ENTROPY GEOMETRY AND DISJOINTNESS FOR ZERO-DIMENSIONAL ALGEBRAIC ACTIONS

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Abstract. We show that many algebraic actions of higher-rank abelian groups on zero-dimensional compact abelian groups are mutually disjoint. The proofs exploit differences in the entropy geometry arising from subdynamics and a form of Abramov–Rokhlin formula for half-space entropies.

We discuss some mutual disjointness properties of algebraic actions of higher-rank abelian groups on zero-dimensional compact abelian groups. The tools used are a version of the half-space entropies introduced by Kitchens and Schmidt [14] and adapted by Einsiedler [7], a basic geometric entropy formula from [7], and the structure of expansive subdynamics for algebraic $\mathbb{Z}^d$-actions due to Einsiedler, Lind, Miles and Ward [9]. We show that any collection of algebraic $\mathbb{Z}^d$-actions on zero-dimensional groups with entropy rank or co-rank one that look sufficiently different are mutually disjoint. The main results are the following (here $N(\cdot)$ denotes the set of non-expansive directions; non-expansive directions and mutual disjointness are defined in Section 1.)

Theorem 5.1. *Let $\mathsf{X}_1,\dots,\mathsf{X}_n$ be a collection of irreducible algebraic zero-dimensional $\mathbb{Z}^d$-actions, all with entropy rank one. If*

$$N(\alpha_j)\setminus\bigcup_{k>j}N(\alpha_k)\neq\emptyset \text{ for } j=1,\dots,n$$

then the systems are mutually disjoint.

The simplest illustration of Theorem 5.1 is the fact that Ledrappier's Example 2.3 and its mirror image are disjoint. This is shown directly in Section 3 to illustrate how the Abramov–Rokhlin formula for half-space entropies may be used.

Theorem 6.2. *Let Y and Z be prime $\mathbb{Z}^d$-actions with entropy co-rank one. If $N(\alpha_Y)\neq N(\alpha_Z)$, then Y and Z are disjoint.*

2000 *Mathematics Subject Classification.* 22D40, 37A15, 52B11.

The first author was supported by the Erwin Schrödinger Stipendium J2090 and the Center for Dynamical Systems at Penn State University.

Once again the simplest illustration of the meaning of this result comes from an example of Ledrappier type: Example 6.3 is a three-dimensional analogue of Ledrappier's example. This is a $\mathbb{Z}^3$-action defined by a 'four-dot' condition which has positive entropy $\mathbb{Z}^2$-subactions; it and its mirror image are disjoint.

Surprisingly, it is not the familiar presence of different non-mixing sets but the entropy and subdynamical geometry of the systems that forces this high level of measurable difference of structure. The methods should extend to entropy rank or co-rank greater than one, but the notational and technical difficulties become more substantial. Related work for $\mathbb{Z}^d$-actions by toral automorphisms has been done by Kalinin and Katok [11], where more refined information is found about joinings and the consequences of the presence of non-trivial joinings. Actions by toral automorphisms automatically have entropy rank not exceeding one.

Our purpose here is to begin to address some of the problems inherent in understanding the joinings between algebraic $\mathbb{Z}^d$-actions. The ultimate goal is to extend results like those of [11] to general algebraic actions, just as the rigidity results have been extended from the toral case in [13], to irreducible actions in [15]. In the rigidity theory, entropy rank one also has a privileged position (see [3], [4] for the details of how entropy rank influences rigidity).

A $\mathbb{Z}^d$-action is called irreducible if it has no closed invariant infinite proper subgroups. Irreducible actions on connected and zero-dimensional groups are extensively studied because they exhibit rigidity for $d \geq 2$ (cf. [11], [12], [15]). The class of actions with entropy rank one is a natural extension of the class of irreducible actions (see [8]).

Irreducible actions on zero-dimensional groups are a natural analogue of irreducible actions on finite-dimensional tori and solenoids, see [8]. In particular, both types of action allow a local description using locally compact fields. While $\mathbb{R}, \mathbb{C}$ and finite extensions of $\mathbb{Q}_p$ are used for the toral and solenoidal cases, for irreducible actions on zero-dimensional groups locally compact fields of positive characteristic are used, namely fields of Laurent series in one variable over a finite field (see [6] and [8] for how this works). Using the local isometry to a product of local fields, one can define Lyapunov exponents and foliations of the spaces just as for the toral case. For our purpose it is simpler to use half-space entropies instead of ultrametric Lyapunov exponents. Half-space entropies were introduced in [14] and adapted to be defined via state partitions in [7]. The notion of entropy geometry for actions of higher-rank groups was introduced by Milnor in [17] in the setting of cellular automata.

A special case showing how the entropy geometry gives insight into joinings is dealt with in Section 3, and this can be read independently of the rest of the paper (up to accepting some plausible results on entropy geometry proved elsewhere).

1. Introduction

An algebraic $\mathbb{Z}^d$-action is an action of $\mathbb{Z}^d$ generated by d commuting automorphisms of a compact abelian metrizable group X. Duality (in the sense of Pontryagin) gives a one–to–one correspondence between countable modules $M, N, \dots$ over the ring $R_d = \mathbb{Z}[u_1^{\pm1}, \dots, u_d^{\pm1}]$ and algebraic $\mathbb{Z}^d$-actions $\mathsf{X}_M = (X_M, \alpha_M), \mathsf{X}_N, \dots$ (see [19] for an overview of how this correspondence has been used to study algebraic dynamical systems). It is convenient to write monomials (units) in R_d in the form $\mathbf{u}^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$.

An algebraic dynamical system $\mathsf{X} = (X, \alpha)$ automatically preserves the Haar measure $\lambda = \lambda_X$ on X; we reserve λ for Haar measures and μ for any α-invariant probability measure.

The results on expansive subdynamics we need come from [9]: If α is a $\mathbb{Z}^d$-action by homeomorphisms of a compact metric space (X, ρ), then $N(\alpha)$ denotes the set of non-expansive vectors $\mathbf{v} \in \mathbb{R}^d \backslash \{0\}$. That is, $\mathbf{v} \in N(\alpha)$ if and only if for every $\epsilon > 0$ there exists a pair of points $x \neq y$ in X with the property that

$$\rho\left(\alpha^{\mathbf{n}} x, \alpha^{\mathbf{n}} y\right) \leq \epsilon \text{ for all } \mathbf{n} \in \{\mathbf{m} \in \mathbb{Z}^d \mid \mathbf{v} \cdot \mathbf{m} < 0\}.$$

The whole action is called expansive if there is an $\epsilon > 0$ with the property that

$$\rho\left(\alpha^{\mathbf{n}} x, \alpha^{\mathbf{n}} y\right) \leq \epsilon \text{ for all } \mathbf{n} \in \mathbb{Z}^d \implies x = y.$$

Let α be an expansive algebraic $\mathbb{Z}^d$-action on a zero-dimensional group X. By [7], Lemma 7.1, such an action is automatically an algebraic Markov shift in the following sense: There are integers q and s and a module of relations $J \subset (R_d/(q))^s$ such that

$$X \cong J^{\perp} \subset \left((\mathbb{Z}/q\mathbb{Z})^s\right)^{\mathbb{Z}^d}, \tag{1}$$

where $\cong$ denotes an algebraic isomorphism of $\mathbb{Z}^d$-actions and $J^{\perp}$ denotes the annihilator of the submodule J in the dual group $\left((\mathbb{Z}/q\mathbb{Z})^s\right)^{\mathbb{Z}^d}$ of the R_d-module $(R_d/(q))^s$. Under the isomorphism in (1), the $\mathbb{Z}^d$-action on X corresponds to the natural shift action on $J^{\perp}$. Having chosen such a presentation of the system, there is an associated (non-canonical) state partition $\xi = \xi(q, s, J)$ comprising the q^s cylinder sets obtained by specifying the $\mathbf{0}$ coordinate (some of these sets may be empty).

Given a $\mathbb{Z}^d$-action α by measure-preserving transformations on (X, μ) and any measurable partition η of X, write

$$\eta^A = \bigvee_{\mathbf{n} \in A \cap \mathbb{Z}^d} \alpha^{-\mathbf{n}} \eta$$

for the join of η over any set $A \subset \mathbb{R}^d$. The conditional entropy of A given B with respect to η and μ is defined to be $H_\mu(\eta^A | \eta^B)$. For a fixed η (for instance the state partition for a fixed presentation), we simply write $H_\mu(A|B)$ for this conditional entropy.

