THE ADJOINT ACTION OF AN EXPANSIVE ALGEBRAIC $\mathbb{Z}^d$-ACTION

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Abstract. Let $d \geq 1$, and let α be an expansive $\mathbb{Z}^d$-action by continuous automorphisms of a compact abelian group X with completely positive entropy. Then the group $\Delta_\alpha(X)$ of homoclinic points of α is countable and dense in X, and the restriction of α to the α-invariant subgroup $\Delta_\alpha(X)$ is a $\mathbb{Z}^d$-action by automorphisms of $\Delta_\alpha(X)$. By duality, there exists a $\mathbb{Z}^d$-action α^* by automorphisms of the compact abelian group $X^* = \widehat{\Delta_\alpha(X)}$: this action is called the *adjoint action* of α.

We prove that α^* is again expansive and has completely positive entropy, and that α and α^* are weakly algebraically equivalent, i.e. algebraic factors of each other.

A $\mathbb{Z}^d$-action α by automorphisms of a compact abelian group X is *reflexive* if the $\mathbb{Z}^d$-action $\alpha^{**} = (\alpha^*)^*$ on the compact abelian group $X^{**} = \widehat{\Delta_{\alpha^*}(X^*)}$ adjoint to α^* is algebraically conjugate to α. We give an example of a non-reflexive expansive $\mathbb{Z}^d$-action α with completely positive entropy, but prove that the third adjoint $\alpha^{***} = (\alpha^{**})^*$ is always algebraically conjugate to α^*. Furthermore, every expansive and ergodic $\mathbb{Z}$-action α is reflexive.

The last section contains a brief discussion of adjoints of certain expansive algebraic $\mathbb{Z}^d$-actions with zero entropy.

1. Introduction

An *algebraic $\mathbb{Z}^d$-action* α on a compact abelian group X is a homomorphism $\alpha\colon \mathbf{n} \mapsto \alpha^{\mathbf{n}}$ from $\mathbb{Z}^d$ into the group $\operatorname{Aut}(X)$ of continuous automorphisms of X. An algebraic $\mathbb{Z}^d$-action α on a compact abelian group X is *expansive* if there exists an open neighbourhood $\mathcal{O} \subset X$ of the identity element $0_X \in X$ with

$$\bigcap_{\mathbf{n}\in\mathbb{Z}^d} \alpha^{-\mathbf{n}}(\mathcal{O}) = \{0_X\}.$$

If α and β are algebraic $\mathbb{Z}^d$-actions on compact abelian groups X and Y, respectively, then β is an *algebraic factor* of α if there exists a continuous surjective group homomorphism $\chi\colon X \longrightarrow Y$ with

$$\chi \circ \alpha^{\mathbf{n}} = \beta^{\mathbf{n}} \circ \chi \text{ for every } \mathbf{n} \in \mathbb{Z}^d. \tag{1.1}$$

The map χ in (1.1) is an *algebraic factor map* from α to β. The actions α and β are *algebraically conjugate* if the factor map $\chi\colon X \longrightarrow Y$ in (1.1) can be chosen to be a continuous group isomorphism, and α and β are *weakly algebraically equivalent* if each of them is an algebraic factor of the other.

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The actions α and β are *finitely equivalent* if each of them is a finite-to-one algebraic factor of the other.

If α is an expansive algebraic $\mathbb{Z}^d$-action on a compact abelian group X, then the set $\Delta_\alpha(X)$ of all homoclinic points of α is a countable α-invariant subgroup of X, and the closure $Y = \overline{\Delta_\alpha(X)}$ is the largest closed α-invariant subgroup of X such that the restriction α_Y of α to Y has completely positive entropy (cf. Definition 3.1, Proposition 4.2 and [3]). Furthermore, the restriction $\alpha|_{\Delta_\alpha(X)}$ of α to $\Delta_\alpha(X)$ is a $\mathbb{Z}^d$-action by automorphisms of $\Delta_\alpha(X)$. We consider $\Delta_\alpha(X)$ as a discrete module and denote by α^* the algebraic $\mathbb{Z}^d$-action on the compact abelian group $X^* = \widehat{\Delta_\alpha(X)}$ dual to $\alpha|_{\Delta_\alpha(X)}$. This action is called the *adjoint* of α. In the case of a single expansive (and hence ergodic) automorphism of the m-torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$, α^* is the automorphism of $\mathbb{T}^m$ defined by the transpose (or adjoint) of the matrix $A \in \mathrm{GL}(m, \mathbb{Z})$ defining the automorphism α (cf. Example 5.6).

In Theorem 4.7 we show that the adjoint α^* of any expansive algebraic $\mathbb{Z}^d$-action α is expansive and has completely positive entropy. Hence we can define inductively the higher-order adjoint actions

$$\alpha^*,\, \alpha^{**} = (\alpha^*)^*,\, \alpha^{***} = (\alpha^{**})^*,\, \dots,\, \alpha^{*^n} = (\alpha^{*^{n-1}})^*,\, \dots,$$

of α, which act on the compact abelian groups

$$X^*,\, X^{**} = \widehat{\Delta_{\alpha^*}(X^*)},\, X^{***} = \widehat{\Delta_{\alpha^{**}}(X^{**})},\, \dots,\, X^{*^n} = (X^{*^{n-1}})^*,\, \dots$$

The $\mathbb{Z}^d$-actions α and α^* have the same entropy by Theorem 4.6, but may otherwise be quite different (especially if α does not have completely positive entropy). However, Theorem 4.7 shows that the sequence $\alpha, \alpha^*, \alpha^{**}, \dots$ is *eventually periodic* in the sense that α^* is algebraically conjugate to α^{***}, and hence α^{*^n} is algebraically conjugate to $\alpha^{*^{n+2}}$ for every $n \geq 1$.

An expansive algebraic $\mathbb{Z}^d$-action α is *reflexive* if it is algebraically conjugate to α^{**}. Example 4.11 shows that an expansive algebraic $\mathbb{Z}^d$-action need not be reflexive, even if it has completely positive entropy. Single expansive and ergodic automorphisms (or, more precisely, the $\mathbb{Z}$-actions generated by them) are, however, always reflexive by Theorem 5.1.

In the last section we consider an expansive (and hence ergodic) irreducible $\mathbb{Z}^d$-action α by commuting automorphisms of a finite-dimensional torus $\mathbb{T}^m$. If $d > 1$, α has zero entropy and no nontrivial homoclinic point by [3]. However, the homoclinic group $\Delta_{\alpha^{\mathbf{n}}}(\mathbb{T}^m)$ of any individual expansive automorphism $\alpha^{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^d$, is again a dense α-invariant subgroup of $\mathbb{T}^m$, and the $\mathbb{Z}^d$-actions obtained by restricting α to $\Delta_{\alpha^{\mathbf{m}}}(\mathbb{T}^m)$ and to $\Delta_{\alpha^{\mathbf{n}}}(\mathbb{T}^m)$ for different expansive elements $\alpha^{\mathbf{m}}$ and $\alpha^{\mathbf{n}}$ are algebraically conjugate, although $\Delta_{\alpha^{\mathbf{m}}}(\mathbb{T}^m) \cap \Delta_{\alpha^{\mathbf{n}}}(\mathbb{T}^m)$ may be trivial. By using duality one can thus again define an *adjoint action* α^* of α. In this example $(\alpha^*)^{\mathbf{n}} = (\alpha^{\mathbf{n}})^*$ for every expansive $\alpha^{\mathbf{n}}$ in the sense of our original definition of adjoint actions, and α^{**} is algebraically conjugate to α by Theorem 5.1. This example points towards a more general theory of adjoint actions of arbitrary expansive algebraic $\mathbb{Z}^d$-actions with not necessarily positive entropy, but our understanding of the general situation is still rather rudimentary.

2. Algebraic $\mathbb{Z}^d$-actions

In this section we give a short summary of the theory of algebraic $\mathbb{Z}^d$-actions on compact abelian groups and of the role of commutative algebra in their description.

Let $R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ be the ring of Laurent polynomials with integral coefficients in the variables $u_1, \dots, u_d$. We write the elements $f \in R_d$ as

$$f = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} u^{\mathbf{n}}$$

with $u^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$ and $f_{\mathbf{n}} \in \mathbb{Z}$ for all $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$, where $f_{\mathbf{n}} = 0$ for all but finitely many $\mathbf{n} \in \mathbb{Z}^d$.

The additively-written (Pontryagin) dual group $M = \widehat{X}$ is a module over the ring R_d with operation

$$f \cdot a = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} \widehat{\alpha^{\mathbf{n}}}(a)$$

for $f \in R_d$ and $a \in M$, where $\widehat{\alpha^{\mathbf{n}}}$ denotes the automorphism of $\widehat{X}$ dual to $\alpha^{\mathbf{n}}$. The module $M = \widehat{X}$ is called the *dual module* of α.

