

# RIGIDITY OF MEASURES INVARIANT UNDER THE ACTION OF A MULTIPLICATIVE SEMIGROUP OF POLYNOMIAL GROWTH ON $\mathbb{T}$

MANFRED EINSIEDLER, ALEXANDER FISH

## 1. INTRODUCTION

In 1967 Furstenberg, in his seminal paper [F], proved that any closed subset of the torus  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  which is invariant under the action of multiplication by 2 and by 3 (mod 1) is either finite or  $\mathbb{T}$ .

For every  $n \in \mathbb{N}$  we denote by  $T_n$  the following map on  $\mathbb{T}$ :

$$T_n(x) = nx \pmod{1}, \quad x \in \mathbb{T}.$$

The structure of  $(T_2, T_3)$ -invariant measures on the torus is still not fully understood.

**Problem** (Furstenberg): *Is it true that a  $(T_2, T_3)$ -invariant ergodic Borel probability measure on  $\mathbb{T}$  is either Lebesgue or has finite support?*

The best known result regarding Furstenberg's measure rigidity problem is due to Rudolph [R] who proved measure rigidity for the  $(T_2, T_3)$  action under the additional assumption of positivity of entropy.

**Theorem 1.1.** (Rudolph) *Let  $\mu$  be a  $(T_2, T_3)$ -invariant ergodic probability Borel measure on  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ . Then either  $h_\mu(T_2) = h_\mu(T_3) = 0$  or  $\mu$  is Lebesgue measure.*

We prove that if a Borel probability measure  $\mu$  on  $\mathbb{T}$  is invariant under the action of a "large" multiplicative semigroup (lower logarithmic density is positive) and the action of the whole semigroup is ergodic then  $\mu$  is either Lebesgue or has finite support, without any entropy assumption.

**Theorem 1.2.** *Let  $\alpha > 0$  and let  $\Sigma \subset \mathbb{N}$  be a multiplicative semigroup with*

$$(1.1) \quad \liminf_{N \rightarrow \infty} \frac{\#(\Sigma \cap [1, N])}{N^\alpha} > 0.$$

*Let  $\mu$  be an ergodic  $\Sigma$ -invariant Borel probability measure on  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ . Then either  $\mu$  is supported on a finite number of points or  $\mu$  is Lebesgue measure.*

We note that the assumption in the theorem is equivalent to positivity of the expression

$$\liminf_{N \rightarrow \infty} \frac{\log \#(\Sigma \cap [1, N])}{\log N}.$$

Also note that if  $\Sigma$  has a positive upper density (which corresponds to the much stronger assumption of  $\alpha = 1$ )

$$\limsup_{N \rightarrow \infty} \frac{\#(\Sigma \cap [1, N])}{N} > 0,$$

then the statement of the theorem follows by an application of the Wiener lemma, see [K, p. 47].

This work is motivated by recent results of Bourgain, Furman, Lindenstrauss and Mozes (see [BFLM]) but it is much simpler. Our case is much easier because the semigroup  $\Sigma$  is abelian. The largeness of  $\Sigma$  makes it possible to deduce that zero entropy implies finite support for  $\mu$ . So, the remaining case is of positive entropy. In this case we use Johnson's result from [J], which we state below and is a generalization of Rudolph's theorem (Thm. 1.1).

This paper shows, in the concrete setting of endomorphisms of  $\mathbb{T}$ , that positivity of entropy is a checkable condition (see Lemma 3.2). The reader is strongly recommended to consult the papers [BL], [EKL], [EL], [ELMV1], [ELMV2], [L] and [MV], to see examples of number-theoretic implications of the checkability of positivity of entropy in more complex situations.

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## 2. MAIN INGREDIENTS

We remind the reader of the Shannon-McMillan-Breiman theorem.