The following terminology comes from [5] and (in this context) [9], and the resulting condition for vanishing entropy, which holds for any invariant measure μ, is the first key observation in our work. In the system $\mathsf{X}_M = (X_M, \alpha_M)$, a set $A \subset \mathbb{R}^d$ codes $B \subset \mathbb{R}^d$ if for every $\mathbf{m} \in B \cap \mathbb{Z}^d$ there exists a polynomial

$$f(\mathbf{u}) = \sum_{\mathbf{n} \in A \cap \mathbb{Z}^d} f_{\mathbf{n}} \mathbf{u}^{\mathbf{n}}$$

such that $(\mathbf{u}^{\mathbf{m}} - f)M = 0_M$. Viewing X_M in the form (1), this means that knowledge of the coordinates $(x_{\mathbf{m}})_{\mathbf{m} \in A}$ of a point $x \in X_M$ determines uniquely the coordinates $(x_{\mathbf{m}})_{\mathbf{m} \in B}$. Notice that

- A codes $B \implies H_\mu(A|B) = 0$;
- A codes $B \implies A + \mathbf{n}$ codes $B + \mathbf{n}$ for every $\mathbf{n} \in \mathbb{Z}^d$;
- A codes $B, A \cup B$ codes $C \implies A$ codes $B \cup C$.

A joining of a finite collection of $\mathbb{Z}^d$-actions

$$\mathsf{X}_i = (X_i, \mu_i, \alpha_i), \quad 1 \leq i \leq n,$$

is a measure μ on $X_1 \times \cdots \times X_n$ invariant under $\alpha_1 \times \cdots \times \alpha_n$ and with the property that the projection of μ onto the ith coordinate is μ_i for each i. Write $J(\mathsf{X}_1, \ldots, \mathsf{X}_n)$ for the collection of all joinings of $\mathsf{X}_1, \ldots, \mathsf{X}_n$. The systems are called mutually disjoint if the only joining is the product measure, so $J(\mathsf{X}_1, \ldots, \mathsf{X}_n) = \{\mu_1 \times \cdots \times \mu_n\}$. For $n = 2$ this property is simply called disjointness.

The major simplifying assumption we make is to restrict the entropy rank: α has entropy rank one if there exists a cyclic subgroup of $\mathbb{Z}^d$ with positive entropy (viewed as a $\mathbb{Z}$-action) but all rank two subgroups of $\mathbb{Z}^d$ act with zero entropy. Similarly, α has entropy rank $k < d$ if there is a rank k subgroup of $\mathbb{Z}^d$ acting with positive entropy (when viewed as a $\mathbb{Z}^k$-action) but all subgroups of rank $(k + 1)$ act with zero entropy; finally α has entropy rank d if it has positive entropy as a $\mathbb{Z}^d$-action. Similarly, α has entropy co-rank k if it has entropy rank $(d - k)$. Entropy rank in this context comes from [9], Sect. 7, and the special properties of rank one systems are studied in [8] and [10].

2. Entropy geometry for $d = 2$

The results from [7] summarized and extended in this section require the entropy co-rank to be one. On the other hand, many technical simplifications are possible when the entropy rank is one. In order to have both conditions, $d = 2$ in this section. We will see in Section 4 that this does not restrict the applications to rigidity for larger values of d. By [9], for such actions every element of the non-expansive set is a scalar multiple of an integer vector. This is illustrated in Example 2.3 below, where the non-expansive set is described explicitly for an example.

Definition 2.1. Let μ be an invariant measure on the zero-dimensional expansive algebraic system $\mathsf{X} = (X, \alpha)$ presented as in (1). Let $\mathbf{v} \in \mathbb{Z}^2 \backslash \{0\}$ be a vector with associated half-space $\mathsf{H}_\mathbf{v} = \{\mathbf{n} \in \mathbb{Z}^2 \mid \mathbf{v} \cdot \mathbf{n} < 0\}$. The half-space entropy of $\mathbf{v}$ is

$$\mathsf{h}_\mu(\mathbf{v}) = H_\mu(\xi^{\mathbf{v}^\perp} | \xi^{\mathsf{H}_\mathbf{v}}) \tag{2}$$

where ξ is the state partition (for a fixed presentation) and

$$\mathbf{v}^\perp = \{\mathbf{t} \in \mathbb{Z}^2 \mid \mathbf{v} \cdot \mathbf{t} = 0\}.$$

If $\mathcal{C}$ is an α-invariant σ-algebra, then similarly define the conditional half-space entropy of $\mathbf{v}$ to be

$$\mathsf{h}_\mu(\mathbf{v}|\mathcal{C}) = H_\mu(\xi^{\mathbf{v}^\perp} | \xi^{\mathsf{H}_\mathbf{v}} \vee \mathcal{C}).$$

For a vector $\mathbf{v} \in \mathbb{Z}^2 \backslash \{0\}$, let $\mathbf{v}^*$ be a primitive vector in $\mathbb{Z}^2$ chosen so that

$$\mathsf{H}_\mathbf{v} + \mathbf{v}^* = \mathsf{H}_\mathbf{v} \cup \mathbf{v}^\perp = \{\mathbf{n} \in \mathbb{Z}^2 \mid \mathbf{v} \cdot \mathbf{n} \leq 0\}$$

and let $\ell(\mathbf{v}, r)$ be chosen so that

$$\mathbf{v}^\perp + (-\ell(\mathbf{v}, r), \ell(\mathbf{v}, r))\mathbf{v}^* \supseteq \mathbf{v}^\perp + B(r),$$

where $B(r)$ denotes the closed Euclidean ball of radius r in $\mathbb{R}^2$ centered at the origin.

The half-space entropy from [7] defined by (2) differs from the entropies used in [14] in that it depends *a priori* on the choice of presentation (1) and only turns out after the event to be invariant under algebraic isomorphism. The more robust half-space entropies in [14] are automatically invariant under measurable isomorphism (under suitable hypotheses rigidity makes measurable and algebraic isomorphism coincide). For Haar measure the two entropies coincide.

Lemma 2.2. *Let $\mathsf{X} = (X, \alpha)$ be a zero-dimensional expansive algebraic $\mathbb{Z}^d$-action. The half-space entropy function $\mathsf{h}_\mu : \mathbb{Z}^2 \backslash \{0\} \to \mathbb{R}_{\geq 0}$*

is independent of the choice of the parameters q, s and the module of relations J in the presentation

$$X \cong J^{\perp} \subset \left((\mathbb{Z}/q\mathbb{Z})^{s}\right)^{\mathbb{Z}^d}$$

of the system X.

Proof. Let (X, α) be an expansive zero-dimensional $\mathbb{Z}^2$-action and assume that

$$X \cong J^{\perp} \subset \left((\mathbb{Z}/q\mathbb{Z})^{s}\right)^{\mathbb{Z}^2}$$

and

$$X \cong I^{\perp} \subset \left((\mathbb{Z}/r\mathbb{Z})^{t}\right)^{\mathbb{Z}^2}$$

are two presentations of the system giving corresponding state partitions ξ and η with corresponding half-space entropy functions h_{μ}^{ξ} and h_{μ}^{η}. This means that there is an R_2-module isomorphism between R_2^s/J and R_2^t/I. Dual to this isomorphism of R_2-modules there is a continuous isomorphism of compact groups from $I^{\perp}$ to $J^{\perp}$: It follows that there exists an $r > 0$ with the property that

$$\xi^{B(r)} \supseteq \eta \text{ and } \eta^{B(r)} \supseteq \xi.$$

Standard properties of entropy and the inclusions

$$\xi^{\mathbf{v}^{\perp}} \subset \eta^{\mathbf{v}^{\perp}+B(r)} \text{ and } \xi^{\mathsf{H}_{\mathbf{v}}} \supset \eta^{\mathsf{H}_{\mathbf{v}}-\ell(\mathbf{v},r)\mathbf{v}^*}$$

imply that

$$H_{\mu}\left(\xi^{\mathbf{v}^{\perp}} \middle| \xi^{\mathsf{H}_{\mathbf{v}}}\right) \leq H_{\mu}\left(\eta^{\mathbf{v}^{\perp}+B(r)} \middle| \eta^{\mathsf{H}_{\mathbf{v}}-\ell(\mathbf{v},r)\mathbf{v}^*}\right).$$

To obtain a sharper statement, notice that the invariance of the measure implies (or use [7], Prop. 8.3, for $d = 2$)

$$\begin{aligned}
H_{\mu}\left(\xi^{\mathbf{v}^{\perp}} \middle| \xi^{\mathsf{H}_{\mathbf{v}}}\right) &= \frac{1}{N} H_{\mu}\left(\xi^{\mathbf{v}^{\perp}+[0,N)\mathbf{v}^*} \middle| \xi^{\mathsf{H}_{\mathbf{v}}}\right) \\
&\leq \frac{1}{N} H_{\mu}\left(\eta^{\mathbf{v}^{\perp}+(-\ell(\mathbf{v},r),N+\ell(\mathbf{v},r))\mathbf{v}^*} \middle| \eta^{\mathsf{H}_{\mathbf{v}}-\ell(\mathbf{v},r)\mathbf{v}^*}\right) \\
&\leq \frac{N+2\ell(\mathbf{v},r)}{N} H_{\mu}\left(\eta^{\mathbf{v}^{\perp}} \middle| \eta^{\mathsf{H}_{\mathbf{v}}}\right).
\end{aligned}$$