Conversely, if M is a module over R_d, then we can define an algebraic $\mathbb{Z}^d$-action α_M on $X_M = \widehat{M}$ by setting

$$\widehat{\alpha_M^{\mathbf{n}}}(a) = u^{\mathbf{n}} \cdot a \tag{2.1}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $a \in M$. Clearly, M is the dual module of α_M.

By using this duality one can express many topological and dynamical properties of X and α in terms of the dual module $M = \widehat{X}$. For example, X is metrizable if and only if M is countable, and X is connected if and only if M is torsion-free as a group. If α is expansive, the dual module M is Noetherian. However, for a characterization of expansiveness and other more subtle dynamical properties we need the notion of an associated prime ideal.

A prime ideal $\mathfrak{p} \subset R_d$ is *associated with an R_d-module M* if $\mathfrak{p} = \{f \in R_d : f \cdot a = 0_M\}$ for some $a \in M$, and the module M is *associated with a prime ideal* $\mathfrak{p} \subset R_d$ if $\mathfrak{p}$ is the only prime ideal associated with M. The set of prime ideals associated with a Noetherian R_d-module M is finite and denoted by $\operatorname{Asc}(M)$.

If α is an algebraic $\mathbb{Z}^d$-action on X, then its topological entropy $h(\alpha)$ coincides with the metric entropy $h_{\lambda_X}(\alpha)$ with respect to the normalized Haar measure λ_X on X. We recall the following results from [8].

Lemma 2.1. *Let M be a Noetherian R_d-module with associated prime ideals $\operatorname{Asc}(M) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$.*

(1) *The following conditions are equivalent.*
   (i) *α_M is expansive;*
   (ii) *$\alpha_{R_d/\mathfrak{p}_j}$ is expansive for every $j = 1, \dots, m$;*
   (iii) *$V_{\mathbb{C}}(\mathfrak{p}_j) \cap \mathbb{S}^d = \varnothing$ for every $j = 1, \dots, m$, where*

$$V_{\mathbb{C}}(\mathfrak{p}_j) = \{\mathbf{z} \in (\mathbb{C}^\times)^d : f(\mathbf{z}) = 0 \text{ for every } f \in \mathfrak{p}_j\},$$

$$\mathbb{C}^\times = \mathbb{C} \smallsetminus \{0\} \textit{ and } \mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}.$$

(2) *The following conditions are equivalent.*
 (i) α_M *is mixing (with respect to Haar measure);*
 (ii) $\alpha_{R_d/\mathfrak{p}_j}$ *is mixing for every* $j = 1, \dots, m$*;*
 (iii) $\mathfrak{p}_j \cap \{u^{\mathbf{n}} - 1 : 0 \neq \mathbf{n} \in \mathbb{Z}^d\} = \varnothing$ *for every* $j = 1, \dots, m$.

(3) *The following conditions are equivalent.*
 (i) α_M *has positive entropy (with respect to Haar measure);*
 (ii) $\alpha_{R_d/\mathfrak{p}_j}$ *has positive entropy for some* $j = 1, \dots, m$*;*
 (iii) $\mathfrak{p}_j$ *is principal and* $\alpha_{R_d/\mathfrak{p}_j}$ *is mixing for some* $j = 1, \dots, m$.

(4) *The following conditions are equivalent.*
 (i) α_M *has completely positive entropy (with respect to Haar measure);*
 (ii) $\alpha_{R_d/\mathfrak{p}_j}$ *has positive entropy for every* $j = 1, \dots, m$*;*
 (iii) $\mathfrak{p}_j$ *is principal and* $\alpha_{R_d/\mathfrak{p}_j}$ *is mixing for every* $j = 1, \dots, m$.

(5) *There exists a Noetherian* R_d*-module* $N \supseteq M$ *with the following properties.*
 (i) $h(\alpha_N) = h(\alpha_M)$*;*
 (ii) $N = N^{(1)} \oplus \dots \oplus N^{(m)}$*, where each of the modules* $N^{(j)}$ *has a finite sequence of submodules* $N^{(j)} = N^{(j)}_{s_j} \supset \dots \supset N^{(j)}_0 = \{0\}$ *with* $N^{(j)}_k / N^{(j)}_{k-1} \cong R_d/\mathfrak{p}_j$ *for* $k = 1, \dots, s_j$*. In particular,* α_N *is expansive (or mixing) if and only if* α_M *is expansive (or mixing).*

For an explicit realization of $\mathbb{Z}^d$-actions of the form $\alpha_{R_d/\mathfrak{p}}$, $\mathfrak{p} \in \mathrm{Asc}(M)$, we refer to Example 5.2 (2) in [8].

3. The homoclinic module

Definition 3.1. Let α be an algebraic $\mathbb{Z}^d$-action on X. An element $x \in X$ is *α-homoclinic* (to the identity element 0_X of X), or simply *homoclinic*, if

$$\lim_{\mathbf{n}\to\infty} \alpha^{\mathbf{n}} x = 0_X.$$

The α-invariant subgroup $\Delta_\alpha(X) \subset X$ of all α-homoclinic points is an R_d-module under the operation

$$f \cdot x = \sum_{\mathbf{n}\in\mathbb{Z}^d} f_{\mathbf{n}} \alpha^{\mathbf{n}} x, \ \ f \in R_d, \ \ x \in \Delta_\alpha(X).$$

The module $\Delta_\alpha(X)$ is called the *homoclinic module.*

For $X = \mathbb{T}^{\mathbb{Z}^d}$ with the usual $\mathbb{Z}^d$-shift action α, the homoclinic module $\Delta_\alpha(X)$ is obviously uncountable. According to [3, Lemma 3.2] this cannot happen for expansive actions.

Lemma 3.2. *The homoclinic module of an expansive algebraic* $\mathbb{Z}^d$*-action is at most countable.*

The next proposition is taken from [3, Theorems 4.1 and 4.2].

Proposition 3.3. *Let* α *be an expansive algebraic* $\mathbb{Z}^d$*-action on a non-trivial compact group* X*. Then* $\Delta_\alpha(X)$ *is nontrivial (resp. dense in* X*) if and only if* α *has positive entropy (resp. completely positive entropy).*

One of the major concerns of this paper is the relation between the associated module $M = \widehat{X}$ and the homoclinic module $\Delta_\alpha(X)$ of an expansive algebraic $\mathbb{Z}^d$-action α on a compact abelian group X. In the simplest case, where $M = R_d/(f)$ for some nonzero $f \in R_d$, this question is answered in [3, Lemma 4.5].

Lemma 3.4. *Let $f \in R_d$ be a (possibly reducible) Laurent polynomial such that the $\mathbb{Z}^d$-action $\alpha = \alpha_{R_d/(f)}$ is expansive and mixing. Then the homoclinic module $\Delta_\alpha(X)$ is countable and dense, and there exists a module isomorphism $\tau\colon R_d/(f) \longrightarrow \Delta_\alpha(X)$.*

Definition 3.5. Let α be an expansive algebraic $\mathbb{Z}^d$-action on a compact group X. A homoclinic point $w \in X$ is *fundamental* if it generates $\Delta_\alpha(X)$ as a module, i.e. if

$$\Delta_\alpha(X) = \{f \cdot w : f \in R_d\}.$$

Remarks 3.6. (1) If an expansive algebraic $\mathbb{Z}^d$-action α on a compact abelian group X has a fundamental homoclinic point w^Δ, then every point of the form $\pm\alpha^{\mathbf{n}} w^\Delta$ with $\mathbf{n} \in \mathbb{Z}^d$ is again fundamental homoclinic.

(2) Recall that an R_d-module M is *cyclic* if there exists an $a \in M$ with $M = R_d \cdot a = \{f \cdot a : f \in R_d\}$. Obviously, α has a fundamental homoclinic point if and only if the homoclinic module $\Delta_\alpha(X)$ is cyclic. Lemma 3.4 shows that expansive algebraic $\mathbb{Z}^d$-actions of the form $\alpha_{R_d/(f)}$ have fundamental homoclinic points.

4. The adjoint action of an expansive algebraic $\mathbb{Z}^d$-action

Definition 4.1. Let α be an expansive algebraic $\mathbb{Z}^d$-action on a compact abelian group X. We view the homoclinic module $\Delta_\alpha(X)$ as a discrete abelian group and define the *adjoint action* α^* of $\mathbb{Z}^d$ on $X^* = \widehat{\Delta_\alpha(X)}$ as $\alpha^* = \alpha_{\Delta_\alpha(X)}$ (cf. (2.1)).

For a polynomial $f \in R_d$ as in Lemma 3.4, the adjoint action α^* on X^* of the action $\alpha = \alpha_{R_d/(f)}$ on $X = X_{R_d/(f)}$ is algebraically conjugate to α. In general this is not true, as the following proposition shows (cf. also Example 4.11).