**Theorem 2.1.** *Let  $(X, \mathbb{B}, \mu, T)$  be a measure preserving system such that  $\mathbb{B}$  is a countably generated  $\sigma$ -algebra, and let  $\xi$  be a countable measurable partition of  $X$  with finite entropy*

$$H_\mu(\xi) = \sum_{A \in \xi} -\log(\mu(A))\mu(A) < \infty.$$

Then

$$\frac{1}{n} I_\mu \left( \bigvee_{i=0}^{n-1} T^{-i}(\xi) \right) (x) \rightarrow h_{\mu_x}(T, \xi)$$

for  $\mu$ -almost every  $x \in X$  and in  $L^1_\mu$ , where  $I_\mu(\xi)$  is the information function of the partition  $\xi$  defined by

$$I_\mu(\xi)(x) = -\log \mu(A), \quad x \in A \in \xi,$$

the measures  $\{\mu_x^\xi\}_{x \in X}$  are the ergodic measures for the transformation  $T$  in the ergodic decomposition of  $\mu$ :

$$\mu = \int_X \mu_x^\xi d\mu,$$

and  $h_{\mu_x^\xi}(T, \xi)$  denotes the entropy of  $T$  with respect to  $\xi$  and the measure  $\mu_x^\xi$ .

The other ingredient is Johnson's theorem ([J]) which is a generalization of Rudolph's theorem 1.1.

We recall the notion of a nonlacunary multiplicative semigroup.

**Definition 2.1.** A multiplicative semigroup  $\Sigma \subset \mathbb{N}$  is called **nonlacunary** if there does not exist  $a \in \mathbb{N}$  such that  $\Sigma \subset \{a^n \mid n \in \mathbb{N} \cup \{0\}\}$ .

**Theorem 2.2.** (Johnson) If  $\Sigma$  is a nonlacunary multiplicative semigroup of integers whose action on  $\mathbb{T}$  as multiplication (mod 1) has an ergodic invariant Borel probability measure  $\mu$ , then either  $\mu$  is Lebesgue measure or the entropy of each map  $T_n$ ,  $n \in \Sigma$ , has  $\mu$ -entropy zero.

### 3. PROOF OF THEOREM 1.2

We start with a reformulation of the Shannon-McMillan-Breiman theorem (Thm. 2.1) in the setting considered here.

**Lemma 3.1.** In the setting of Theorem 1.2, if  $h_\mu(T_p) = 0$  for some  $p \in \Sigma$ , then for any  $\varepsilon > 0$  and any  $\beta > 0$ , there exist  $\delta_0 > 0$  and  $X' \subset \mathbb{T}$  of measure  $\mu(X') > 1 - \varepsilon$ , such that for every positive  $\delta \leq \delta_0$  and every  $x \in X'$ ,

$$\mu(B_\delta(x)) > \delta^\beta,$$

where  $B_\delta(x)$  is the ball of radius  $\delta$  with center at  $x$ .

Vaguely speaking, the conclusion of the lemma is that  $\mu$  has zero dimension.

*Proof.* Let  $p \in \Sigma$  such that  $h_\mu(T_p) = 0$ . Take the partition

$$\xi = \left\{ \left(0, \frac{1}{p}\right], \left(\frac{1}{p}, \frac{2}{p}\right], \dots, \left(\frac{p-1}{p}, 1\right] \right\}$$

of  $\mathbb{T}$ . Then  $\bigvee_{i=0}^{n-1} T_p^{-i}(\xi) = \{(0, \frac{1}{p^n}], \dots, (\frac{p^n-1}{p^n}, 1]\}$ . Therefore, for every  $x \in (\frac{k}{p^n}, \frac{k+1}{p^n}]$  and  $k \in \{0, 1, \dots, p^n - 1\}$  we have

$$I_\mu \left( \bigvee_{i=0}^{n-1} T_p^{-i}(\xi) \right) (x) = -\log \mu \left( \left( \frac{k}{p^n}, \frac{k+1}{p^n} \right] \right).$$

Let  $\beta > 0$ , without loss of generality we may assume  $\beta \in (0, 1)$ .