It follows that

$$\mathsf{h}_{\mu}^{\xi}(\mathbf{v}) = H_{\mu}\left(\xi^{\mathbf{v}^{\perp}} \middle| \xi^{\mathsf{H}_{\mathbf{v}}}\right) \leq H_{\mu}\left(\eta^{\mathbf{v}^{\perp}} \middle| \eta^{\mathsf{H}_{\mathbf{v}}}\right) = \mathsf{h}_{\mu}^{\eta}(\mathbf{v}),$$

so by symmetry $\mathsf{h}_{\mu}^{\xi}(\mathbf{v}) = \mathsf{h}_{\mu}^{\eta}(\mathbf{v})$. □

A similar argument shows that the half-space entropy remains well-defined when conditioned on an invariant σ-algebra: If $\mathcal{C}$ is a σ-algebra in $J^{\perp}$ (in the notation of the proof of Lemma 2.2) with $\mathcal{C}'$ its image under the isomorphism, then

$$\mathsf{h}_{\mu}^{\xi}(\mathbf{v}|\mathcal{C}) = \mathsf{h}_{\mu}^{\eta}(\mathbf{v}|\mathcal{C}'). \tag{3}$$

Example 2.3. The archetypal example of a zero-dimensional system with entropy rank one is due to Ledrappier [16]: Let

$$X_1 = \left\{x \in \mathbb{F}_2^{\mathbb{Z}^2} \mid x_{\mathbf{n}} + x_{\mathbf{n}+\mathbf{e}_1} + x_{\mathbf{n}+\mathbf{e}_2} = 0 \text{ for all } \mathbf{n} \in \mathbb{Z}^2\right\},$$

with α_1 the $\mathbb{Z}^2$-action defined by the natural shift action, and $\lambda = \lambda_{X_1}$ the Haar measure. Then (cf. [9], Ex. 5.6) $\mathbf{v} \in N(\alpha_1)$ if and only if $\mathbf{v}$ is parallel to an outward normal of the convex hull of the set

$$L = \{(0,0),(0,1),(1,0)\}.$$

Similarly, the half-space entropy $\mathsf{h}_{\lambda}(\mathbf{v})$ is positive if and only if $\mathbf{v}$ is parallel to an outward normal of the convex hull of the set L.

For a polynomial $f \in R_2$ with $f(\mathbf{u}) = \sum_{\mathbf{n}\in\mathbb{Z}^2} f_{\mathbf{n}}\mathbf{u}^{\mathbf{n}}$, the Newton polygon $\mathcal{N}(f)$ of f is the convex hull of the support $\{\mathbf{n} \mid f_{\mathbf{n}} \neq 0\}$.

In Example 2.3 it is not a coincidence that the set of points whose convex hull determines the non-expansive directions is exactly the support of the polynomial $1 + u_1 + u_2$ generating the module of relations. The same holds more generally when the entropy co-rank is one – see [9] for the details.

The following properties hold for any expansive $\mathbb{Z}^2$-action α_M on a zero-dimensional group X_M with entropy rank one, presented as in (1), and for any $\alpha = \alpha_M$-invariant measure μ on X_M. It is useful to talk in terms of directions: a vector $\mathbf{v} \in \mathbb{R}^2\backslash\{0\}$ defines a ray

$$r(\mathbf{v}) = \{t\mathbf{v} \mid t \in [0,\infty)\};$$

vectors $\mathbf{v}$ and $\mathbf{w}$ are in the same direction if their rays coincide, and a vector $\mathbf{v}$ is in a rational direction if there is a vector $\mathbf{w} \in \mathbb{Q}^d$ with $r(\mathbf{v}) = r(\mathbf{w})$.

- There is an annihilating polynomial $f \in R_d$ with the property that $fM = 0_M$ and each vertex coefficient of f is coprime to q.
- For every direction $\mathbf{v}$, $\mathsf{h}_{\mu}(\mathbf{v}) < \infty$.
- If $\mathbf{v}$ is not an outward normal vector to an edge of $\mathcal{N}(f)$, then $\mathsf{h}_{\mu}(\mathbf{v}) = 0$.
- Hence, $\mathsf{h}_{\mu}(\mathbf{v}) > 0$ only for $\mathbf{v}$ in finitely many directions, all of them rational.

The entropy formula in Theorem 2.4 relates the half-space or geometric entropies $\mathsf{h}(\cdot)$ defined by (2) to the dynamical entropies $h(\cdot)$ of individual elements. In the case of higher entropy rank, an analogous formula relates the entropy of subactions of the appropriate rank to geometric entropies of the same rank.

Theorem 2.4. *Let (X, α) be a zero-dimensional algebraic $\mathbb{Z}^2$-action with entropy rank one, let μ be any α-invariant measure on X, and let $\mathcal{C}$ be any α-invariant σ-algebra. Then*

$$h_\mu(\alpha^{\mathbf{n}}|\mathcal{C}) = \sum_{\mathbf{v}\cdot\mathbf{n}>0} (\mathbf{v}\cdot\mathbf{n})\mathsf{h}_\mu(\mathbf{v}|\mathcal{C}) \tag{4}$$

where the sum is taken over all primitive integer vectors $\mathbf{v}$ with $\mathbf{v}\cdot\mathbf{n} > 0$.

The unconditioned version of this is proved in [7]; making the obvious modifications to that proof shows Theorem 2.4. Notice that the left-hand side is the usual dynamical (conditional) entropy of the measure-preserving transformation $\alpha^{\mathbf{n}}$ while the right-hand side involves only the half-space or geometrical (conditional) entropies.

The half-space entropies also obey a form of Abramov–Rokhlin entropy addition formula (cf. [1], [20]). This result will only be needed under the additional assumption that the map ϕ is a group homomorphism.

Theorem 2.5. *Let $\phi : \mathsf{X} \to \mathsf{Y}$ be a continuous surjective map between zero-dimensional expansive entropy rank one algebraic $\mathbb{Z}^2$-systems. Assume that ϕ sends the invariant measure μ on X to the invariant measure ν on Y. Then, for any non-zero vector $\mathbf{v} \in \mathbb{Z}^2$,*

$$\mathsf{h}_\mu(\mathbf{v}) = \mathsf{h}_\nu(\mathbf{v}) + \mathsf{h}_\mu(\mathbf{v}|\phi^{-1}(\mathcal{B}_Y)) \tag{5}$$

where $\mathcal{B}_Y$ denotes the Borel σ-algebra on Y.

Proof. Assume that X and Y have been presented in the form (1), with corresponding state partitions ξ and η. In (5), $\mathsf{h}_\mu(\cdot)$, $\mathsf{h}_\nu(\cdot)$ are defined using ξ, η respectively. Since the half-space entropies are independent of the chosen presentation of the system we can assume without loss of generality that $\phi^{-1}\eta \subset \xi$. Then

$$\begin{aligned}
\mathsf{h}_\mu(\mathbf{v}) &= \frac{1}{N} H_\mu\left(\xi^{\mathbf{v}^\perp+[0,N)\mathbf{v}^*} \big| \xi^{\mathsf{H}_\mathbf{v}}\right) \\
&= \frac{1}{N} H_\mu\left((\phi^{-1}\eta)^{\mathbf{v}^\perp+[0,N)\mathbf{v}^*} \big| \xi^{\mathsf{H}_\mathbf{v}}\right) \\
&\qquad + \frac{1}{N} H_\mu\left(\xi^{\mathbf{v}^\perp+[0,N)\mathbf{v}^*} \big| \xi^{\mathsf{H}_\mathbf{v}} \vee (\phi^{-1}\eta)^{\mathbf{v}^\perp+[0,N)\mathbf{v}^*}\right)
\end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{1}{N}\sum_{n=0}^{N-1} H_\mu\left((\phi^{-1}\eta)^{\mathbf{v}^\perp+n\mathbf{v}^*}\big|\xi^{\mathsf{H}_\mathbf{v}}\vee(\phi^{-1}\eta)^{\mathbf{v}^\perp+[0,n)\mathbf{v}^*}\right)\\
&+\frac{1}{N}\sum_{n=0}^{N-1} H_\mu\left(\xi^{\mathbf{v}^\perp+n\mathbf{v}^*}\big|\xi^{\mathsf{H}_\mathbf{v}\cup\left(\mathbf{v}^\perp+[0,n)\mathbf{v}^*\right)}\vee(\phi^{-1}\eta)^{\mathbf{v}^\perp+[0,N)\mathbf{v}^*}\right).
\end{aligned}
$$