Proposition 4.2. *Let α be an expansive algebraic $\mathbb{Z}^d$-action α on a compact abelian group X, and let $Y = \overline{\Delta_\alpha(X)} \subset X$ be the closure of the homoclinic group of X. Then Y is the biggest closed α-invariant subgroup such that the restriction α_Y of α to Y has completely positive entropy, $\Delta_{\alpha_Y}(Y) = \Delta_\alpha(X)$, and hence $\alpha^* = \alpha_Y^*$. Furthermore $h(\alpha) = h(\alpha_Y)$.*

Proof. The equality of $\Delta_{\alpha_Y}(Y)$ and $\Delta_\alpha(X)$ follows from the definition of Y, and the first statement is clear from Proposition 3.3.

By [4] there exists a subgroup Y' such that $\alpha_{Y'}$ has completely positive entropy and $h(\alpha_{X/Y'}) = 0$. From above we get $Y' \subseteq Y$ and therefore $h(\alpha_{X/Y}) = 0$. Yuzvinskii's addition formula (cf. [4] or [8, Theorem 14.1]) shows that $h(\alpha) = h(\alpha_Y)$. □

Proposition 4.3. *Let α be an expansive algebraic $\mathbb{Z}^d$-action on a compact abelian group X with completely positive entropy. Then the adjoint $\mathbb{Z}^d$-action α^* on $X^* = \widehat{\Delta_\alpha(X)}$ is expansive, mixing and has completely positive entropy.*

Furthermore there exists a nonzero polynomial $f \in R_d$ with the following properties.

(1) $f \cdot M = \{0\}$, *where $M = \widehat{X}$ is the dual module of α;*
(2) *The $\mathbb{Z}^d$-action α_R is expansive, where $R = R_d/(f)$ with $(f) = fR_d$ (cf. (2.1));*
(3) *The homoclinic module $\Delta_\alpha(X)$ is Noetherian and isomorphic to the module* $\mathrm{Hom}_{R_d}(M, R)$ *of all R_d-module homomorphisms $\chi\colon M \longrightarrow R$.*

Proof. Since α is expansive and has completely positive entropy, Lemma 2.1 shows that the dual module $M = \widehat{X}$ is Noetherian, its set $\mathrm{Asc}(M)$ of associated prime ideals is finite, and every prime ideal $\mathfrak{p} \in \mathrm{Asc}(M)$ is principal and nonzero. We set $\mathrm{Asc}(M) = \{(f_1), \dots, (f_k)\}$ and use [8, Proposition 6.1] to find a prime filtration

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M \tag{4.1}$$

such that, for every $i = 1, \dots, n$, $M_i/M_{i-1} \cong R_d/\mathfrak{q}_i$ for some prime ideal $\mathfrak{q}_i \subset R_d$ with $\mathfrak{q}_i \supset (f_{j_i})$ for some $j_i \in \{1, \dots, k\}$. This shows that there exists an $n \ge 1$ with $f \cdot M = 0$, where

$$f = \Bigl(\prod_{i=1}^{k} f_i\Bigr)^n. \tag{4.2}$$

and that M is thus a module over the ring

$$R = R_d/(f).$$

Note that the polynomial f in (4.2) is not unique.

From Lemma 2.1 (1) and the expansiveness of α it is clear that $V_{\mathbb{C}}(f) \cap \mathbb{S}^d = \varnothing$, and hence that the action α_R on $X_R = \widehat{R} = \widehat{R_d/(f)}$ is expansive and mixing. Lemma 3.4 allows us to choose and fix a fundamental homoclinic point $w^\Delta \in X_R$ of α_R and to define a map $\chi\colon \mathrm{Hom}_{R_d}(M, R) \longrightarrow X$ by setting

$$\langle \chi(v^+), a \rangle = \langle w^\Delta, v^+(a) \rangle \tag{4.3}$$

for every $a \in M$ and $v^+ \in \mathrm{Hom}_{R_d}(M, R)$. Here the left-hand side denotes the value of the character $\chi(v^+) \in X = \widehat{M}$ at the point $a \in M$, and the right-hand side the value of $w^\Delta \in X_R = \widehat{R}$ at $v^+(a) \in R$.

We claim that $\chi(v^+) \in \Delta_\alpha(X)$ for every $v^+ \in \mathrm{Hom}_{R_d}(M, R)$, and that the resulting map

$$\chi\colon \mathrm{Hom}_{R_d}(M, R) \longrightarrow \Delta_\alpha(X) \tag{4.4}$$

is a module isomorphism (here we are abusing notation by using the same symbol χ to denote the maps in (4.3)–(4.4) with different ranges).

In order to see that $\chi(\mathrm{Hom}_{R_d}(M, R)) \subset \Delta_\alpha(X)$ we recall that the automorphism $\alpha^{\mathbf{n}}$ of $X = \widehat{M}$ is dual to multiplication with $u^{\mathbf{n}}$ on M for every $\mathbf{n} \in \mathbb{Z}^d$, so that

$$\begin{aligned}
\lim_{\mathbf{n}\to\infty} \langle \alpha^{\mathbf{n}} \chi(v^+), a \rangle &= \lim_{\mathbf{n}\to\infty} \langle \chi(v^+), u^{\mathbf{n}} \cdot a \rangle \\
&= \lim_{\mathbf{n}\to\infty} \langle w^\Delta, v^+(u^{\mathbf{n}} \cdot a) \rangle = \lim_{\mathbf{n}\to\infty} \langle w^\Delta, u^{\mathbf{n}} \cdot v^+(a) \rangle = 1
\end{aligned}$$

for every $a \in M$ and $v^+ \in \mathrm{Hom}_{R_d}(M,R)$. Since the topology on $X = \widehat{M}$ is that of pointwise convergence, we obtain that

$$\lim_{\mathbf{n}\to\infty} \alpha^{\mathbf{n}}\chi(v^+) = 0_X$$

and hence $\chi(v^+) \in \Delta_\alpha(X)$ for every $v^+ \in \mathrm{Hom}_{R_d}(M,R)$.

For $\mathbf{n} \in \mathbb{Z}^d$ and $v_1^+, v_2^+ \in \mathrm{Hom}_{R_d}(M,R)$,

$$\langle \chi(u^{\mathbf{n}}v_1^+ + v_2^+), a\rangle = \langle w^\Delta, u^{\mathbf{n}} \cdot v_1^+(a))\rangle \cdot \langle w^\Delta, v_2^+(a)\rangle = \langle \alpha^{\mathbf{n}}\chi(v_1^+) + \chi(v_2^+), a\rangle,$$

which shows that $\chi\colon \mathrm{Hom}_{R_d}(M,R) \longrightarrow \Delta_\alpha(X)$ is a module homomorphism.

Next we claim that $\chi(\mathrm{Hom}_{R_d}(M,R)) = \Delta_\alpha(X)$. For every $w \in \Delta_\alpha(X)$ and $a \in M$, the map $h \in R \mapsto \langle w, h \cdot a\rangle \in \mathbb{S}$ defines a homoclinic point $w_a \in \Delta_{\alpha_R}(X_R) \subset X_R = \widehat{R}$ with

$$\langle w_a, h\rangle = \langle w, h \cdot a\rangle$$

for every $h \in R$. Since $w^\Delta \in \Delta_{\alpha_R}(X_R)$ is a fundamental homoclinic point of α_R, there exists a unique $g_a \in R$ with $w_a = g_a \cdot w^\Delta$, and

$$\langle w_{u^{\mathbf{n}}a_1 + a_2}, h\rangle = \langle w, u^{\mathbf{n}}h \cdot a_1 + h \cdot a_2\rangle = \langle \alpha^{\mathbf{n}} w_{a_1} + w_{a_2}, h\rangle$$

for $\mathbf{n} \in \mathbb{Z}^d$, $a_1, a_2 \in M$ and $h \in R$. This shows that

$$g_{u^{\mathbf{n}}a_1 + a_2} = u^{\mathbf{n}} g_{a_1} + g_{a_2}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $a_1, a_2 \in M$, and hence that the map $v^+\colon a \in M \mapsto v^+(a) = g_a \in R$ is an element of $\mathrm{Hom}_{R_d}(M,R)$ with

$$\langle \chi(v^+), a\rangle = \langle w^\Delta, v^+(a)\rangle = \langle w^\Delta, g_a\rangle = \langle g_a \cdot w^\Delta, 1\rangle = \langle w_a, 1\rangle = \langle w, a\rangle$$

for every $a \in M$. It follows that $\chi(\mathrm{Hom}_{R_d}(M,R)) = \Delta_\alpha(X))$.

Since $\langle w^\Delta, u^{\mathbf{n}} g\rangle = 1$ for all $\mathbf{n} \in \mathbb{Z}^d$ if and only if $0 = g \in R$, (4.3) shows that the map χ is injective, which completes the proof of (1).

