When does a submodule of $(\mathbb{R}[x_1, \ldots, x_k])^n$ contain a positive element?

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Abstract

We characterize for modules consisting of tuples of Laurent polynomials with real coefficients whether such a module contains a positive element. The two conditions needed are numerical and directional positivity. The proof applies universal Gröbner bases.

1 Introduction

Let $S_k = \mathbb{R}[x_1, \ldots, x_k]$, $S_k^+ = \mathbb{R}^+[x_1, \ldots, x_k]$ and $S_k^{++} = S_k^+ \backslash \{0\}$. Considering Laurent polynomials, similarly write $R_k = \mathbb{R}[x_1^{\pm}, \ldots, x_k^{\pm}]$, $R_k^+ = \mathbb{R}^+[x_1^{\pm}, \ldots, x_k^{\pm}]$ and $R_k^{++} = R_k^+ \backslash \{0\}$. When there is no need to make the number of variables explicit we will simply write S, S^+, S^{++}, R, R^+ and R^{++} for these objects. The elements of R^{++} and S^{++} we call *positive polynomials*.

If $u = (u(1), \ldots, u(k)) \in \mathbb{Z}^k$, put $x^u = x_1^{u(1)} \cdots x_k^{u(k)}$ and denote the coefficient of x^u in $p \in R$ by p_u. Then $p = \sum_{u \in \mathbb{Z}^k} p_u x^u$ and the Newton polytope $N(p)$ of p is the convex hull of the finite set $\mathrm{Log}(p) = \{u \in \mathbb{Z}^k : p_u \neq 0\}$. For $v \in \mathbb{R}^k$, let $\mathrm{in}_v(p)$ be the sum of $p_u x^u$ over those $u \in \mathrm{Log}(p)$ for which the dot product $u \cdot v$ is maximal, with the convention that $\mathrm{in}_v(0) = 0$. The polynomial $\mathrm{in}_v(p)$ is the 'face polynomial' of p in the direction of v – see Figure 1. Clearly if p is a positive polynomial $\mathrm{in}_v(p)$ is positive as well. For an ideal $I \subset R$ and $v \in \mathbb{R}^k$ we have the initial ideal $\mathrm{in}_v(I) = \langle \mathrm{in}_v(p) : p \in I \rangle \subset R$. It was proved in [ET] that an ideal I of R intersects R^{++} if and only if for every $v \in \mathbb{R}^k$ and $a \in (0, \infty)^k$ there exists $f \in \mathrm{in}_v(I)$ such that $f(a) > 0$. (It clearly suffices to consider unit v and $v = 0$.) We will extend this result to R-submodules of R^n.

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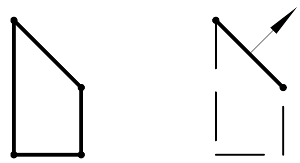

Figure 1: The Newton polytope and the face polynomial $\mathrm{in}_v(f) = xy + y^2$ for the single polynomial $f = 1 + x + xy + y^2$ and direction $v = (1, 1)$.

For $f = (f(1), \ldots, f(n)) \in R^n$ and $v \in \mathbb{R}^k$, we define $\mathrm{in}_v(f) \in R^n$ by letting $\mathrm{in}_v(f)(i) = \mathrm{in}_v(f(i))$. If $\alpha = (\alpha(1), \ldots, \alpha(n)) \in \mathbb{R}^n$, let

$$\mathrm{m}_{v,\alpha}(f) = \max_{1 \leq i \leq n} \{\max\{N(f(i)) \cdot v\} + \alpha(i)\}$$

and define $\mathrm{in}_{v,\alpha}(f) \in R^n$ by letting the i-th component

$$\mathrm{in}_{v,\alpha}(f)(i) = \begin{cases} \mathrm{in}_v(f(i)) & \text{if } \max\{N(f(i)) \cdot v\} + \alpha(i) = \mathrm{m}_{v,\alpha}(f), \\ 0 & \text{if } \max\{N(f(i)) \cdot v\} + \alpha(i) < \mathrm{m}_{v,\alpha}(f). \end{cases}$$

Note that if $f(i) = 0$ then $\mathrm{in}_{v,\alpha}(f)(i) = 0$, since we define $\max\{\emptyset\} = -\infty$. For a module $M \subset R^n$ we have the initial module $\mathrm{in}_{v,\alpha}(M) = \langle \mathrm{in}_{v,\alpha}(f) : f \in M \rangle \subset R^n$.

For a vector $f \in (R^{++})^n$ the 'face part' $\mathrm{in}_{v,\alpha}(f)$ for direction v and 'offset' α might or might not belong to $(R^{++})^n$ – see Figure 2.