Now

$$
\begin{aligned}
&\frac{1}{N}\sum_{n=0}^{N-1} H_\mu\left((\phi^{-1}\eta)^{\mathbf{v}^\perp+n\mathbf{v}^*}\big|\xi^{\mathsf{H}_\mathbf{v}}\vee(\phi^{-1}\eta)^{\mathbf{v}^\perp+[0,n)\mathbf{v}^*}\right)\\
&\quad\le\frac{1}{N}\sum_{n=0}^{N-1} H_\mu\left((\phi^{-1}\eta)^{\mathbf{v}^\perp+n\mathbf{v}^*}\big|(\phi^{-1}\eta)^{\mathsf{H}_\mathbf{v}}\vee(\phi^{-1}\eta)^{\mathbf{v}^\perp+[0,n)\mathbf{v}^*}\right)\\
&\quad= H_\mu\left((\phi^{-1}(\eta)^{\mathbf{v}^\perp}\big|(\phi^{-1}\eta)^{\mathsf{H}_\mathbf{v}}\right)\\
&\quad= H_\nu\left(\eta^{\mathbf{v}^\perp}\big|\eta^{\mathsf{H}_\mathbf{v}}\right)\\
&\quad= \mathsf{h}_\nu(\mathbf{v}).
\end{aligned}
$$

On the other hand, for fixed n

$$
\begin{aligned}
H_\mu&\left(\xi^{\mathbf{v}^\perp+n\mathbf{v}^*}\big|\xi^{\mathsf{H}_\mathbf{v}\cup\left(\mathbf{v}^\perp+[0,n)\mathbf{v}^*\right)}\vee(\phi^{-1}\eta)^{\mathbf{v}^\perp+[0,N)\mathbf{v}^*}\right)\\
&\qquad\to H_\mu\left(\xi^{\mathbf{v}^\perp}\big|\xi^{\mathsf{H}_\mathbf{v}}\vee\phi^{-1}(\mathcal{B}_Y)\right)
\end{aligned}
$$

as $N\to\infty$ by Martingale convergence. It follows that

$$
\begin{aligned}
&\frac{1}{N}\sum_{n=0}^{N-1} H_\mu\left(\xi^{\mathbf{v}^\perp+n\mathbf{v}^*}\big|\xi^{\mathsf{H}_\mathbf{v}\cup\left(\mathbf{v}^\perp+[0,n)\mathbf{v}^*\right)}\vee(\phi^{-1}\eta)^{\mathbf{v}^\perp+[0,N)\mathbf{v}^*}\right)\\
&\qquad\to H_\mu\left(\xi^{\mathbf{v}^\perp}\big|\xi^{\mathsf{H}_\mathbf{v}}\vee\phi^{-1}(\mathcal{B}_Y)\right)=\mathsf{h}_\mu(\mathbf{v}|\phi^{-1}(\mathcal{B}_Y)).
\end{aligned}
$$

This shows that

$$\mathsf{h}_\mu(\mathbf{v})\le\mathsf{h}_\nu(\mathbf{v})+\mathsf{h}_\mu(\mathbf{v}|\phi^{-1}(\mathcal{B}_Y)). \tag{6}$$

On the other hand, by the classical Abramov–Rokhlin entropy addition formula,

$$h_\mu(\alpha^\mathbf{n})=h_\nu(\alpha^\mathbf{n})+h_\mu(\alpha^\mathbf{n}|\phi^{-1}(\mathcal{B}_Y)). \tag{7}$$

Equation (4) for the trivial σ-algebra and the σ-algebra $\mathcal{C}=\phi^{-1}\mathcal{B}_Y$ together with (6) and (7) show that

$$\mathsf{h}_\mu(\mathbf{v})=\mathsf{h}_\nu(\mathbf{v})+\mathsf{h}_\mu(\mathbf{v}|\phi^{-1}(\mathcal{B}_Y)).$$

□

3. A simple example

In this section we show how to use the entropy geometry of Section 2 to prove that Ledrappier's Example 2.3,

$$X_1 = \left\{x \in \mathbb{F}_2^{\mathbb{Z}^2} \mid x_{\mathbf{n}} + x_{\mathbf{n}+\mathbf{e}_1} + x_{\mathbf{n}+\mathbf{e}_2} = 0 \text{ for all } \mathbf{n} \in \mathbb{Z}^2\right\},$$

and its close sibling

$$X_2 = \left\{x \in \mathbb{F}_2^{\mathbb{Z}^2} \mid x_{\mathbf{n}} + x_{\mathbf{n}+\mathbf{e}_1} + x_{\mathbf{n}-\mathbf{e}_2} = 0 \text{ for all } \mathbf{n} \in \mathbb{Z}^2\right\},$$

are disjoint. That is, if α_i denotes the natural shift action on X_i, and $\mathsf{X}_i = (X_i, \alpha_i)$, then $J(\mathsf{X}_1, \mathsf{X}_2) = \{\lambda_{X_1} \times \lambda_{X_2}\}$. Let $\mathsf{X} = \mathsf{X}_1 \times \mathsf{X}_2$, and write α for the Cartesian product of the two $\mathbb{Z}^2$ shift actions. Let μ be a joining of the two systems.

A polynomial which annihilates the module corresponding to X is the product

$$(1 + u_1 + u_2)(1 + u_1 + u_2^{-1}) = u_2^{-1} + u_1 u_2^{-1} + u_1^2 + u_2 + u_1 u_2,$$

with Newton polygon shown in Figure 1. Write $\mathcal{B}_i$ for the Borel σ-

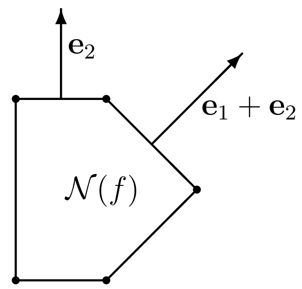

Figure 1. The Newton polygon of the annihilating polynomial

algebra and $\mathcal{N}_i$ for the trivial σ-algebra on X_i, ξ_i for the state partition in X_i for $i = 1, 2$, and $\xi = \xi_1 \times \xi_2$ for the state partition in X.

Part of our purpose here is to show how the half-space entropies and the Abramov–Rokhlin formula for half-space entropies in Theorem 2.5 allow joinings to be understood. The first proof below uses the classical Abramov-Rokhlin formula and the entropy formula Theorem 2.4. The second, much shorter, proof uses Theorem 2.5.

3.1. Proof of disjointness using Theorem 2.4.

By Section 2,

$$h_\mu(\alpha^{\mathbf{e}_2}) = \mathsf{h}_\mu(\mathbf{e}_2) + \mathsf{h}_\mu(\mathbf{e}_1 + \mathbf{e}_2). \tag{8}$$

On the other hand, projecting onto X_1 gives a factor of α, so by the Abramov–Rokhlin formula and Theorem 2.4

$$h_\mu(\alpha^{\mathbf{e}_2}) \;=\; h_{\lambda_1}(\alpha_1^{\mathbf{e}_2}) + h_\mu(\alpha^{\mathbf{e}_2} | \mathcal{B}_1 \times \mathcal{N}_2)$$

$$= h_{\lambda_1}(\alpha_1^{\mathbf{e}_2}) + \mathsf{h}_\mu(\mathbf{e}_2|\mathcal{B}_1 \times \mathcal{N}_2) + \mathsf{h}_\mu(\mathbf{e}_1+\mathbf{e}_2|\mathcal{B}_1 \times \mathcal{N}_2). \tag{9}$$

Since $\xi_1^{\mathbb{R}\times(-\infty,0)} = \mathcal{B}_1$,

$$\begin{aligned}\mathsf{h}_\mu(\mathbf{e}_2|\mathcal{B}_1 \times \mathcal{N}_2) &= H_\mu(\xi^{\mathbb{R}\times\{0\}}|\xi^{\mathbb{R}\times(-\infty,0)} \vee \mathcal{B}_1 \times \mathcal{N}_2)\\ &= H_\mu(\xi^{\mathbb{R}\times\{0\}}|\xi^{\mathbb{R}\times(-\infty,0)})\\ &= \mathsf{h}_\mu(\mathbf{e}_2).\end{aligned}$$

Similarly,

$$\mathsf{h}_\mu(\mathbf{e}_1+\mathbf{e}_2|\mathcal{B}_1 \times \mathcal{N}_2) = 0, \tag{10}$$

and so by comparing (8), (9) and (10),

$$h_{\lambda_1}(\alpha_1^{\mathbf{e}_2}) = \mathsf{h}_\mu(\mathbf{e}_1+\mathbf{e}_2). \tag{11}$$

Projecting onto X_2 gives a different factor of α and a similar argument shows that

$$h_{\lambda_2}(\alpha_2^{\mathbf{e}_2}) = \mathsf{h}_\mu(\mathbf{e}_2). \tag{12}$$

Theorem 2.4, (11) and (12) together show that

$$\begin{aligned}h_\mu(\alpha^{\mathbf{e}_1+\mathbf{e}_2}) &= \mathsf{h}_\mu(\mathbf{e}_2) + \mathsf{h}_\mu(\mathbf{e}_1+\mathbf{e}_2)\\ &= h_{\lambda_2}(\alpha_2^{\mathbf{e}_2}) + h_{\lambda_1}(\alpha_1^{\mathbf{e}_2})\\ &= \log 4\\ &= h_\lambda(\alpha^{\mathbf{e}_1+\mathbf{e}_2}).\end{aligned}$$

That is, the joining measure μ is a measure of maximal entropy for the transformation $\alpha^{\mathbf{e}_1+\mathbf{e}_2}$. Since $\alpha^{\mathbf{e}_1+\mathbf{e}_2}$ is itself an ergodic automorphism of a compact group with finite entropy, it follows from [2] that

$$\mu = \lambda = \lambda_{X_1} \times \lambda_{X_2}.$$

Thus the systems X_1 and X_2 are disjoint.