By the a.e. convergence in the Shannon-McMillan-Breiman theorem it follows that for every  $\varepsilon > 0$  there exists a measurable set  $X' \subset \mathbb{T}$  with  $\mu(X') > 1 - \varepsilon$  and an  $N$  such that for every  $n \geq N$  we have

$$\frac{1}{n} I_\mu \left( \bigvee_{i=0}^{n-1} T_p^{-i}(\xi) \right) (x) < \frac{\beta}{2} \log p$$

for every  $x \in X'$ . Take  $\delta_1 = p^{-N}$  and let  $\delta$  be any positive number with  $\delta \leq \delta_1$ . Let  $n$  be the smallest integer such that  $p^{-n} < \delta$  ( $n \geq N$ ). If  $x \in A = (\frac{k}{p^n}, \frac{k+1}{p^n}] \in \bigvee_{i=0}^{n-1} T_p^{-i}(\xi)$  then  $A \subset B_\delta(x)$ . Therefore, for every  $x \in X'$ , we have

$$I_\mu \left( \bigvee_{i=0}^{n-1} T_p^{-i}(\xi) \right) (x) = -\log \mu(A) \geq -\log \mu(B_\delta(x)).$$

So we have, for every  $x \in X'$ ,

$$\frac{1}{n} (-\log \mu(B_\delta(x))) \leq \frac{1}{n} I_\mu \left( \bigvee_{i=0}^{n-1} T_p^{-i}(\xi) \right) (x) < \frac{\beta}{2} \log p.$$

The latter implies that

$$\mu(B_\delta(x)) > e^{-n \frac{\beta}{2} \log p} \geq \delta^{\frac{\beta}{2}} p^{-\frac{\beta}{2}}.$$

as  $p^{-n+1} \geq \delta$  by choice of  $n$ . Rewriting the right hand side, we have equivalently

$$\mu(B_\delta(x)) > \delta^\beta (\delta^{-\frac{\beta}{2}} p^{-\frac{\beta}{2}})$$

This gives the desired conclusion once  $\delta < \delta_0 = \min(\delta_1, \frac{1}{p})$ , for then the last term in the parenthesis on the right is  $\geq 1$ .  $\square$

In the following we will make use of the assumption of positive logarithmic density of  $\Sigma$ .

**Lemma 3.2.** *Under the assumptions of Theorem 1.2, if  $h_\mu(T_p) = 0$  for some  $p \in \Sigma$ , then the measure  $\mu$  has finite support.*

*Proof.* Recall that  $\alpha > 0$  satisfies the condition (1.1). We define  $\beta = \frac{\alpha}{20}$ . Also by (1.1) there exists  $M_0$  such that for  $M \geq M_0$  we have  $\#(\Sigma \cap [1, M]) > M^{\frac{\alpha}{2}}$ . We also define  $\delta = \delta(M) = M^{-5}$ .

Assume that  $h_\mu(T_p) = 0$  for some  $p \in \Sigma$ . From Lemma 3.1 it follows that for every  $\varepsilon > 0$  —  $\varepsilon = \frac{1}{2}$  will do — there exists a set  $X' \subset \mathbb{T}$  such that  $\mu(X') > 1 - \varepsilon$  and there exists  $\delta_0 > 0$  such that for every positive  $\delta < \delta_0$  and every  $x \in X'$ ,

$$\mu(B_\delta(x)) > \delta^\beta.$$

We may assume  $\delta(M) \leq \delta_0$  for  $M \geq M_0$  (by adjusting  $M_0$  if necessary).

At this point we use the invariance<sup>1</sup> of  $\mu$  under  $\Sigma$ . Fix some  $x \in X'$  and  $\delta \leq \delta_0$ . For every  $q \in \Sigma \cap [1, M]$  we write  $A_q = qB_\delta(x)$ . Then  $B_\delta(x) \subset T_q^{-1}(A_q)$ , and so by invariance of  $\mu$  under the action of  $T_q$  we get

$$\mu(A_q) = \mu(qB_\delta(x)) = \mu(T_q^{-1}(A_q)) \geq \mu(B_\delta(x)) > \delta^\beta.$$

We vary  $q \in \Sigma \cap [1, M]$  and note that

$$M^{\frac{\alpha}{2}} \delta^\beta = M^{\frac{\alpha}{2} - 5\frac{\alpha}{20}} = M^{\frac{\alpha}{4}} > 1.$$

To summarize, if we restrict ourselves to  $q \in \Sigma \cap [1, M]$  — and there are at least  $M^{\frac{\alpha}{2}}$  many such choices, every image interval  $A_q = qB_\delta(x)$  has  $\mu$ -measure at least  $\delta^\beta$ . In total this would contradict the assumption of having a probability space unless the various sets  $A_q$  are not all disjoint.

