global solutions have mean-convex free boundaries (see, e.g., [7, Lemma 3.2]), and mean curvature is a main component in the stability inequality (1.2). In fact, finite total (mean) curvature in two dimensions is precisely the condition used in [4] to show that free boundaries of finite index solutions in $\mathbb{R}^2$ are finite unions of curves, reducing the classification result there to [47].

The goal of this section is to show that finite total mean curvature alone is a sufficient condition to obtain axial symmetry in all dimensions⁶.

Theorem 5.3. *Let $n \geq 3$, and let $u : \mathbb{R}^n \rightarrow [0, \infty)$ be a classical solution to the Bernoulli problem which is not one-dimensional. If $\text{FB}(u)$ has finite total mean curvature, then the exact same conclusions as in Theorem 1.2 follow.*

The properties of $\mathbb{H}$ that we will need are precisely the following:

Lemma 5.4. *The total mean curvature satisfies the following properties.*

- **Scaling invariance:** $\mathbb{H}(\lambda\Sigma, \lambda U) = \mathbb{H}(\Sigma, U)$ for any $\lambda > 0$ and $U \subset \mathbb{R}^n$.
- **Monotonicity:** $\mathbb{H}(\Sigma, U) \leq \mathbb{H}(\Sigma, V)$ if $U \subset V \subset \mathbb{R}^n$.
- **Definiteness for the Bernoulli problem:** Let U be a connected domain, and let $u : U \rightarrow [0, \infty)$ be a classical solution to the Bernoulli problem, with $\text{FB}(u) \cap U \neq \emptyset$ and $|\nabla u| \leq 1$. If $\mathbb{H}(\text{FB}(u), U) = 0$, then $|D^2u| \equiv 0$ in $\{u > 0\} \cap U$.

Proof. The first two properties are a standard consequence of the definition. The definiteness follows from [7, Lemma 3.12] and its proof, since it shows that $H_{\text{FB}(u)}$ can vanish at some $x_0 \in \text{FB}(u) \cap U$ only if there are some $e \in \mathbb{S}^{n-1}$ and $r > 0$ such that $u \equiv ((x - x_0) \cdot e)_+$ in $B_r(x_0)$, and the result follows by unique continuation. $\square$

We now obtain a local regularity result under smallness of the total mean curvature, in the spirit of [2].

Lemma 5.5. *Let $u : B_1 \rightarrow [0, \infty)$ be a classical solution to the Bernoulli problem with $|\nabla u| \leq 1$ and $0 \in \text{FB}(u)$. Given $\eta > 0$, there exists $\delta_0 > 0$ such that, if $\mathbb{H}(\text{FB}(u), B_1) \leq \delta_0$, then $|D^2u| \leq \eta$ in $\{u > 0\} \cap B_{1/2}$.*

We first need the following lemma, which says that the regularity of the free boundary alone implies the regularity of u .

Lemma 5.6. *Let $u : B_1 \rightarrow [0, \infty)$ be a classical solution. If $|\Pi_{\text{FB}(u)}| \leq C_0$ in B_1 , then $|D^2u| + |D^3u| \leq C$ in $\{u > 0\} \cap B_{1/2}$, for some $C = C(n, C_0)$.*

Proof. By a scaling, translation, a finite covering argument, and interior harmonic estimates, it suffices to find some $r = r(C_0, n)$ such that $|D^2u| + |D^3u| \leq C$ in $U_r \cap B_{r/2}$, where U_r is the connected component of $\{u > 0\} \cap B_r$ with $0 \in \overline{U_r}$.

Now, for $r \ll 1$ small enough we can ensure that $\text{FB}(u) \cap \overline{U_r}$ is a union of at most two “parallel” graphs in B_r . Then, density estimates for U_r (see [7, Lemma 3.5]) ensure that such graphs are at a uniform distance, so that up to making r smaller (depending only on C_0 and n) we have that $\text{FB}(u) \cap B_r$ consists of only one graphical component. Finally, a classical solution in B_r whose free boundary is a single C^2 graph enjoys uniform C^3 estimates in $\{u > 0\} \cap B_{r/2}$, by e.g. [36]. $\square$

⁶In this way, total mean curvature seems to play a similar role to total curvature $\mathbb{K}(\Sigma, U) := \int_{\Sigma \cap U} |\Pi_\Sigma|^{n-1} d\mathcal{H}^{n-1}$ in minimal surface theory. Finite total curvature is known to imply finite index for global minimal hypersurfaces, as well as a strong regularity property for their ends, see [2, 8].