If $a_1, \dots, a_l \in M$ is a set of generators of M over the ring R (or, equivalently, over R_d), then an element $v^+ \in \mathrm{Hom}_{R_d}(M,R)$ is uniquely determined by the vector $v = (v^+(a_1), \dots, v^+(a_l)) \in R^l$. Therefore $\mathrm{Hom}_{R_d}(M,R)$ is isomorphic to a submodule of R^l. This shows that $\mathrm{Hom}_{R_d}(M,R)$ is itself a Noetherian R_d-module, and that every prime ideal associated to $\mathrm{Hom}_{R_d}(M,R)$ is also associated to R^l. The latter prime ideals are, in turn, given by the associated prime ideals $(f_1), \dots, (f_k)$ of R. By Lemma 2.1, the action α^* on X^* is expansive, mixing, and has completely positive entropy. □

Next we show that the entropies of α and α^* coincide. In the case of an expansive algebraic $\mathbb{Z}^d$-action α with completely positive entropy we prove the stronger statement that α and α^* are weakly algebraically equivalent. To do this we need a strengthening of Lemma 2.1 (5).

Proposition 4.4. *Let α be an expansive algebraic $\mathbb{Z}^d$-action on a compact abelian group X with completely positive entropy, and let $M = \widehat{X}$ be the dual module with associated prime ideals*

$$\mathrm{Asc}\, M = \{(f_1), \dots, (f_k)\}.$$

Then there exists a module N of the form

$$N=\prod_{i=1}^{k}\prod_{j\in F^{(i)}}\left[R_d/(f_i^j)\right]^{n_{i,j}} \tag{4.5}$$

for some finite sets $F^{(i)}\subset\mathbb{N}=\{1,2,3,\dots\}$ and some positive integers $n_{i,j}$, $i=1,\dots,k$, $j\in F^{(i)}$, and injective R_d-module homomorphisms

$$\psi_1\colon M\longrightarrow N,\qquad \psi_2\colon N\longrightarrow M.$$

Furthermore, if $\beta=\alpha_N$ is the algebraic $\mathbb{Z}^d$-action on $Y=\widehat{N}$ defined as in (2.1)*, then α and β are weakly algebraically equivalent.*

Proof. The set $S=\bigcap_{i=1}^{k}R_d\smallsetminus(f_i)$ is multiplicatively closed. We let $\tilde{R}=S^{-1}R_d$ be the ring of fractions with denominators in S. Our assumptions imply that $sm\neq 0$ for every $m\in M\smallsetminus\{0\}$ and $s\in S$, so that the obvious embedding $\psi\colon M\longrightarrow S^{-1}M$ is injective.

We claim that $\tilde{R}$ is a principal ideal domain. Indeed, let P be a nontrivial prime ideal in $\tilde{R}$, and let $g\in P\smallsetminus\{0\}$. Since R_d is a unique factorization domain, the same applies to $\tilde{R}$, and we can write g as $g=\frac{s'}{s}\prod_{i=1}^{k}f_i^{a_i}$ with $s,s'\in S$ and $a_i\geq 0$ for $i=1,\dots,k$. As $g\in P$, one of the factors of g must lie in P as well. Since s,s' are units in $\tilde{R}$, we have that $f_i\in P$ for some $i\in\{1,\dots,k\}$ with $a_i>0$, and hence that $(f_i)\subset P$.

If $(f_i)\subsetneq P$ we take $g\in P\smallsetminus(f_i)$. Since

$$s''=g\prod_{j\neq i}f_j+f_i\in P,$$

but $s''\notin(f_j)$ for $j\in\{1,\dots,k\}$, we have that $s''\in S$ and $P=\tilde{R}$. This contradiction shows that every prime ideal in $\tilde{R}$ is equal to one of the principal ideals $(f_i)=f_i\tilde{R}$ with $i\in\{1,\dots,k\}$.

Let J be an arbitrary ideal in $\tilde{R}$. Then the greatest common divisor g of the elements in J is well-defined up to multiplication by a unit, and $g^{-1}J$ is either an ideal in, or equal to, $\tilde{R}$. If $g^{-1}J=\tilde{R}$, the ideal J is principal and equal to $(g)=g\tilde{R}$. If $g^{-1}J\subsetneq\tilde{R}$, $g^{-1}J$ must be contained in a prime ideal P of $\tilde{R}$. However, by the above discussion, $P=(f_i)$ for some $i\in\{1,\dots,k\}$, which shows that $g^{-1}J\subset(f_i)$ and contradicts the fact that g was the greatest common divisor. This shows that $S^{-1}R_d$ is a principal ideal domain.

The module $S^{-1}M$ is torsion over the principal ideal domain $\tilde{R}$ (i.e. the prime ideal $\{0\}\subset\tilde{R}$ is not associated with $S^{-1}M$ as a module over $\tilde{R}$), and the only prime ideals in $\tilde{R}$ are the principal ideals $(f_i)=f_i\tilde{R}$, $i=1,\dots,k$. The structure theorem for modules over principal ideal domains (cf. e.g. [2]) proves that $S^{-1}M$ is a direct product of cyclic submodules $\tilde{N}_j$, $j=1,\dots,m$, say, each of which has the form $\tilde{R}/(f_i^{m_i})$ for some f_i and $m_i\geq 1$. For every $j=1,\dots,m$ we choose a generator a_j of the cyclic submodule $\tilde{N}_j$, and we set

$$N=\prod_{j=1}^{m}R_d\cdot a_i\subset S^{-1}M.$$

Then N can obviously be written as a product of the form (4.5). Since N and $\psi(M)$ both generate $S^{-1}M$ over $\tilde{R}$ and are finitely generated over R_d, one can find $s, s' \in S$ such that

$$\psi(M) \subset \tfrac{1}{s}N, \qquad N \subset \tfrac{1}{s'}\psi(M),$$

and the module R_d-module homomorphism $\psi_1 \colon M \longrightarrow N$, $\psi_2 \colon N \longrightarrow M$, defined by $\psi_1(a) = s\psi(a)$ and $\psi_2(b) = \psi^{-1}(s'b)$ for $a \in M$ and $b \in N$, are injective.

By duality, the homomorphisms $\widehat{\psi}_1 \colon Y \longrightarrow X$, $\widehat{\psi}_2 \colon X \longrightarrow Y$, are surjective, and the actions α and β satisfy that $\alpha^{\mathbf{n}} \circ \widehat{\psi}_1 = \widehat{\psi}_1 \circ \beta^{\mathbf{n}}$ and $\beta^{\mathbf{n}} \circ \widehat{\psi}_2 = \widehat{\psi}_2 \circ \alpha^{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{Z}^d$, i.e. α and β are algebraic factors of each other. $\square$

Lemma 4.5. *Let α_i be expansive algebraic $\mathbb{Z}^d$-actions on compact abelian groups X_i $(i = 1, 2)$. If*

$$\chi_1 \colon X_1 \longrightarrow X_2, \qquad \chi_2 \colon X_2 \longrightarrow X_1,$$

are algebraic factor maps, then $h(\alpha_1) = h(\alpha_2)$, and the restrictions

$$\chi_1|_{\Delta_{\alpha_1}(X_1)} \colon \Delta_{\alpha_1}(X_1) \longrightarrow \Delta_{\alpha_2}(X_2), \quad \chi_2|_{\Delta_{\alpha_2}(X_2)} \colon \Delta_{\alpha_2}(X_2) \longrightarrow \Delta_{\alpha_1}(X_1),$$

are injective.

Proof. Since the χ_i are factor maps, it follows that $h(\alpha_1) \geq h(\alpha_2)$ and $h(\alpha_2) \geq h(\alpha_1)$, and hence that $h(\alpha_i) = h(\alpha_2)$. Yuzvinskii's addition formula shows that the restrictions $\alpha_i|_{\ker(\chi_i)}$ have zero entropy, and Proposition 3.3 implies that the invariant subgroup $\ker(\chi_i)$ cannot contain any nonzero homoclinic points, so that $\ker(\chi_i) \cap \Delta_{\alpha_i}(X_i) = \{0\}$ and $\chi_i|_{\Delta_{\alpha_i}(X_i)}$ is injective. $\square$

Theorem 4.6. *Let α be an expansive algebraic $\mathbb{Z}^d$-action on a compact abelian group X with completely positive entropy. Then α and α^* are weakly algebraically equivalent and $h(\alpha) = h(\alpha^*)$.*

Proof. Let N be the module appearing in Proposition 4.4, let $Y = \widehat{N}$ and let $\beta = \alpha_N$ be the $\mathbb{Z}^d$-action (2.1) on Y. Since β and α are algebraic factors of each other, Lemma 4.5 shows that $\Delta_\beta(Y)$ and $\Delta_\alpha(X)$ can be embedded injectively into each other. This shows that β^* on Y^* and α^* on X^* are algebraic factors of each other.