Hence there exist for any given  $x \in X'$

$$q_1, q_2 \in \Sigma \cap [1, M], \quad q_1 > q_2$$

with

$$q_1 B_\delta(x) \cap q_2 B_\delta(x) \neq \emptyset.$$

Thus there exist  $i_1, i_2 \in B_\delta(x)$  such that  $q_1 i_1 - q_2 i_2 = k \in \mathbb{Z}$ .

We now think of  $x \in [0, 1)$  and  $i_1, i_2 \in \mathbb{R}$  as real numbers instead of as cosets belonging to  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ . The formulae

$$k = q_1(i_1 - x + x) - q_2(i_2 - x + x) = q_1(i_1 - x) - q_2(i_2 - x) + (q_1 - q_2)x$$

and

$$|q_1(i_1 - x) - q_2(i_2 - x)| \leq 2\delta M = 2M^{-4}$$

imply that

$$x = \frac{k}{q_1 - q_2} + \kappa, \quad |\kappa| \leq 2M^{-4}.$$

Let  $\ell = q_1 - q_2$ , so  $\ell < M$ . We summarize: We have shown that  $x$  is close to a rational number  $\frac{k}{\ell}$  with related bounds on the denominator  $\ell$  and on how close  $x$  is to  $\frac{k}{\ell}$ . Actually the reader should be surprised

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<sup>1</sup>We will use invariance in the ‘wrong way’ by using that the image of an interval has at least the measure of the original interval.

here on how good a bound on the error  $\kappa$  we have achieved — this will be important.

For a given  $x \in X'$  let us denote the rational number  $\frac{k}{\ell}$  obtained above with the given  $M$  as  $r(M) = \frac{k}{\ell}$ . Take some  $M_1 \geq M_0$ . and denote by  $\frac{k_1}{\ell_1}$  the rational  $r(M_1)$  obtained with the above properties. Similarly denote by  $\frac{k_2}{\ell_2}$  the rational  $r(M_1^2)$ . So, we have

$$\left| x - \frac{k_2}{\ell_2} \right| \leq 2M_1^{-8}$$

and

$$\ell_2 \leq M_1^2.$$

If we suppose that  $\frac{k_1}{\ell_1} \neq \frac{k_2}{\ell_2}$  then

$$\left| \frac{k_1}{\ell_1} - \frac{k_2}{\ell_2} \right| \geq \frac{1}{\ell_1 \ell_2} \geq M_1^{-3}$$

On the other hand,

$$\left| \frac{k_1}{\ell_1} - \frac{k_2}{\ell_2} \right| \leq \left| \frac{k_1}{\ell_1} - x \right| + \left| \frac{k_2}{\ell_2} - x \right| \leq 2(M_1^{-4} + M_1^{-8}).$$

For  $M_1$  sufficiently large (there exists positive  $M'_0 \geq M_0$  such that  $M_1$  should satisfy  $M_1 \geq M'_0$ ), we get a contradiction. Thus  $\frac{k_1}{\ell_1} = \frac{k_2}{\ell_2}$ . Repeating the argument for the given  $x \in X'$  infinitely often (continuing with  $M_1^2, M_1^4, \dots$  replacing  $M_1$  in that order) improves the accuracy of the approximation without changing the rational number and so shows that  $x = \frac{k_1}{\ell_1}$ .

The last argument proves that  $X'$  has only finitely many points but has  $\mu$ -measure at least  $\frac{1}{2}$ . So  $\mu$  has atoms. By ergodicity, we get that  $\mu$  has a finite support.  $\square$

Theorem 1.2 follows from A.Johnson theorem cited in Section 2 combined with Lemma 3.2.

#### 4. FURTHER DISCUSSION

As equidistribution and rigidity of invariant measures often take place simultaneously, one may believe that Theorem 1.2 holds because of some kind of equidistribution phenomenon. We indicate in this section that in the case at hand, at least naively, this is not true.