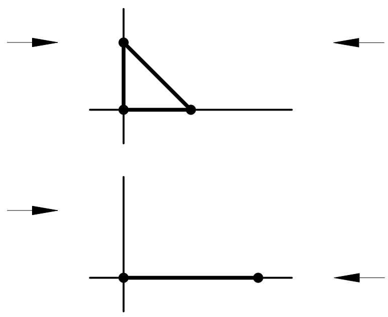

Figure 2: The figure shows the Newton polytopes of the components of $f = (1+x+y, 1+x^2)$. The arrows on the left indicate the levels which could contribute to the initial part for $v = (0, 1)$ and $\alpha_1 = (0, 0)$, on the right for $\alpha_2 = (0, 1)$. The initial part $\mathrm{in}_{v,\alpha_1}(f)$ for $v = (0, 1)$ and $\alpha_1 = (0, 0)$ is trivial in the second component, because the Newton polytope of $f(2)$ does not reach as far as the one for $f(1)$. The vector $\alpha_2 = (0, 1)$ compensates for this, and here $\mathrm{in}_{v,\alpha_2}(f) \in (R^{++})^2$.

Let B_k be the closed unit ball in $\mathbb{R}^n$. The first answer to the question in the title is the following.

(1.1) Theorem *For a submodule M of R^n we have $M \cap (R^{++})^n \neq \emptyset$ if and only if there exists a continuous function $v \mapsto \alpha$ from B_k to $\mathbb{R}^k$ such that for every $v \in B_k$ and $a \in (0,\infty)^k$ there exists $f \in \mathrm{in}_{v,\alpha}(M)$ with $f(a) > 0$.*

We will write $\alpha = \alpha_v$ when we wish to make explicit the dependence of α on v.

Note that the origin $v = 0$ plays a special role in the theorem. The module $\mathrm{in}_{0,\alpha}(M)$ is generated by $\mathrm{in}_{0,\alpha}(f)$, and $\mathrm{in}_{0,\alpha}(f) = f(i)$ for $\alpha(i) = \max_{i'} \alpha(i')$ and $f_\alpha(i) = 0$ otherwise. So if the function α_v satisfies the desired property, α_0 has to be constant vector, and $\mathrm{in}_{0,\alpha_0}(M)$ is M itself.

It is easy to see that the condition in Theorem 1.1 is necessary: Suppose $f \in M \cap (R^{++})^n$. For $v \in \mathbb{R}^k$ and $i \in \{1, \ldots, n\}$ put

$$\alpha(i) = \alpha_v(i) = -\max\{N(f(i)) \cdot v\}.$$

Then $\mathrm{m}_{v,\alpha}(f) = 0$ and $\mathrm{in}_{v,\alpha}(f)(i) = \mathrm{in}_v(f(i)) \in R^{++}$, so that $\mathrm{in}_{v,\alpha}(f)(a) > 0$ for all $a \in (0,\infty)^k$. It is clear from the definition of α_v that $v \mapsto \alpha_v$ is continuous.

Let D_k denote the unit sphere in $\mathbb{R}^k$. Let us refer to an element of D_k as a *direction* and say that $v \in D_k$ is *rational* if $tv \in \mathbb{Q}^k$ for some $t \in \mathbb{R}$. For $c \in \mathbb{R}^n$ let us agree to write $c > 0$ if and only if $c \in (0,\infty)^n$.

We obtain a slightly stronger version of the above theorem if we restrict the condition to rational directions $\widetilde{v}$, and the requirement on α_v to a neighborhood of $\widetilde{v}$.

(1.2) Theorem *For a submodule M of R^n we have $M \cap (R^{++})^n \neq \emptyset$ if and only if the following two conditions hold.*

(a) For every $a \in (0,\infty)^k$ there exists $f \in M$ such that $f(a) > 0$.

(b) For every rational $\widetilde{v} \in D_k$ we have a neighbourhood U of $\widetilde{v}$ in D_k and a continuous map $v \mapsto \alpha : U \to \mathbb{R}^n$ such that for every $v \in U$ and $a \in (0,\infty)^k$ there exists $f \in \mathrm{in}_{v,\alpha}(M)$ with $f(a) > 0$.

As an immediate corollary we have the analogous result for S: An S-submodule M of S^n intersects $(S^{++})^n$ if and only if (a) and (b) of the theorem hold. (Here, $\mathrm{in}_{v,\alpha}(M)$ may be replaced by the S-submodule of S^n generated by $\mathrm{in}_{v,\alpha}(f)$, $f \in M$.)

An example highlighting the importance of the continuity requirement of (b) may be found at the end of section 3. The sufficiency of (a) and (b) is proved in sections 2–5. In sections 2 and 3 we use universal Gröbner bases and an induction on the number k of variables to reduce (1.2) to the following.

(1.3) Theorem *For a submodule M of R^n we have $M \cap (R^{++})^n \neq \emptyset$ if and only if the following two conditions hold.*

(a) For every $a \in (0,\infty)^k$ there exists $f \in M$ such that $f(a) > 0$.