Since N is a product of cyclic modules, $\Delta_\beta(Y)$ is isomorphic to N by Lemma 3.4. Therefore the $\mathbb{Z}^d$-action β^* on Y^* is algebraically conjugate to β on Y. Together with the previous paragraph we see that α^* is a factor of β^*, which is conjugate to β and therefore a factor of α. Similarly one obtains that α is a factor of α^*, which forces their entropies to coincide. $\square$

We denote by $\Delta_{\alpha^*}(X^*)$ the homoclinic module of the expansive algebraic $\mathbb{Z}^d$-action α^* on X^* and define the *bi-adjoint* (or *second adjoint*) $\mathbb{Z}^d$-action $\alpha^{**} = (\alpha^*)^*$ on the compact abelian group $X^{**} = \widehat{\Delta_{\alpha^*}(X^*)}$ as above with α^* and X^* replacing α and X. According to Proposition 4.3, α^{**} is again expansive, mixing, and has completely positive entropy.

In this manner we obtain a sequence of expansive algebraic $\mathbb{Z}^d$-actions $\alpha, \alpha^*, \alpha^{**}, \alpha^{***}, \dots$ on compact abelian groups $X, X^*, X^{**}, X^{***}, \dots$ Theorem 4.7 on the following page shows that this sequence is eventually periodic in the sense that α^* and α^{***} are algebraically conjugate.

Theorem 4.7. *Let α be an expansive algebraic $\mathbb{Z}^d$-action on a compact abelian group X, and let $Y = \overline{\Delta_\alpha(X)} \subset X$ be the closure of the homoclinic module as a subset of X. Then the adjoint action α^* of $\mathbb{Z}^d$ on $X^* = Y^*$ is an expansive algebraic $\mathbb{Z}^d$-action with completely positive entropy. Furthermore there exists a canonical algebraic factor map*

$$\phi\colon X^{**} \longrightarrow Y \subset X \tag{4.6}$$

from the bi-adjoint X^{**} to $Y \subset X$. Finally, the corresponding algebraic factor map

$$\phi^*\colon X^{***} \longrightarrow X^*$$

from the third adjoint group $X^{***} = (X^{**})^*$ onto X^* is an isomorphism of X^* and X^{***}.

For the proof of Theorem 4.7 we use the following elementary facts. If $\psi\colon G \longrightarrow H$ is a continuous homomorphism of locally compact abelian groups, then ψ is injective if and only if its dual homomorphism $\widehat{\psi}\colon \widehat{H} \longrightarrow \widehat{G}$ has dense image. In the case where either both groups are compact or both discrete, ψ is injective if and only if $\widehat{\psi}$ is surjective. Let

$$\imath\colon \Delta_\alpha(X) \xrightarrow{\imath_1} Y \xrightarrow{\imath_2} X \tag{4.7}$$

be the inclusion of $\Delta_\alpha(X)$ into X. The first inclusion $\imath_1$ in (4.7) has dense range by the definition of $Y = \overline{\Delta_\alpha(X)}$ in Theorem 4.7.

If $M = \widehat{X}$ is the dual module of X and $M' = \widehat{Y}$ the dual of Y, then the dual homomorphism

$$\hat{\imath}\colon M \xrightarrow{\hat{\imath}_2} M' \xrightarrow{\hat{\imath}_1} X^* \tag{4.8}$$

is again the composition of two maps. The map $\hat{\imath}_2$ in (4.8) is surjective, and the map $\hat{\imath}_1$ is injective with dense range. Furthermore, the maps $\imath$ and $\hat{\imath}$ satisfy that

$$\imath(u^{\mathbf{n}} \cdot w) = \alpha^{\mathbf{n}}(\imath(w)), \qquad \hat{\imath}(u^{\mathbf{n}} \cdot a) = (\alpha^*)^{\mathbf{n}}(\hat{\imath}(a)),$$

for every $\mathbf{n} \in \mathbb{Z}^d$, $w \in \Delta_\alpha(X)$ and $a \in M$.

Lemma 4.8. *Let α be an expansive algebraic $\mathbb{Z}^d$-action. Then $\hat{\imath}(M) \subset \Delta_{\alpha^*}(X^*)$.*

Proof. For every $a \in M$ and $w \in \Delta_\alpha(X)$,

$$\langle w, (\alpha^*)^{\mathbf{n}}(\hat{\imath}(a))\rangle = \langle u^{\mathbf{n}} \cdot w, \hat{\imath}(a)\rangle = \langle \imath(u^{\mathbf{n}} \cdot w), a\rangle = \langle \alpha^{\mathbf{n}}\imath(w), a\rangle \to 1$$

as $\mathbf{n} \to \infty$, since $\imath(w) \in X$ is a homoclinic point. Hence $\hat{\imath}(a) \in \Delta_{\alpha^*}(X^*)$ for every $a \in M$. □

Proof of Theorem 4.7. The first part of the theorem follows from Proposition 4.3.

Equation (4.8) and Lemma 4.8 show that $\hat{\imath}$ is the composition of three maps

$$\hat{\imath}\colon M \xrightarrow{\hat{\imath}_2} M' \xrightarrow{\hat{\imath}_1'} \Delta_{\alpha^*}(X^*) \xrightarrow{\hat{\imath}_1''} X^*, \tag{4.9}$$

where $\hat{\imath}_1''\colon \Delta_{\alpha^*}(X^*) \longrightarrow X^*$ is the inclusion map, $\hat{\imath}_1 = \hat{\imath}_1'' \circ \hat{\imath}_1'$, and $\hat{\imath}_1'$ is injective. By duality,

$$\imath\colon \Delta_\alpha(X) \xrightarrow{\imath_1''} X^{**} \xrightarrow{\imath_1'} Y \xrightarrow{\imath_2} X, \tag{4.10}$$

where the map $\phi = \imath_1'\colon X^{**} \longrightarrow Y$ is surjective.

According to the Propositions 4.2–4.3, the $\mathbb{Z}^d$-action $\alpha^* = \alpha_Y^*$ has completely positive entropy. Hence the closure $Y^* = \overline{\Delta_{\alpha^*}(X^*)}$ of the homoclinic module of α^* coincides with X^*, and the analogues of the relations (4.9)–(4.10) simplify to

$$\begin{aligned} \hat{\imath}^*\colon \Delta_\alpha(X) &\xrightarrow{\hat{\jmath}_1'} \Delta_{\alpha^{**}}(X^{**}) \xrightarrow{\hat{\jmath}_1''} X^{**}, \\ \imath^*\colon \Delta_{\alpha^*}(X^*) &\xrightarrow{\jmath_1''} X^{***} \xrightarrow{\jmath_1'} X^*, \end{aligned} \tag{4.11}$$

where $\hat{\jmath}_1''\colon \Delta_{\alpha^{**}}(X^{**}) \longrightarrow X^{**}$ is the inclusion map and

$$\phi^* = \jmath_1'\colon X^{***} \longrightarrow X^*$$

is surjective.

In order to simplify notation we set

$$(M')^+ = \operatorname{Hom}_{R_d}(M', R), \qquad (M')^{++} = \operatorname{Hom}_{R_d}(\operatorname{Hom}_{R_d}(M', R), R),$$
$$(M')^{+++} = \operatorname{Hom}_{R_d}(\operatorname{Hom}_{R_d}(\operatorname{Hom}_{R_d}(M', R), R), R).$$

Proposition 4.3 yields an isomorphism $\chi\colon (M')^+ \longrightarrow \Delta_\alpha(X)$ with

$$\langle \chi(v^+), \hat{\imath}_1(a)\rangle = \langle w^\Delta, v^+(a)\rangle \tag{4.12}$$

for every $v^+ \in (M')^+$ and $a \in M'$, and a polynomial $f \in R_d$ which annihilates M'. Since f also annihilates the R_d-module $(M')^+ = \operatorname{Hom}_{R_d}(M', R)$ and this module again satisfies the hypotheses of Proposition 4.3, we can find an isomorphism

$$\chi^*\colon (M')^{++} \longrightarrow \Delta_{\alpha^*}(X^*)$$

with

$$\langle \chi^*(v^{++}), \imath_1''(\chi(v^+))\rangle = \langle w^\Delta, v^{++}(v^+)\rangle \tag{4.13}$$

for every $v^+ \in (M')^+$ and $v^{++} \in (M')^{++}$. We write $\psi\colon M' \longrightarrow (M')^{++}$ for the inclusion map defined by

$$\psi(a)(v^+) = v^+(a) \tag{4.14}$$

with $v^+ \in (M')^+$ and $a \in M'$. Then

$$\begin{aligned} \langle \hat{\imath}_1'(a), \imath_1''(\chi(v^+))\rangle &= \langle \hat{\imath}_1(a), \chi(v^+)\rangle = \langle w^\Delta, v^+(a)\rangle \\ &= \langle w^\Delta, \psi(a)(v^+)\rangle = \langle \chi^*(\psi(a)), \imath_1''(\chi(v^+))\rangle \end{aligned} \tag{4.15}$$

for every $a \in M'$ and $v^+ \in (M')^+$. Here the first bracket pairs the element $\hat{\imath}_1'(a) \in \Delta_{\alpha^*}(X^*)$ with $\imath_1''(\chi(v^+)) \in \imath_1''(\Delta_\alpha(X)) \subset X^{**}$. The equality of the second and third brackets in (4.15) follows from (4.12), that of the the third and fourth bracket from (4.14), and the last term uses (4.13).