It was pointed out to us by Fedor Nazarov that for every  $\alpha \notin \mathbb{Q}$  one can construct a multiplicative semigroup  $\Sigma = \{\sigma_1 < \sigma_2 < \dots < \sigma_n < \dots\}$  of positive lower density such that the sequence  $\{\sigma_n \alpha\}$  is not equidistributed in  $\mathbb{T}$ . We thank Fedor Nazarov for allowing us to reproduce the construction here.

Our semigroup  $\Sigma$  will satisfy that

$$d_*(\Sigma) = \liminf_{N \rightarrow \infty} \frac{\#(\Sigma \cap \{1, \dots, N\})}{N} \geq \frac{1}{200}.$$

Choose  $\alpha \notin \mathbb{Q}$ . We construct  $\Sigma$  by an iterative process. Let  $S \subset \mathbb{N}$ . For any  $N \in \mathbb{N}$  denote by  $S_N = S \cap \{1, \dots, N\}$ . Let  $N_0 \in \mathbb{N}$  be such that for any  $n \geq N_0$  we have

$$(4.1) \quad \frac{n}{8} - \frac{n}{1000} < \left| \left\{ k \leq n \mid 0 < k\alpha \pmod{1} < \frac{1}{8} \right\} \right| < \frac{n}{8} + \frac{n}{1000}.$$

We choose  $N_1 = 2N_0$  and let  $B_{N_1}$  to be the subset of size at least  $\frac{1}{10}N_0$  of  $\{N_0 + 1, \dots, N_1\}$  such that for every  $k \in B_{N_1}$  we have

$$0 < k\alpha \pmod{1} < \frac{1}{8}.$$

If we take the semigroup  $\Sigma_1$  generated by  $B_{N_1}$  then there exists a smallest  $N'_2 = N_1 2^{\ell_1}$  (with  $\ell_1 \in \mathbb{N}$ ) such that

$$(4.2) \quad |(\Sigma_1)_{2N'_2}| \leq \frac{2N'_2}{100}.$$

Let  $N_2 = 2N'_2$ . We use the lower bound of (4.1) in the interval  $\{1, \dots, 2N'_2\}$ , the upper bound of (4.1) in the interval  $\{1, \dots, N'_2\}$  and the inequality (4.2) to conclude that we can find  $\frac{1}{10}N'_2$  elements  $k$  in the set  $\{N'_2 + 1, \dots, 2N'_2\} \setminus \Sigma_1$  with  $0 < k\alpha \pmod{1} < \frac{1}{8}$ , denote this set by  $A_2$ . We define  $B_{N_2} = (\Sigma_1)_{N_2} \cup A_2$ .

Then we can repeat that process infinitely many times. We will get a sequence of times  $N_1 < N_2 < \dots < N_m < \dots$  and the sequence of finite sets  $B_{N_1} \subset B_{N_2} \subset \dots$  such that for every  $k \in \mathbb{N}$  we have

$$(4.3) \quad \operatorname{Re} \left( \frac{1}{|B_{N_k}|} \sum_{\sigma \in B_{N_k}} \exp(2\pi i \sigma \alpha) \right) \geq \frac{1}{20} \frac{\sqrt{2}}{2} - \frac{1}{100} > 0.$$

Denote by  $\Sigma = \cup_k B_{N_k}$  and notice that by construction  $\Sigma_{N_k} = B_{N_k}$  for any  $k \geq 1$ .

It is now relatively straight forward to check that  $\Sigma$  is a multiplicative semigroup, and that (by the minimality of the  $\ell_k$  in the construction) the lower density of  $\Sigma$  is at least  $\frac{1}{200}$  as claimed. Clearly (4.3) implies that  $\Sigma\alpha$  is not uniformly distributed.

By a similar construction, for any choice of countably many irrational numbers  $\alpha_1, \dots, \alpha_n, \dots$  there exists a multiplicative semigroup of a linear growth (of positive lower density)  $\Sigma$  such that for every  $k \in \mathbb{N}$  we have  $\Sigma\alpha_k$  is not uniformly distributed.

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