3.2. **Proof of disjointness using Theorem 2.5.** By the Abramov–Rokhlin formula for half-space entropies,

$$\mathsf{h}_\mu(\mathbf{e}_2) = \mathsf{h}_{\lambda_2}(\mathbf{e}_2) + \mathsf{h}_\mu(\mathbf{e}_2|\mathcal{N}_1 \times \mathcal{B}_2) \geq \log 2,$$

where we use the fact that $\mathsf{h}_{\lambda_2}(\mathbf{e}_2) = h_{\lambda_2}(\alpha^{\mathbf{e}_2}) = \log 2$. Similarly

$$\mathsf{h}_\mu(\mathbf{e}_1+\mathbf{e}_2) = \mathsf{h}_{\lambda_1}(\mathbf{e}_1+\mathbf{e}_2) + \mathsf{h}_\mu(\mathbf{e}_1+\mathbf{e}_2|\mathcal{B}_1 \times \mathcal{N}_2) \geq \log 2,$$

so by Theorem 2.4 the entropy of the map $\alpha^{\mathbf{e}_2}$ satisfies

$$h_\mu(\alpha^{\mathbf{e}_2}) = \mathsf{h}_\mu(\mathbf{e}_2) + \mathsf{h}_\mu(\mathbf{e}_1+\mathbf{e}_2) \geq \log 4 = h_\lambda(\alpha^{\mathbf{e}_2}).$$

That is, the joining measure μ is maximal for the transformation $\alpha^{\mathbf{e}_2}$. Since $\alpha^{\mathbf{e}_2}$ is itself an ergodic automorphism of a compact group with finite entropy, it follows again from [2] that $\mu = \lambda = \lambda_{X_1} \times \lambda_{X_2}$. Thus the systems X_1 and X_2 are disjoint.

4. Reduction step

In this section we give a corollary to the considerations in Section 2, allowing mutual disjointness for entropy rank one examples to be shown inductively. Recall that an algebraic $\mathbb{Z}^d$-action on a zero-dimensional group is expansive if and only if the corresponding R_d-module is Noetherian (see [19]). Throughout this section X will be an expansive system.

Recall from [5] and [9], Sect. 2, the notion of expansiveness for subsets, and more specifically for half-spaces $\mathsf{H}_{\mathbf{v}}$. Parameterize half-spaces by the outward normal vector $\mathbf{v}$, and write $N(\alpha)$ for the finite set (see [9], Th. 4.9 and [8], Th. 7.2) of non-expansive half-spaces.

Theorem 4.1. *Let $\mathsf{Y} = (Y, \alpha_Y, \lambda_Y)$ and $\mathsf{Z} = (Z, \alpha_Z, \mu_Z)$ be expansive zero-dimensional algebraic $\mathbb{Z}^d$-actions with entropy rank one, and let μ be in $J(\mathsf{Y}, \mathsf{Z})$. If there is an integer vector $\mathbf{v} \in N(\alpha_Y) \backslash N(\alpha_Z)$, then μ is invariant under translation by an infinite subgroup $Y_0 \subset Y$. In the case $d = 2$,*

$$Y_0 = \{y \in Y \mid y_{\mathbf{n}} = 0 \textit{ for } \mathbf{n} \in \mathsf{H}_{\mathbf{v}}\}.$$

Translation in $X = Y \times Z$ by an element $y' \in Y$ means translation of the form $(y, z) \mapsto (y + y', z)$. Notice that μ_Z is any α_Z-invariant measure, not necessarily Haar measure.

Proof. The first step is to restrict the action to a $\mathbb{Z}^2$-subaction without losing the hypotheses. By [9], Prop. 7.3, there exists an element $\alpha^{\mathbf{n}}$ which acts expansively on $X = Y \times Z$. Let $\mathbf{m} \in \mathbb{Z}^d$ be linearly independent to $\mathbf{n}$, and write P for the plane in $\mathbb{R}^d$ spanned by $\mathbf{m}$ and $\mathbf{n}$. Write β for the $\mathbb{Z}^2$-subaction generated by $\alpha^{\mathbf{k}}$ with $\mathbf{k} \in P \cap \mathbb{Z}^d$. Similarly, write β_Y, β_Z for the two factors of β on Y and on Z. Then β, β_Y and β_Z are each expansive $\mathbb{Z}^2$-actions. We claim the normal vectors to non-expansive half-spaces for β are obtained by projecting the normal vectors to non-expansive half-spaces for α onto the plane P along the orthogonal complement. Thus a half-space in the plane P is non-expansive if and only if it is contained in a non-expansive half-space for α. This can be seen by a coding argument similar to the proof of [5], Th. 3.6 (replacing subspaces by half-spaces). Perturbing the plane P slightly does not affect the expansiveness of the subaction by [5], Lemma 3.4. By a small perturbation, one can ensure that those pairs of normal vectors in the finite set $N(\alpha)$ which define different half-spaces do so in the plane as well. This ensures that there is a vector $\mathbf{v} \in N(\beta_Y) \backslash N(\beta_Z)$. So, without loss of generality assume now that α is a $\mathbb{Z}^2$-action.

Write $\pi_Y : X \to Y$ and $\pi_Z : X \to Z$ for the canonical projection maps. Then (writing as before $\mathcal{B}_W$, $\mathcal{N}_W$, ξ_W for the Borel σ-algebra, trivial σ-algebra and state partition in $W = Y$ or $W = Z$ respectively)

$$\begin{aligned} \mathsf{h}_\mu(\mathbf{v}) &= \mathsf{h}_{\lambda_Y}(\mathbf{v}) + \mathsf{h}_\mu(\mathbf{v}|\pi_Y^{-1}(\mathcal{B}_Y)) \\ &= \mathsf{h}_{\mu_Z}(\mathbf{v}) + \mathsf{h}_\mu(\mathbf{v}|\pi_Z^{-1}(\mathcal{B}_Z)). \end{aligned}$$

Now

$$\mathsf{h}_\mu(\mathbf{v}|\pi_Y^{-1}(\mathcal{B}_Y)) = 0 \text{ and } \mathsf{h}_{\mu_Z}(\mathbf{v}) = 0$$

since $\mathbf{v} \notin N(\alpha_Z)$. It follows that

$$\mathsf{h}_{\lambda_Y}(\mathbf{v}) = \mathsf{h}_\mu(\mathbf{v}|\pi_Z^{-1}(\mathcal{B}_Z)),$$

so

$$H_{\lambda_Y}\left(\xi_Y^{\mathbf{v}^\perp}|\xi_Y^{\mathsf{H}_\mathbf{v}}\right) = H_\mu\left(\xi_X^{\mathbf{v}^\perp}|\xi_X^{\mathsf{H}_\mathbf{v}} \vee \pi_Z^{-1}(\mathcal{B}_Z)\right). \tag{13}$$

We will show that this is the maximal possible value for this half-space entropy, and deduce the desired translation invariance property.

Let

$$Y_0 = \{y \in Y \mid y_\mathbf{n} = 0 \text{ for } \mathbf{n} \in \mathsf{H}_\mathbf{v}\}$$

and write $\pi : Y_0 \to ((\mathbb{Z}/q\mathbb{Z})^s)^{\mathbf{v}^\perp \cap \mathbb{Z}^2}$ for the projection map onto the coordinates in $\mathbf{v}^\perp \cap \mathbb{Z}^2$ (Y is presented in the form (1) with state partition ξ_Y as usual). Let

$$\eta_Y = \xi_Y^{\mathsf{H}_\mathbf{v}} \text{ and } \zeta_Y = \xi_Y^{\mathsf{H}_\mathbf{v} \cup \mathbf{v}^\perp}. \tag{14}$$

For a measure ν and partition κ write $[x]_\kappa$ for the atom of the partition κ containing x, and $\nu_{x,\kappa}$ for the associated conditional measure (characterised by $\int f \mathrm{d}\nu_{x,\kappa} = \mathbb{E}_\nu(f|\kappa)(x)$ for $f \in L^1(\mu)$). By definition of η_Y and ζ_Y the atom $[y]_{\eta_Y}$ is a union of atoms $[y + y_0]_{\zeta_Y}$ with $y_0 \in Y_0$, where

$$\pi(y_0) = \pi(y_0') \implies [y + y_0]_{\zeta_Y} = [y + y_0']_{\zeta_Y}.$$

For the Haar measure λ_Y all those ζ_Y-atoms have the same weight with respect to λ_{y,η_Y}, so that

$$H_{\lambda_Y}(\zeta_Y|\eta_Y) = \log|\pi(Y_0)|$$

is finite. The finiteness follows from entropy rank one, see Section 2.