From (4.15) it is clear that the maps $\widehat{\phi} = \hat{\imath}_1'\colon M' \longrightarrow \Delta_{\alpha^*}(X^*)$ and $\chi^* \circ \psi\colon M' \longrightarrow \Delta_{\alpha^*}(X^*)$ coincide.

Since the map $\imath_1'$ is surjective, the homomorphisms $\widehat{\phi} = \hat{\imath}_1'$ and ψ have to be injective.

In order to complete the proof of the theorem we apply the above argument to the adjoint module $\Delta_\alpha(X) \cong (M')^+$ and prove that the map $\psi^+ : (M')^+ \longrightarrow (M')^{+++}$ corresponding to (4.14) is a bijection.

Define $\rho : (M')^{+++} \longrightarrow (M')^+$ by $\rho(c) = c \circ \psi$ for $c \in (M')^{+++}$. Then ρ is a left inverse of ψ^+. To see this, let $a \in M'$ and $v^+ \in (M')^+$. Then

$$\big((\rho \circ \psi^+)(v^+)\big)(a) = \big(\psi^+(v^+)\big)(\psi(a)) = \psi(a)(v^+) = v^+(a),$$

which shows that $\rho \circ \psi^+$ is the identity on $(M')^+$.

We claim that ρ is injective. Assume to the contrary that $0 \neq c \in (M')^{+++}$, but that $\rho(c) = 0$. Then $c : (M')^{++} \longrightarrow R$ is an R_d-module homomorphism with $c(\psi(M')) = 0$ and thus induces a well-defined map $c' \colon (M')^{++}/\psi(M') \longrightarrow R$. If $b \in (M')^{++}/\psi(M')$ with $c'(b) \neq 0$, then $\mathrm{Ann}(b) \subset \mathrm{Ann}(c'(b))$. Since $\mathrm{Ann}(c'(b))$ is a principal ideal whose generator is a product of the polynomials $f_1, \dots, f_l$, the $\mathbb{Z}^d$-action $\alpha_{(M')^{++}/\psi(M')}$ on the dual of $(M')^{++}/\psi(M')$ has positive entropy. However, we can use the isomorphism χ^* to see that

$$(M')^{++}/\psi(M') \cong \Delta_{\alpha^*}(X^*)/\widehat{\phi}(M'),$$

and that the dual group $\widehat{(M')^{++}/\psi}(M')$ is therefore isomorphic to $\ker(\phi) \subseteq X^{**}$. Since α^{**} and α_Y have the same entropy by Proposition 4.6 and $\phi \colon X^{**} \longrightarrow Y$ is a factor map, Yuzvinskii's addition formula shows that $\alpha^{**}|_{\ker(\phi)}$ has zero entropy.

This contradiction shows that ρ is injective and inverse to ψ^+. Hence ψ^+ is a bijection, as claimed. $\square$

Motivated by Theorem 4.7 we introduce the following definition.

Definition 4.9. The action α on X is *reflexive* if α and α^{**} are algebraically conjugate.

Remarks 4.10. (1) By Theorem 4.7 the adjoint action α^* on X^* is reflexive.

(2) Although our definition of reflexivity only assumes the existence of an algebraic conjugacy between α and α^{**}, it is equivalent to the condition that the factor map ϕ in (4.6) is an isomorphism. This follows from the last assertion in Theorem 4.7, since in this case α is conjugate to $\alpha^{**} = (\alpha^*)^*$, which in return is reflexive in the strong sense.

The next example shows that an algebraic $\mathbb{Z}^d$-action need not be reflexive, even if it has completely positive entropy.

Example 4.11. We give an example of an algebraic $\mathbb{Z}^2$-action α on a compact abelian group X with completely positive entropy such that $M = \widehat{X}$ is not a cyclic module, but $\Delta_\alpha(X)$ is cyclic. The $\mathbb{Z}^d$-actions α and α^* are thus not algebraically conjugate, whereas α^*, α^{**} and α^{***} are algebraically conjugate by Lemma 3.4 and Theorem 4.7. In particular, α and α^{**} are not algebraically conjugate.

Let $f = 4+u_1-u_2$. Then $\alpha_{R_2/(f)}$ is an expansive $\mathbb{Z}^2$-action with completely positive entropy by Lemma 2.1. We define $R = R_2/(f)$ and $M = (2, 1+u_1) = 2R+(1+u_1)R \subset R$. The ring R is a unique factorization domain and M is a non-principal ideal in R. To see this, we first note that u_2 can be expressed in terms of u_1 (modulo (f)), and hence

$$R \cong \mathbb{Z}\big[u_1^{\pm 1}, \tfrac{1}{u_1+4}\big]$$

is isomorphic to a localization of the unique factorization domain $\mathbb{Z}[u_1]$. Since 2 and $1 + u_1$ are different primes in R, the ideal M must either be non-principal or equal to R. The evaluation map

$$\mathrm{ev}_{(1,1)} \colon R_2 \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

$$u_1 \mapsto 1, \qquad u_2 \mapsto 1,$$

induces an R_2-module homomorphism $\mathrm{ev}_{(1,1)}\colon R \longrightarrow \mathbb{F}_2$, since $\mathrm{ev}_{(1,1)}(f) = 0$. As $\mathrm{ev}_{(1,1)}(R) = \mathbb{F}_2$ but $\mathrm{ev}_{(1,1)}(M) = \{0\}$, we see that M is nonprincipal and therefore not a cyclic R_2-module. As R is associated to the prime ideal $(f) = fR_2$ as an R_2-module, the same holds for the R_2-module M.

Let $X = \widehat{M}$ and $\alpha = \alpha_M$. Then α is expansive, has completely positive entropy by Lemma 2.1, and Proposition 4.3 shows that

$$\Delta_\alpha(X) \cong \mathrm{Hom}_{R_d}(M, R).$$

We claim that

$$\mathrm{Hom}_{R_d}(M, R) \cong R,$$

where $g \in R$ corresponds to the map $\psi_g\colon M \longrightarrow R$ defined by $\psi_g(a) = ga$ for every $a \in M$.

It is easy to see that the map $g \mapsto \psi_g$ from R to $\mathrm{Hom}_{R_d}(M, R)$ defines an injective R_2-module homomorphism. To see that this homomorphism is also surjective we take an arbitrary element $\psi \in \mathrm{Hom}_{R_d}(M, R)$. Then there exist elements $a, b \in R$ with

$$\psi(2) = a, \qquad \psi(1 + u_1) = b,$$
$$\psi(2(1 + u_1)) = 2b = (1 + u_1)a.$$

Therefore $2 \mid a$, $(1 + u_1) \mid b$ and

$$\tfrac{a}{2} = \tfrac{b}{1+u_1} = g \in R,$$

which shows that $\psi = \psi_g$. We have proved that

$$\Delta_\alpha(X) \cong R = R_2/(f)$$

is cyclic, whereas M is not a cyclic module over R_2. As explained at the beginning of this example, this proves that $\alpha = \alpha_M$ is not reflexive.

5. The adjoint of a single automorphism

Theorem 5.1. *Let α be an expansive and ergodic automorphism of a compact abelian group X. Then α is reflexive.*

For the proof of Theorem 5.1 we fix an expansive and ergodic automorphism α of a compact abelian group X. Since α is ergodic, it has completely positive entropy by Lemma 2.1, and $\Delta_\alpha(X)$ is dense in X by Proposition 3.3. We define the adjoint and bi-adjoint automorphism α^* and α^{**} on $X^* = \widehat{\Delta_\alpha(X)}$ and $X^{**} = (X^*)^* = \widehat{\Delta_{\alpha^*}(X^*)}$ as in Definition 4.1 and note that these automorphisms are again expansive and ergodic by Theorem 4.7.

Lemma 5.2. *Let β be an expansive automorphism of a compact abelian group Y. If $h_{\mathrm{top}}(\beta) = 0$ then Y is finite.*

Proof. We denote by $M = \hat{Y}$ the dual module of Y and claim that every prime ideal $\mathfrak{p} \in \mathrm{Asc}(M)$ is nonprincipal. Indeed, if $\mathfrak{p} = (f)$ for some $f \in R_1$, then f does not have roots of absolute value 1 by Lemma 2.1 (1). When combined with Lemma 2.1 (2)–(3) this shows that $\alpha_{R_1/(f)}$ and finally β has positive entropy.