⁷The condition $|\nabla u| \leq 1$ is always true if u is a restriction of a global classical solution; see [7, Lemma 3.2].

Proof of Lemma 5.5. Step 1. We first show that $|\Pi_{\text{FB}(u)}| \leq \eta$ in $B_{1/2}$ as long as $\delta_0 > 0$ is small enough.

We argue in the spirit of [2]. Assume for contradiction that for some $\eta > 0$ this were false. Then there would be u_i as in the statement, with $\mathbb{H}(\text{FB}(u_i), B_1) \xrightarrow{i \rightarrow \infty} 0$, but with $|\Pi_{\text{FB}(u_i)}(x_i)| > \eta$ for some $x_i \in B_{3/4}$. In particular

$$\max_{y \in B_{4/5}} \text{dist}(y, \partial B_{4/5}) |\Pi_{\text{FB}(u_i)}| \geq \frac{1}{20} \eta,$$

and we can find a sequence y_i with

$$\max_{y \in B_{4/5}} \text{dist}(y, \partial B_{4/5}) |\Pi_{\text{FB}(u_i)}|(y) = \text{dist}(y_i, \partial B_{4/5}) |\Pi_{\text{FB}(u_i)}|(y_i) =: r_i \geq \frac{1}{20} \eta.$$

Put $\lambda_i := |\Pi_{\text{FB}(u_i)}|(y_i)$, and consider $\tilde{u}_i = \lambda_i u_i (\lambda_i^{-1} x + y_i) : B_{r_i/2} \rightarrow [0, \infty)$, which satisfies now $|\Pi_{\text{FB}(\tilde{u}_i)}| \leq 2$ and $|\Pi_{\text{FB}(\tilde{u}_i)}|(0) = 1$. By Lemma 5.6 and Arzelà–Ascoli, there is $r \in (0, \frac{1}{40} \eta)$ such that: A subsequence converges to a limit $u_\infty : B_r \rightarrow [0, \infty)$, and there is a connected component U of $\{u_\infty > 0\} \cap B_r$ such that $v := u_\infty \mathbf{1}_U$ is a classical solution, $0 \in \text{FB}(v)$, and $|\Pi_{\text{FB}(v)}|(0) = 1$.

On the other hand, by the first two bullets in Lemma 5.4, $\mathbb{H}(\text{FB}(v), B_r) = 0$. By the third bullet there, then $|\Pi_{\text{FB}(v)}|(0) = 0$, which yields a contradiction.

Step 2. Conclusion.

By Step 1, Lemma 5.6 and a simple covering argument, for $\delta_0 > 0$ small enough we have $|D^2 u| + |D^3 u| \leq C$ in $\{u > 0\} \cap B_{3/4}$ for some $C = C(n)$. The claim that—up to making $\delta_0 > 0$ smaller—we have $|D^2 u| \leq \eta$ as well follows: Otherwise, the C^3 estimates and Arzelà–Ascoli allow us to pass to a solution with flat free boundary but nonzero Hessian, and the third bullet in Lemma 5.4 yields a contradiction again. $\square$

We then obtain small asymptotic curvatures (with the correct scaling rate) for global solutions with finite total mean curvature:

Lemma 5.7. *Let $n \geq 3$, and let u be a classical solution to the Bernoulli problem in $\mathbb{R}^n$ with $\mathbb{H}(\text{FB}(u), \mathbb{R}^n) < \infty$. Let $\eta > 0$. Then, there is $R_0 = R_0(u, \eta)$ such that $|D^2 u| \leq \frac{\eta}{|x|}$ in $\{u > 0\} \cap \mathbb{R}^n \setminus B_{R_0}$.*

Proof. Given $\delta_0 > 0$, there is $R_0 = R_0(u, \delta_0)$ such that $\mathbb{H}(\text{FB}(u), \mathbb{R}^n \setminus B_{R_0}) \leq \delta_0$, just since $\mathbb{H}(\text{FB}(u), \mathbb{R}^n) < \infty$. Moreover, $|\nabla u| \leq 1$ by [7, Lemma 3.2]. Applying Lemma 5.5—appropriately rescaled, on balls $B_{|x|/2}(x)$ for $x \in \mathbb{R}^n \setminus B_{2R_0}$ —and making R_0 larger gives the result. $\square$

We can then find the exact analogue of Lemma 3.7:

Lemma 5.8 (Compactness toward a two-plane configuration). *For every $\varepsilon \in (0, 1)$ there exists $\delta = \delta(n, \varepsilon) > 0$ such that the following holds.*

Let $n \geq 3$, and let u be a classical solution in $\mathbb{R}^n$. Assume that

$$[W(u, 2) - W(u, 1)] + \int_{\text{FB}(u) \cap (\mathbb{R}^n \setminus B_\delta)} |\mathbb{H}_{\text{FB}(u)}|^{n-1} \leq \delta$$

and

$$\text{FB}(u) \cap (B_2 \setminus \overline{B_1}) \neq \emptyset.$$

Then there exists a unit vector $e \in \mathbb{S}^{n-1}$ such that either

$$\|u - (e \cdot x)_+\|_{L^\infty(B_{16})} \leq \varepsilon \quad \text{or} \quad \|u - |e \cdot x|\|_{L^\infty(B_{16})} \leq \varepsilon. \quad (5.1)$$

Moreover, in the second case, letting $\mathcal{R}_e : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a rotation sending e to e_n , we have:

$$\mathcal{R}_e(\{u > 0\}) = \{x_n > \tilde{g}^{(+)}(x')\} \cup \{x_n < \tilde{g}^{(-)}(x')\} \quad \text{in } B_8 \setminus \overline{B_{1/2}},$$

where $\tilde{g}^{(\pm)} : B'_8 \rightarrow \mathbb{R}$, $\tilde{g}^{(-)} < \tilde{g}^{(+)}$, and there is $C = C(n) > 0$ such that

$$\|\tilde{g}^{(\pm)}\|_{L^\infty(B'_8)} + C^{-1}\|D^2\tilde{g}^{(\pm)}\|_{L^\infty(B'_8)} \leq \varepsilon \leq 1$$

for some C depending only on n .

In particular,

$$\text{FB}(u) \cap \partial B_r \neq \emptyset \quad \text{for every } r \in (1, 4).$$

Proof. The proof starts exactly as that of Lemma 3.7. The Hessian and nondegeneracy estimates (3.5)–(3.8) follow from Lemma 5.7 and a Harnack chain argument via interior harmonic estimates.

To see (5.1) we argue by contradiction, considering $\delta_i = \frac{1}{i}$ and solutions u_i contradicting the thesis, as in the proof of Lemma 3.7. The corresponding subsequential limit is again a one-homogeneous solution. Moreover, by the vanishing curvature estimates from Lemma 5.7—recall that we can make η arbitrarily small there up to making δ_0 smaller—we find that $|\text{II}_{\text{FB}(u_i)}| \rightarrow 0$ in C^0_{loc} . Then, by $C^2_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ convergence of the free boundaries (given by the C^3 estimates from Lemma 5.6 plus Arzelà–Ascoli), we conclude that $\text{FB}(u_\infty) = \{e \cdot x = 0\}$ for some $e \in \mathbb{S}^{n-1}$ —without ever requiring stability. If there is a tangential collapse of two sheets at some $x \in \text{FB}(u_\infty) \setminus \{0\}$, we find again that $u_\infty = |e \cdot x|$. Otherwise, u_∞ is a classical solution and thus of the form $u_\infty = (e \cdot x)_+$. In both cases we reach a contradiction, as eventually (5.1) is satisfied by the u_i .

The graphicality follows by Lemma 3.6, just as in Lemma 3.7, concluding the proof. $\square$

Finally, we can give:

Proof of Theorem 5.3. With Lemma 5.8 at hand, we can follow almost verbatim the arguments in the rest of the article. First, exactly as in Proposition 3.1 it follows that either $\text{FB}(u)$ is bounded, or there is some $R_0 = R_0(u)$ such that $\text{FB}(u) \cap \partial B_R \neq \emptyset$ for every $R \geq R_0$. In the first case, Proposition 4.1 gives radial symmetry. In the second one, we obtain the decomposition $u = u_+ + u_-$, away from a ball, where u and the $u_\pm$ satisfy the asymptotics in Proposition 3.1 and Corollary 3.8: the same proofs go through, with the finite index condition replaced by having finite total mean curvature, since they only use the properties from Lemma 3.7 (which we can replace by Lemma 5.8 in our case). Sections 4.2 and 4.3 can then be replicated exactly, as they exclusively use the asymptotics in Corollary 3.8 plus the fact that u is a global classical solution. We then conclude by putting everything together as in the proof of Theorem 1.2. $\square$

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