We return to the study of μ on $X = Y \times Z$. Let η_X and ζ_X be defined similarly to (14), using the state partition $\xi_X = \xi_Y \times \xi_Z$. Let

$$\eta = \eta_X \vee \pi_Z^{-1}\mathcal{B}_Z \text{ and } \zeta = \zeta_X \vee \pi_Z^{-1}\mathcal{B}_Z.$$

Then each atom $[x]_\eta$ is a finite union of atoms $[x + y_0]_\zeta$ with $y_0 \in Y_0$, where the sum is defined by $x + y = x + (y, 0)$. As before,

$$\pi(y_0) = \pi(y_0') \implies [x + y_0]_\zeta = [x + y_0']_\zeta.$$

By definition, the information function is

$$I_\mu(\zeta|\eta) = -\log \mu_{x,\eta}[x]_\zeta$$

and the entropy is its integral

$$\begin{aligned} H_\mu(\zeta|\eta) &= \int I_\mu(\zeta|\eta)\mathrm{d}\mu \\ &= \int \sum_{y_0\in\pi(Y_0)} -\mu_{x,\eta}([x+y_0]_\zeta)\log \mu_{x,\eta}([x+y_0]_\zeta)\mathrm{d}\mu. \end{aligned}$$

The maximum value of the integral is $\log|\pi(Y_0)|$, which is achieved by (13). This happens only when $\mu_{x,\eta}$ restricted to the partition

$$\{[x+y_0]_\zeta \mid y_0 \in Y_0\}$$

of the atom $[x]_\eta$ is a uniform distribution almost surely. Since translation by $y_0 \in Y_0$ permutes the ζ-atoms inside a fixed η-atom, we deduce that $\mu(A) = \mu(A+y)$ for any $A \in \zeta$ and $y \in Y_0$. This argument may be repeated for the next layers, using

$$\eta' = \eta \text{ and } \zeta' = \xi_X^{\mathsf{H}_{\mathbf{v}}+\mathbf{n}},$$

for some $\mathbf{n} \in \mathbb{Z}^d\backslash\mathsf{H}_{\mathbf{v}}$. As before a restricted version of translation invariance for any $A \in \zeta'$ can be shown. Since this holds for all $\mathbf{n} \in \mathbb{Z}^d$, it follows that μ is invariant under translation by any $y \in Y_0$. Since $\mathbf{v} \in \mathsf{N}(\alpha_Y)$, the subgroup Y_0 is infinite and the theorem follows. □

5. Applications to disjointness

The results of Section 4 suggest the following approach to mutual disjointness for systems of this kind. Given a joining $\mu \in J(\mathsf{X}_1,\dots,\mathsf{X}_n)$ of several algebraic systems $\mathsf{X}_1,\dots,\mathsf{X}_n$, look for a vector $\mathbf{v}$ that is non-expansive for X_1 but expansive for $\mathsf{X}_2\times\cdots\times\mathsf{X}_n$. The proof of Theorem 4.1 gives an equality between two half-space entropies, and then shows that μ is invariant under translation by a subgroup. If the group is large enough, this may be enough to deduce that for almost every $x \in X_2\times\cdots\times X_n$, the conditional measure μ_x is Haar measure λ_{X_1}. This shows that $\mu = \lambda_{X_1}\times\mu_1$ for some Borel probability μ_1 on $X_2\times\cdots\times X_n$. If the process can be repeated with μ_1, then it shows that the systems are mutually disjoint.

This approach needs two things to happen. First, the non-expansive sets of the systems must differ enough to keep producing suitable candidate vectors $\mathbf{v}$. Second – a more subtle problem – the translation invariance provided by Theorem 4.1 may only give partial information about the measures. To avoid the latter problem we assume that the systems are irreducible.

Theorem 5.1. *Let $\mathsf{X}_1, \dots, \mathsf{X}_n$ be a collection of irreducible algebraic zero-dimensional $\mathbb{Z}^d$-actions, all with entropy rank one. If*

$$N(\alpha_j)\setminus \bigcup_{k>j} N(\alpha_k) \neq \emptyset \text{ for } j = 1, \dots, n$$

then the systems are mutually disjoint.

Proof. Let $\mu \in J(\mathsf{X}_1, \dots, \mathsf{X}_n)$ be a joining. Assume by induction that for some $r \geq 1$ we know that

$$\mu = \lambda_{X_1} \times \cdots \times \lambda_{X_{r-1}} \times \mu_r,$$

and let $\mathbf{v}$ be a vector in $N(\alpha_r)\setminus \bigcup_{k>r} N(\alpha_k)$. Then $\mu_r \in J(\mathsf{X}_r, \dots, \mathsf{X}_n)$. Apply Theorem 4.1 with $Y = X_r$ and $Z = \prod_{j\neq r} X_j$. Since the actions are irreducible, the subgroup Y_0 is dense, so the translation invariance shows that each fibre of μ along Y must be Haar measure on Y. That is, $\mu_r = \lambda_{X_r} \times \mu_{r+1}$, and

$$\mu = \lambda_{X_1} \times \cdots \times \lambda_{X_r} \times \mu_{r+1},$$

showing that $\mu = \prod_j \lambda_{X_j}$ by induction. □

Since there is a large collection of irreducible polynomials in $R_2/(p)$ for any fixed prime number p, Theorem 5.1 gives the following corollary.

Corollary 5.2. *There is an infinite family of algebraic $\mathbb{Z}^2$-actions (on zero-dimensional groups) with the property that the members of any finite subcollection are mutually disjoint.*

Recall that a system $\mathsf{X} = (X, \alpha)$ is irreducible if X has no infinite closed α-invariant subgroups. An irreducible component of a system Y is a closed infinite irreducible invariant subgroup.

Theorem 5.3. *Let $\mathsf{Y} = (Y, \alpha_Y)$ and $\mathsf{Z} = (Z, \alpha_Z)$ be ergodic expansive $\mathbb{Z}^d$-actions with entropy rank one on zero-dimensional groups. Assume that for any irreducible component of Y, there is a vector that is non-expansive on that component, but expansive for α_Z. Then Y and Z are disjoint.*

Proof. Let $\mathsf{X} = \mathsf{Y} \times \mathsf{Z}$ and let $\mu \in J(\mathsf{Y}, \mathsf{Z})$. Define

$$H_Y = \{y \in Y \mid \mu \text{ is invariant under translation by } y\}.$$

Notice that H_Y is a closed α-invariant subgroup of Y since μ is an α-invariant Borel measure. If $H_Y = Y$, the measure μ must be the trivial joining. So assume $H_Y \neq Y$ and consider the factors $Y' = Y/H_Y$ and $Y' \times Z$ of Y and X respectively; the factor measure μ' is a joining between the Haar measure $\lambda_{Y'}$ and λ_Z. Since Y' is non-trivial and carries an action of entropy rank one, it must be infinite and therefore contains a non-trivial irreducible component of entropy rank one. Furthermore, the irreducible components of Y' are also irreducible components of Y. So the assumptions of the theorem remain valid. However, by construction the subgroup

$$H_{Y'} = \{y \in Y' \mid \mu' \text{ is invariant under translation by } y\}$$

must be trivial. Without loss of generality, we may pick a vector $\mathbf{v} \in \mathbb{Z}^d$ that is non-expansive for $\alpha_{Y'}$ but expansive for α_Z. By Theorem 4.1 the measure μ is invariant under translation by an infinite subgroup $Y_0 \subset Y'$. This contradiction concludes the proof. □

6. Entropy co-rank one in higher dimensions

In this section we assume that the actions have entropy co-rank one, allow $d \geq 2$, and show disjointness for such actions. The following replacement for the property of irreducibility is needed. Call an algebraic $\mathbb{Z}^d$-action prime if it is of the form X_M for a module $M = R_d/\mathfrak{p}$ with $\mathfrak{p}$ a prime ideal in R_d.

Lemma 6.1. *Let Y be a prime $\mathbb{Z}^d$-action with entropy rank $k \geq 1$. Let $Y' \subset Y$ be a closed α_Y-invariant subgroup such that the restriction $\alpha_{Y'}$ of the action to Y' still has entropy rank k. Then $Y' = Y$.*

That is, there are no non-trivial closed invariant subgroups on which the entropy rank is k.

Proof. This is shown in the proof of Theorem 1.2 in Section 6 of [7]. □

Theorem 6.2. *Let Y and Z be prime $\mathbb{Z}^d$-actions with entropy co-rank one. If $N(\alpha_Y) \neq N(\alpha_Z)$, then Y and Z are disjoint.*

In this setting the non-expansive sets are the set of directions $\mathbf{v}$ with the property that the corresponding half-space $\mathsf{H}_{\mathbf{v}}$ is non-expansive (see [9], Sect. 2). In contrast to the case of entropy rank one, these sets may be infinite. The next example is the analogue of Section 3 for $d = 3$.