This contradiction shows that every $\mathfrak{p} \in \mathrm{Asc}(M)$ is indeed nonprincipal. Since $R_1/\mathfrak{p}$ is finite for every nonprincipal prime ideal $\mathfrak{p} \subset R_1$, the prime filtration for M as in (4.1) implies that M and $Y = \widehat{M}$ are finite. □

Lemma 5.3. *Let β_1, β_2 be expansive and ergodic automorphisms of compact abelian groups Y_1, Y_2 with $h(\beta_1) = h(\beta_2)$. Then every algebraic factor map $\chi\colon Y_1 \longrightarrow Y_2$ from β_1 to β_2 is finite-to-one.*

Proof. From Yuzvinskii's addition formula it follows that the entropy of the restriction $\beta_1|_{\ker(\phi)}$ is equal to zero, since $h(\beta_1) = h(\beta_1|_{\ker(\phi)}) + h(\beta_2) < \infty$. By Lemma 5.2, $\ker(\phi)$ is finite. □

Lemma 5.4. *Let β be an expansive and ergodic automorphism of a compact abelian group Y. Then there exists, for every pair of points $y^+, y^- \in Y$, a point $z \in Y$ with*

$$\lim_{n\to\infty} \beta^n(y^+ - z) = \lim_{n\to-\infty} \beta^n(y^- - z) = 0.$$

Proof. This is a special case of [3, Remark 5.6]. □

Lemma 5.5. *Let β be an expansive and ergodic automorphism of a compact abelian group Y, $Z \subset Y$ a finite β-invariant subgroup, $\pi\colon Y \longrightarrow Y/Z$ the quotient map, and $\beta_{Y/Z}$ the automorphism of Y/Z induced by β. Then the restriction of π to $\Delta_\beta(Y)$ is injective, $\pi(\Delta_\beta(Y)) \subset \Delta_{\beta_{Y/Z}}(Y/Z)$, and $\Delta_{\beta_{Y/Z}}(Y/Z)/\pi(\Delta_\beta(Y)) \cong Z$.*

Proof. It is clear that $\Delta_\beta(Y) \cap Z = \{0\}$ and that the restriction of π to $\Delta_\beta(Y)$ is therefore injective. According to Lemma 5.4 we can find, for every $z \in Z$, a point $y(z) \in Y$ with

$$\lim_{n\to-\infty} \beta^n(y(z) - z) = \lim_{n\to\infty} \beta^n y(z) = 0.$$

As β is continuous, there exist, for every $y \in \pi^{-1}(\Delta_{\beta_{Y/Z}}(Y/Z))$, elements $z^\pm(y) \in Z$ with

$$\lim_{n\to-\infty} \beta^n(z^-(y) - y) = \lim_{n\to\infty} \beta^n(z^+(y) - y) = 0.$$

Hence

$$\pi^{-1}(\Delta_{\beta_{Y/Z}}(Y/Z)) = \{y(z) : z \in Z\} + \Delta_\beta(Y) + Z,$$

and the restriction of π to $\Delta' = \{y(z) : z \in Z\} + \Delta_\beta(Y)$ is a bijection of Δ' and $\Delta_{\beta_{Y/Z}}(Y/Z)$. It follows that

$$\Delta_{\beta_{Y/Z}}(Y/Z)/\pi(\Delta_\beta(Y)) \cong \Delta'/\Delta_\beta(Y) \cong Z. \qquad \square$$

Proof of Theorem 5.1. According to (4.10), the inclusion map $\imath\colon \Delta_\alpha(X) \longrightarrow X$ is a composition of homomorphisms

$$\imath\colon \Delta_\alpha(X) \xrightarrow{\imath_1''} X^{**} \xrightarrow{\imath_1'} X,$$

where the map $\phi = \imath_1'\colon X^{**} \longrightarrow X$ is an algebraic factor map. As $h(\alpha) = h(\alpha^{**})$ by Theorem 4.6, Lemma 5.3 shows that $Z = \ker(\phi)$ is a finite α^{**}-invariant subgroup of X^{**}. If $Z \neq \{0\}$, then Lemma 5.5 implies that $\phi(\Delta_{\alpha^{**}}(X^{**}))$ is a proper subgroup of $\Delta_\alpha(X)$, which violates the facts that

$\imath_1''(\Delta_\alpha(X)) \subset \Delta_{\alpha^{**}}(X^{**})$ and $\imath(\Delta_\alpha(X)) = \Delta_\alpha(X)$ in (4.10). This contradiction shows that ϕ is injective and hence an algebraic conjugacy of α^{**} and α. $\square$

Example 5.6. *Examples of adjoint automorphisms* ([3]). Let $m \geq 2$, and let α be the automorphism of $X = \mathbb{T}^m$ defined by a hyperbolic matrix $A \in \mathrm{GL}(m, \mathbb{R})$ (this means that A has no eigenvalues of absolute value 1). We denote by $\pi \colon \mathbb{R}^m \longrightarrow \mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ the quotient map and write $\mathbb{R}^m = V_e \oplus V_c$ for the decomposition of $\mathbb{R}^m$ into the contracting and expanding subspaces of A (i.e. $\lim_{n\to\infty} \|A^n \mathbf{v}\| = \lim_{n\to-\infty} \|A^n \mathbf{w}\| = 0$ for every $\mathbf{v} \in V_c$ and $\mathbf{w} \in V_e$). Then $V_c \cap \mathbb{Z}^m = V_e \cap \mathbb{Z}^m = \{\mathbf{0}\}$, since $\mathbb{Z}^m$ has no contracting automorphisms. Hence π is injective on V_c and V_e, and $\pi(V_c)$ and $\pi(V_e)$ are dense in $\mathbb{T}^m$ (otherwise we would obtain an A-invariant subtorus of $\mathbb{T}^m$ on which A cannot have determinant ± 1, which is impossible).

For every $\mathbf{n} \in \mathbb{Z}^d$, $V_c \cap (V_e + \mathbf{n}) = \{v_\mathbf{n}\}$ for some unique $v_\mathbf{n} \in \mathbb{R}^m$, and $x_\mathbf{n} = \pi(v_\mathbf{n})$ is homoclinic for α. Conversely, if $x \in \Delta_\alpha(X)$ and $\mathbf{x} \in \pi^{-1}(\{x\}) \subset \mathbb{R}^m$, then the continuity of A implies that there exist points $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$ with $\lim_{n\to\infty} A^n(\mathbf{x} - \mathbf{m}) = 0$ and $\lim_{n\to-\infty} A^n(\mathbf{x} - \mathbf{n}) = 0$. Hence $\mathbf{x} \in (V_c + \mathbf{m}) \cap (V_e + \mathbf{n})$, and $x = x_{\mathbf{n}-\mathbf{m}}$.

This shows that $\Delta_\alpha(X) \cong \mathbb{Z}^m = \hat{X}$. However, the action of α on $\Delta_\alpha(X) \cong \mathbb{Z}^m$ is given by the matrix A, and not by the transpose matrix $A^\top$ corresponding to the dual automorphism $\hat{\alpha}$ on $\hat{X} = \mathbb{Z}^m$. This shows that α^*, the automorphism of $\mathbb{T}^m = \widehat{\Delta_\alpha(X)}$ dual to the restriction of α to $\Delta_\alpha(X)$, is determined by the matrix $A^\top$, whereas α is determined by A.

If A is the companion matrix

$$C_h = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -h_0 & -h_1 & \dots & -h_{m-2} & -h_{m-1} \end{pmatrix} \in \mathrm{GL}(m, \mathbb{Z})$$

of its characteristic polynomial $h = h_0 + \cdots + h_{m-1}u^{m-1} + u^m$, then A and $A^\top$ are easily seen to be algebraically conjugate in $\mathrm{GL}(m, \mathbb{Z})$, and hence α is conjugate to its adjoint α^* (in fact, both are conjugate to $\alpha_{R_1/(h)}$ by Lemma 3.4). In general, however, the matrices A and $A^\top$ are not conjugate in $\mathrm{GL}(m, \mathbb{Z})$, and the automorphisms α and α^* are not algebraically conjugate. For example, if $A = \left(\begin{smallmatrix} 19 & 5 \\ 4 & 1 \end{smallmatrix}\right)$, then A is not conjugate to $A^\top$ in $\mathrm{GL}(2, \mathbb{Z})$ by [6, p.81]. Hence α and α^* are not algebraically conjugate in this case.

6. The adjoint of an algebraic $\mathbb{Z}^d$-action with zero entropy

Expansive algebraic $\mathbb{Z}^d$-actions with zero entropy have no nontrivial homoclinic points by Proposition 4.2. However, some lower-rank sub-actions may be expansive and have positive entropy, and may therefore have nonzero homoclinic points. Since the homoclinic groups of different subactions can have zero intersection, the situation is obviously much more complicated than in the completely positive entropy case, and a complete picture is not yet available.