Example 6.3. Let

$$X_1 = \left\{x \in \mathbb{F}_2^{\mathbb{Z}^3} \mid x_{\mathbf{n}} + x_{\mathbf{n}+\mathbf{e}_1} + x_{\mathbf{n}+\mathbf{e}_2} + x_{\mathbf{n}+\mathbf{e}_3} = 0 \text{ for all } \mathbf{n} \in \mathbb{Z}^3\right\}$$

and

$$X_2 = \left\{x \in \mathbb{F}_2^{\mathbb{Z}^3} \mid x_{\mathbf{n}} + x_{\mathbf{n}-\mathbf{e}_1} + x_{\mathbf{n}+\mathbf{e}_2} + x_{\mathbf{n}+\mathbf{e}_3} = 0 \text{ for all } \mathbf{n} \in \mathbb{Z}^3\right\},$$

with associated shift $\mathbb{Z}^3$-actions α_1 and α_2. These two systems are associated to the modules $R_3/(2, f_1)$ and $R_3/(2, f_2)$ where $f_1(\mathbf{u}) = 1 + u_1 + u_2 + u_3$ and $f_2(\mathbf{u}) = 1 + u_1^{-1} + u_2 + u_3$. By [9] these systems have entropy co-rank one, and by [5], Ex. 2.9, $N(\alpha_i)$ is the 1-skeleton of the spherical dual to the Newton polytope $\mathcal{N}(f_i)$ for $i = 1, 2$. The vector $\mathbf{e}_1$ lies in $N(\alpha_2)\backslash N(\alpha_1)$, so Theorem 6.2 shows that X_1 and X_2 are disjoint.

The assumption that the entropy co-rank is one in Theorem 6.2 does not seem to be the whole story, since in Sections 4 and 5 we dealt with general $\mathbb{Z}^d$-actions with entropy rank one. Certainly some condition on the entropy rank is required: If it is allowed to be d, then the actions have factors that are measurably isomorphic to Bernoulli shifts by [18], and so have a large space of joinings. The geometric picture for entropy rank $k > 1$ is more complex. To find a restriction of the action to a $\mathbb{Z}^{k+1}$-subactions without losing the assumptions, one needs a more detailed description of $N(\alpha)$ – relating its structure to the entropy rank of the action – which is not yet available.

Before we start the proof of Theorem 6.2 we describe the structure of prime actions with entropy co-rank one, and give some definitions from [7]. If Y is a zero-dimensional prime action with entropy co-rank one, then Y is the dual group of $R_d/(p, f)$ for some prime number p and polynomial f which is irreducible when considered in $R_d/(p)$ (that the prime ideal defining the module must have this form when the entropy co-rank is one follows from [9], Prop. 7.3, which states that the entropy rank of $\alpha_{R_d/\mathfrak{p}}$ is equal to the Krull dimension of $R_d/\mathfrak{p}$ if the characteristic is positive). Clearly f is defined modulo p, so it is natural to assume that p does not divide any nonzero coefficient of f. In the proof of Theorem 6.2 we may assume that Z is defined in the same way by a prime number p' and a polynomial f'.

Applying a $GL(d, \mathbb{Z})$ coordinate change (this may be thought of as a 'time change' in the acting group) for the $\mathbb{Z}^d$-actions if necessary, we can make the following simplifying assumptions. Without loss of generality, $-\mathbf{e}_1$ lies in $N(\alpha_Y)\backslash N(\alpha_Z)$, and $f \in \mathbb{Z}[u_1, u_2^{\pm 1}, \dots, u_d^{\pm 1}]$ is non-zero modulo u_1. The condition that $-\mathbf{e}_1 \in N(\alpha_Y)$ translates to the property that $f = f_0 + f_1 u_1$ for some $f_1 \in \mathbb{Z}[u_1, u_2^{\pm 1}, \dots, u_d^{\pm 1}]$ and $f_0 \in \mathbb{Z}[u_2^{\pm 1}, \dots, u_d^{\pm 1}]$ which is not a monomial by [9], Th. 4.9 and Ex. 5.7. Moreover, we can assume that $f \in \mathbb{Z}[u_1, \dots, u_{d-1}, u_d^{\pm 1}]$, and $f(0, \dots, 0, u_d) = f_0(0, \dots, 0, u_d) \in \mathbb{Z}[u_d]$ is not a monomial (cf. [7], Lemma 9.9). For Z we can assume that $f' \in \mathbb{Z}[u_1, \dots, u_d]$, and

$$f'(0, \dots, 0, u_d) \neq 0$$

is a multiple of a single monomial.

We recall a special case of the notion of lexicographical half-space entropy for an action of entropy co-rank one from [7]. Let

$$\Lambda = \mathbb{Z}^{d-1} \times \{0\} \tag{15}$$

be the subgroup generated by the first $(d-1)$ standard basis vectors. Define lexicographical orders

$$\begin{aligned} \mathbf{m} \prec_{\mathbf{e}_d} \mathbf{n} \quad &\text{if} \quad (m_1, \ldots, m_{d-1}) \prec_{\text{lex}} (n_1, \ldots, n_{d-1}), \\ \mathbf{m} \prec \mathbf{n} \quad &\text{if} \quad \mathbf{m} \prec_{\mathbf{e}_d} \mathbf{n} \text{ and } m_d = n_d, \end{aligned}$$

where $\prec_{\text{lex}}$ is the usual lexicographical order defined by

$$\mathbf{m} \prec_{\text{lex}} \mathbf{n} \text{ if } m_1 = n_1, \ldots, m_{i-1} = n_{i-1}, m_i < n_i \text{ for some } i \leq d.$$

Then the lexicographical half-space entropy is defined by

$$\mathsf{h}_\mu(\mathbf{e}_1, \ldots, \mathbf{e}_{d-1}; \mathbf{e}_d) = H_\mu(\xi^{\mathbb{R}\mathbf{e}_d} | \xi^{S+\mathbb{R}\mathbf{e}_d}),$$

where ξ is the state partition and

$$S = \{\mathbf{n} \in \mathbb{Z}^d \mid \mathbf{n} \succ \mathbf{0}\} \subset \Lambda.$$

By the remarks above,

$$\mathsf{h}_{\lambda_Y}(\mathbf{e}_1, \ldots, \mathbf{e}_{d-1}; \mathbf{e}_d) > 0$$

and

$$\mathsf{h}_{\lambda_Z}(\mathbf{e}_1, \ldots, \mathbf{e}_{d-1}; \mathbf{e}_d) = 0$$

since $S + \mathbb{R}\mathbf{e}_d$ does not code $\mathbb{R}\mathbf{e}_d$ for α_Y, but does for α_Z, and $S + \mathbb{R}\mathbf{e}_d$ contains $\mathbf{e}_1 + H_{-\mathbf{e}_1}$.

Having established these simplifying adjustments and notations, we turn to the proof of Theorem 6.2.

Proof. Let μ be a joining measure, and let f, f', p, p' be chosen as above. One can change the coefficients of f by multiples of p to ensure that the non-zero coefficients are all coprime to pp', and similarly for f'. The product ff' annihilates the R_d-module $R_d/(p, f) \oplus R_d/(p', f')$ dual to $X = Y \times Z$, and every extremal coefficient of ff' is coprime to pp'. Thus X together with ff' and pp' satisfy the hypotheses of [7], Lemma 8.2, and hence the entropy formula in [7], Prop. 8.3 holds for the system: There are only finitely many directions $\mathbf{w} \notin \mathbb{R}^{d-1} \times \{0\}$ with positive half-space entropies $h_\mu(\mathbf{e}_1, \ldots, \mathbf{e}_{d-1}; \mathbf{w})$, moreover the sum of these half-space entropies equals the dynamical entropy $\mathsf{h}_\mu(\alpha_\Lambda)$ of the action of the subgroup Λ defined in (15). Moreover, this entropy formula remains valid when conditioned by an invariant σ-algebra. Just as in Theorem 2.5, factoring onto Y gives

(16) $\mathsf{h}_\mu(\mathbf{e}_1, \ldots, \mathbf{e}_{d-1}; \mathbf{e}_d) =$

$$\mathsf{h}_{\lambda_Y}(\mathbf{e}_1,\dots,\mathbf{e}_{d-1};\mathbf{e}_d)+\mathsf{h}_\mu(\mathbf{e}_1,\dots,\mathbf{e}_{d-1};\mathbf{e}_d|\mathcal{B}_Y\times\mathcal{N}_Z)$$

and factoring onto Z gives

$$\mathsf{h}_\mu(\mathbf{e}_1,\dots,\mathbf{e}_{d-1};\mathbf{e}_d)= \mathsf{h}_{\lambda_Z}(\mathbf{e}_1,\dots,\mathbf{e}_{d-1};\mathbf{e}_d)+\mathsf{h}_\mu(\mathbf{e}_1,\dots,\mathbf{e}_{d-1};\mathbf{e}_d|\mathcal{N}_Y\times\mathcal{B}_Z). \tag{17}$$