In this section we describe a particularly simple example of this situation: irreducible $\mathbb{Z}^d$-actions generated by commuting hyperbolic automorphisms of some finite-dimensional torus $\mathbb{T}^m$ (cf. Definition 6.1 on the next page). If

α is such an action on $X=\mathbb{T}^m$, then the homoclinic group $\Delta_{\alpha^{\mathbf{n}}}(X)$ is dense in X for every $\mathbf{n}\in\mathbb{Z}^d$ with $\alpha^{\mathbf{n}}$ hyperbolic (i.e. expansive), and is again a module over R_d with respect to the action

$$f\cdot w=f(\alpha)(w)=\sum_{\mathbf{m}\in\mathbb{Z}^d}a^{\mathbf{m}}w$$

for every $f=\sum_{\mathbf{n}\in\mathbb{Z}^d}f_{\mathbf{n}}u^{\mathbf{n}}$ and $w\in\Delta_{\alpha^{\mathbf{n}}}(X)$. As before we call $\Delta_{\alpha^{\mathbf{n}}}(X)$ the *homoclinic module* of the hyperbolic automorphism $\alpha^{\mathbf{n}}$. Although this module is — a priori — associated with the expansive automorphism $\alpha^{\mathbf{n}}$ and not with the $\mathbb{Z}^d$-action α, Theorem 6.2 shows that, for irreducible and expansive $\mathbb{Z}^d$-actions, different choices of $\alpha^{\mathbf{n}}$ lead to the same module. One can thus regard $\Delta_{\alpha^{\mathbf{n}}}(X)$ as a homoclinic module of α (rather than of $\alpha^{\mathbf{n}}$) and use it to define an algebraic $\mathbb{Z}^d$-action α^* dual to the restriction of α to $\Delta_{\alpha^{\mathbf{n}}}(X)$ for any expansive $\alpha^{\mathbf{n}}$.

Definition 6.1 ([5])**.** An algebraic $\mathbb{Z}^d$-action on a compact abelian group X is *irreducible* if every closed, α-invariant subgroup $Y\subsetneq X$ is finite.

Theorem 6.2. *Let $d\ge 1$, and let α be an irreducible and expansive $\mathbb{Z}^d$-action by automorphisms of $X=\mathbb{T}^m$ with $m\ge 2$, and let $\bar\alpha$ be the linear $\mathbb{Z}^d$-action on $\mathbb{R}^m$ defined by α. Then the set*

$$E_\alpha=\{\mathbf{n}\in\mathbb{Z}^d:\alpha^{\mathbf{n}}\ \text{ is expansive}\}$$

spans $\mathbb{Z}^d$. Furthermore the following conditions are satisfied.

(1) *If $\mathbf{m}\in\mathbb{Z}^d\smallsetminus E_\alpha$, then $\Delta_{\alpha^{\mathbf{m}}}(X)=\{0\}$;*
(2) *If $\mathbf{m}\in E_\alpha$, then the homoclinic module $\Delta_{\alpha^{\mathbf{m}}}(X)$ of $\alpha^{\mathbf{m}}$ is dense in X and isomorphic to $\mathbb{Z}^m$, where R_d acts on $\mathbb{Z}^m$ by*

$$f\cdot\mathbf{k}=f(\bar\alpha)(\mathbf{k})=\sum_{\mathbf{n}\in\mathbb{Z}^d}f_{\mathbf{n}}\bar\alpha^{\mathbf{n}}\mathbf{k}$$

for every $f=\sum_{\mathbf{n}\in\mathbb{Z}^d}f_{\mathbf{n}}u^{\mathbf{n}}$ and $\mathbf{k}\in\mathbb{Z}^m$;
(3) *If $\mathbf{m},\mathbf{n}\in E_\alpha$, then $\Delta_{\alpha^{\mathbf{m}}}(X)$ is either equal to $\Delta_{\alpha^{\mathbf{n}}}(X)$, or $\Delta_{\alpha^{\mathbf{m}}}(X)\cap\Delta_{\alpha^{\mathbf{n}}}(X)=\{0\}$.*

Proof. Since α is irreducible, the matrices $\alpha^{\mathbf{n}}\in\mathrm{GL}(m,\mathbb{Z})$ are simultaneously diagonalizable (cf. [5]), and we obtain a decomposition $\mathbb{R}^m\cong\mathbb{R}^p\times\mathbb{C}^q$ of $\mathbb{R}^m$ into p real and q complex eigenspaces of the linear action $\bar\alpha$ of $\mathbb{Z}^d$ on $\mathbb{R}^m$ induced by α. For every eigenvector $v\in\mathbb{R}^m$ of $\bar\alpha$ the *eigenvalue* of v is a group homomorphism $\omega_v\colon\mathbb{Z}^d\longrightarrow\mathbb{C}^\times$ with

$$\bar\alpha^{\mathbf{n}}v=\omega_v(\mathbf{n})v$$

for every $\mathbf{n}\in\mathbb{Z}^d$. If $\omega_{v_1},\dots,\omega_{v_p}$ and $\omega_{v_{p+1}},\dots,\omega_{v_{p+q}}$ are the real and complex eigenvalues of $\bar\alpha$, then

$$E_\alpha=\{\mathbf{m}\in\mathbb{Z}^d:\alpha^{\mathbf{m}}\ \text{ is expansive}\}=\mathbb{Z}^d\smallsetminus\bigcup_{i=1}^{p+q}\{\mathbf{n}\in\mathbb{Z}^d:|\omega_i(\mathbf{n})|=1\}$$

generates $\mathbb{Z}^d$.

Example 5.6 shows that $\Delta_{\alpha^{\mathbf{n}}}(X)$ is dense in X and isomorphic to $\mathbb{Z}^m$ for every $\mathbf{n}\in E_\alpha$. The remaining assertions follow from [5]. □

Remarks 6.3. (1) As described at the beginning of this section, we define the adjoint $\mathbb{Z}^d$-action α^* of α by setting $\alpha^* = \alpha_{\Delta_{\alpha^{\mathbf{n}}}(X)}$ for any $\mathbf{n} \in E_\alpha$ (cf. (2.1) — this definition is independent of $\mathbf{n}$ by Theorem 6.2). Then the element $(\alpha^*)^{\mathbf{n}}$ of α^* coincides with the adjoint automorphism $(\alpha^{\mathbf{n}})^*$ of $\alpha^{\mathbf{n}}$ for every $\mathbf{n} \in E_\alpha$, and Example 5.6 shows that $(\alpha^*)^{\mathbf{n}}$ is defined by the transpose of the matrix $A^{\mathbf{n}} \in \mathrm{GL}(m, \mathbb{Z})$ corresponding to $\alpha^{\mathbf{n}}$. If we repeat this construction with α^* replacing α we see that the bi-adjoint α^{**} of α is equal to α.

(2) If α is an expansive algebraic $\mathbb{Z}^d$-action with completely positive entropy, then α^* has completely positive entropy by Theorem 4.7, $h(\alpha) = h(\alpha^*)$ by Theorem 4.6, and [7] shows that α and α^* are Bernoulli and hence measurably conjugate. In contrast, if α is an irreducible, expansive and mixing $\mathbb{Z}^d$-action by automorphisms of a finite-dimensional torus $X = \mathbb{T}^m$, and if $d \geq 2$, then α and α^* are generally not measurably conjugate (cf. Example 5.6 and [1]).

References

[1] A. Katok, S. Katok and K. Schmidt, *Rigidity of measurable structure for algebraic actions of higher-rank abelian groups*, ESI-Preprint: ftp://ftp.esi.ac.at/pub/Preprints/esi850.ps.

[2] S. Lang, *Algebra*, 2nd edn., Addison-Wesley, Reading, Massachusetts, 1984.

[3] D. Lind and K. Schmidt, *Homoclinic points of algebraic $\mathbb{Z}^d$-actions*, J. Amer. Math. Soc. **12** (1999), 953–980.

[4] D. Lind, K. Schmidt and T. Ward, *Mahler measure and entropy for commuting automorphisms of compact groups*, Invent. Math. **101** (1990), 593–629.

[5] A.K. Manning and K. Schmidt, *Common homoclinic points of commuting toral automorphisms*, Israel J. Math. **114** (1999), 189–299.

[6] W. Parry and S. Tuncel, *Classification problems in ergodic theory*, London Math. Soc. Lecture Notes, vol. 67, Cambridge University Press, Cambridge, 1982.

[7] D.J. Rudolph and K. Schmidt, *Almost block independence and Bernoullicity of $\mathbb{Z}^d$-actions by automorphisms of compact groups*, Invent. Math. **120** (1995), 455–488.

[8] K. Schmidt, *Dynamical systems of algebraic origin*, Birkhäuser Verlag, Basel-Berlin-Boston, 1995.

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