Coding arguments show that

$$\begin{aligned}\mathsf{h}_{\lambda_Z}(\mathbf{e}_1,\dots,\mathbf{e}_{d-1};\mathbf{e}_d) &= 0 \text{ and}\\ \mathsf{h}_\mu(\mathbf{e}_1,\dots,\mathbf{e}_{d-1};\mathbf{e}_d|\mathcal{B}_Y\times\mathcal{N}_Z) &= 0.\end{aligned}$$

The first equation was noted above; the second follows similarly. Equations (16) and (17) imply that

$$\mathsf{h}_{\lambda_Y}(\mathbf{e}_1,\dots,\mathbf{e}_{d-1};\mathbf{e}_d)=\mathsf{h}_\mu(\mathbf{e}_1,\dots,\mathbf{e}_{d-1};\mathbf{e}_d|\mathcal{N}_Y\times\mathcal{B}_Z). \tag{18}$$

We use this 'maximality property' of the half-space entropy to deduce a restricted version of translation invariance. Fix $\ell\ge 1$; let

$$\begin{aligned}U_\ell &= [0,\ell-1]^{d-1}\times\{0\} \text{ and}\\ S_\ell &= \ell\{\mathbf{m}\in\Lambda\mid \mathbf{m}\succ\mathbf{0}\}+U_\ell.\end{aligned}$$

Define measurable partitions $\eta=\eta_\ell=\xi^{S_\ell+\mathbb{R}\mathbf{e}_d}$ and $\zeta=\zeta_\ell=\xi^{U_\ell+\mathbb{R}\mathbf{e}_d}\vee\eta$. By [7], Prop. 9.3,

$$H_\mu(\zeta|\eta\vee\mathcal{N}_Y\times\mathcal{B}_Z)=\ell^{d-1}\mathsf{h}_\mu(\mathbf{e}_1,\dots,\mathbf{e}_{d-1};\mathbf{e}_d|\mathcal{N}_Y\times\mathcal{B}_Z)$$

with a similar expression for the lexicographic half-space entropy with respect to Haar measure λ_Y.

By the adjustments made before the proof, $\mathsf{h}_\mu(\mathbf{e}_1,\dots,\mathbf{e}_{d-1};\mathbf{e}_d)>0$. Let

$$Y_\ell=\{y\in Y\mid y_{\mathbf{n}}=0 \text{ for all } \mathbf{n}\in S_\ell+\mathbb{R}\mathbf{e}_d\},$$

and let $\pi: Y_\ell\to(\mathbb{Z}/p\mathbb{Z})^{(U_\ell+\mathbb{R}\mathbf{e}_d)\cap\mathbb{Z}^d}$ be the projection map onto the coordinates in $U_\ell+\mathbb{R}\mathbf{e}_d$. The atom $[x]_{\eta\vee\mathcal{N}_Y\times\mathcal{B}_Z}$ containing the point $x=(x_1,x_2)\in Y\times Z$ is a subset of $Y\times\{x_2\}$ that splits into many atoms $[x+y]_{\zeta\vee\mathcal{N}_Y\times\mathcal{B}_Z}$ with $y\in Y_\ell$ (as before $x+y=(x_1+y,x_2)$). Two such atoms for $y,y'\in Y_\ell$ coincide if and only if $\pi(y)=\pi(y')$, so there are $|\pi(Y_\ell)|$ such atoms. This gives the upper bound $\log|\pi(Y_\ell)|$ for the lexicographical half-space entropy, which is achieved if and only if the conditional measure $\mu_{x,\eta\vee\mathcal{N}_Y\times\mathcal{B}_Z}$ restricted to the partition

$$\{[x+y]_{\zeta\vee\mathcal{N}_Y\times\mathcal{B}_Z}\mid y\in Y_\ell\}$$

is the uniform distribution for μ-almost every $x\in Y\times Z$. Since this holds for λ_Y, the same is true for μ by (18). This implies a restricted translation invariance property

$$\mu(A+y)=\mu(A) \text{ for } A\in\eta_\ell \text{ and } y\in Y_\ell. \tag{19}$$

Let

$$Q_\ell = \{\mathbf{n} \in \mathbb{Z}^d \mid n_i \geq -\ell \text{ for all } i < d\},$$

and $\mathbf{m} = \ell\mathbf{e}_1 + \cdots + \ell\mathbf{e}_{d-1}$. Then $Q_\ell + \mathbf{m} \subset (S_\ell \cup U_\ell) + \mathbb{R}\mathbf{e}_d$, and so $\alpha^{-\mathbf{m}}\xi^{Q_\ell} \subset \eta_\ell$.

Let

$$\begin{aligned} T &= \{\mathbf{n} \in \mathbb{Z}^d \mid n_i \geq 0 \text{ for some } i < d\} \text{ and} \\ Y_0 &= \{y \in Y \mid y_{\mathbf{n}} = 0 \text{ for all } \mathbf{n} \in T\}. \end{aligned}$$

As above $\alpha^{-\mathbf{m}}Y_0 \subset Y_\ell$. Therefore α-invariance of the measure allows us to reformulate (19) as

$$\mu(A + y) = \mu(A) \text{ for } A \in \xi^{Q_\ell} \text{ and } y \in Y_0. \tag{20}$$

However, $\bigcup_\ell Q_\ell = \mathbb{Z}^d$, and so (20) implies that $\mu(A + y) = \mu(A)$ for $y \in Y_0$ and every measurable $A \subset Y \times Z$.

To complete the proof of the theorem, we need to show that μ is in fact invariant under translation by all $y \in Y$. Let $Y' \subset Y$ be the closure of the group generated by the orbit of Y_0 under the action, and let $\alpha_{Y'}$ be the restriction of the action to the invariant subgroup $Y' \subset Y$. The invariance of μ under α and under translation by Y_0 implies that μ is invariant under translation by Y'. We claim that the subaction $(\alpha_{Y'})_\Lambda$ has positive entropy; this shows that $\alpha_{Y'}$ has entropy rank $(d-1)$, and Lemma 6.1 shows that $Y' = Y$.

Suppose Y_0 is the trivial subgroup. Then the restriction map

$$\varphi : Y \to (\mathbb{Z}/(p))^T$$

to the coordinates in T is injective (that is, the dual groups $R_d/(p, f)$ and $\mathbb{Z}[\mathbf{u}^{\mathbf{n}} \mid \mathbf{n} \in T]/(p)$ are equal). Therefore for $\mathbf{m} = -\mathbf{e}_1 - \cdots - \mathbf{e}_{d-1}$ there exists a polynomial $g \in \mathbb{Z}[\mathbf{u}^{\mathbf{n}} \mid \mathbf{n} \in T]/(p)$ with

$$\mathbf{u}^{\mathbf{m}} - g \in (p, f).$$

We will show that this contradicts the special geometry of f and T. In the following the equations are meant modulo p, so suppose $\mathbf{u}^{\mathbf{m}} - g = hf$ for some polynomial h. Split h into a sum $h = h' + h''$ with $h'' \in \mathbb{Z}[\mathbf{u}^{\mathbf{n}} \mid \mathbf{n} \in T]/(p)$ and $h' \in \mathbb{Z}[\mathbf{u}^{\mathbf{n}} \mid n_i < 0 \text{ for all } i < d]$. Taking the product and using $f \in \mathbb{Z}[u_1, \dots, u_d]$ gives $hf = h'f + h''f$ and $h''f \in \mathbb{Z}[\mathbf{u}^{\mathbf{n}} \mid \mathbf{n} \in T]/(p)$. Since $g \in \mathbb{Z}[\mathbf{u}^{\mathbf{n}} \mid \mathbf{n} \in T]/(p)$, we must have $h'f \in \mathbf{u}^{\mathbf{m}} + \mathbb{Z}[\mathbf{u}^{\mathbf{n}} \mid \mathbf{n} \in T]/(p)$. Let $h'_{\min}$ be the sum of those coefficients of h' whose exponent $\mathbf{n}$ of $\mathbf{u}$ is minimal with respect to $\prec_{\mathbf{e}_d}$. Let $f_{\min}$ be the analogous polynomial for f. Then, by the assumption on f, the polynomial $f_{\min} \in \mathbb{Z}[u_d^{\pm 1}]$ cannot be a single monomial. The terms of $h'f$ whose exponents are minimal are exactly the terms in $h'_{\min} f_{\min}$. Since the latter is contained in the ring

$$\mathbb{Z}[\mathbf{u}^{\mathbf{n}} \mid n'_i < 0 \text{ for all } i < d],$$

it must be equal to $\mathbf{u}^{\mathbf{m}}$, which is a contradiction since $f_{\min}$ is not a monomial.

By the above, $Y_0 \subset Y'$ is nontrivial, which implies that

$$\mathsf{h}_{\lambda_{Y'}}(\mathbf{e}_1, \ldots, \mathbf{e}_{d-1}; \mathbf{e}_d) > 0$$

and so by the entropy formula [7], Prop. 8.3, $h_{\lambda_{Y'}}\big((\alpha_{Y'})_\Lambda\big) > 0$ as claimed. □

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