(b) For every rational $v \in D_k$ there exists $f \in M$ such that $\mathrm{in}_v(f) \in (R^{++})^n$.

The word 'rational' may be omitted and there is an analogous result for S. The proof of (1.3) is given in sections 4 and 5. In section 6 we show that it suffices to check (1.3)(b) for finitely many v. We then describe a procedure for determining whether M contains an element of $(R^{++})^n$ or not, and for finding such an element.

We mention that our interest in the questions addressed in [ET] and the present paper was kindled by our work in ergodic theory [MT].

We want to thank the referee for his careful reading of the manuscript, and his historical comments which we summarize here. The question in the title and its answer in the above theorems is strongly related or a direct generalization of earlier work by various authors. Poincaré considered in the paper [P] from 1883 the case of an ideal $\langle f \rangle$ generated by a single polynomial $f \in R_1$ and proved that there exists an integer N with $(1+x)^N f \in R_1^{++}$ if and only if $f(a) > 0$ for all $a > 0$. Adler and Gale considered in [AG] the problem of finding a sum $h_1 f_1 + \cdots h_m f_m \in R_1^{++}$ with $h_i \in R_1^{++}$, such polynomials exist if and only if $\max\{h_1(a), \ldots, h_m(a)\} > 0$ for all $a > 0$. Handelman answered in [H] the question in the case of an ideal $\langle f \rangle$ generated by a single polynomial $f \in R_k$, our proof relies on this statement (see section 5). The general case of an ideal in R_k was considered by Einsiedler and Tuncel in [ET].

2 Universal Gröbner bases

Let $e_1, \ldots, e_n$ be the standard basis of S^n. A *monomial* of S^n is an element of the form $x^u e_i$ for some $u \in (\mathbb{Z}^+)^k$ and $i \in \{1, \ldots, n\}$. A *term order* on the monomials of S^n is a total order $\prec$ satisfying the following two conditions:

(i) $e_i \prec x^u e_i$ for every $i \in \{1, \ldots, n\}$ and nonzero $u \in (\mathbb{Z}^+)^k$,

(ii) $x^u e_i \prec x^{u'} e_{i'}$ implies $x^{u+w} e_i \prec x^{u'+w} e_{i'}$ for all $i, i' \in \{1, \ldots, n\}$ and $u, u', w \in (\mathbb{Z}^+)^k$.

Let N be a submodule of S^n. An element $f \in S^n$ can be written uniquely as a sum

$$\sum_{u,i} f_{u,i}\, x^u e_i \,,$$

with coefficients $f_{u,i}$ in $\mathbb{R}$. Among the finitely many monomials of S^n that have nonzero coefficients in this sum, that which is maximal according to the term order $\prec$ is denoted $\mathrm{in}_\prec(f)$. From N we obtain the submodule $\mathrm{in}_\prec(N) = \langle \mathrm{in}_\prec(f) : f \in N \rangle$ of S^n, called the *initial module* of N with respect to $\prec$. Elements $f_1, \ldots, f_l \in N$ form a *Gröbner basis* for N with respect to $\prec$ if $\mathrm{in}_\prec(N) = \langle \mathrm{in}_\prec(f_j) : j = 1, \ldots, l \rangle$.

The basic facts we use from the theory of Gröbner bases can all be found in chapters 1 and 3 of [AL].

A *universal Gröbner basis* of N is given by elements $f_1, \ldots, f_l \in N$ that form a Gröbner basis of N with respect to every term order. The existence of universal bases for submodules N of S^n is just like that [W, S] for ideals of S: One verifies that N has finitely many initial modules by following the proof of (1.2) of [S], using (3.6.4) of [AL] in place of (1.1) of [S].

Fix a submodule M of R^n. Find $f_1, \ldots, f_l \in M$ that generate M as an R-module. Let $\delta = (\delta(1), \ldots, \delta(k)) \in \{-1, 1\}^k$. Pick $u \in \mathbb{Z}^k$ such that $x^u f_1, \ldots, x^u f_l \in (\mathbb{R}[x_1^{\delta(1)}, \ldots, x_k^{\delta(k)}])^n$, and let $f_{\delta,1}, \ldots, f_{\delta,l(\delta)}$ be a universal Gröbner basis for the $\mathbb{R}[x_1^{\delta(1)}, \ldots, x_k^{\delta(k)}]$-submodule of $(\mathbb{R}[x_1^{\delta(1)}, \ldots, x_k^{\delta(k)}])^n$ generated by $x^u f_1, \ldots, x^u f_l$. List the union of

$\{f_{\delta,1},\ldots,f_{\delta,l(\delta)}\}$ over $\delta\in\{-1,1\}^k$ as $g_1,\ldots,g_m$. We call $g_1,\ldots,g_m$ a *super Gröbner basis*; we will make use of it throughout the paper.

(2.1) Lemma *If* $v\in\mathbb{R}^k$ *and* $\alpha\in\mathbb{R}^n$ *then* $\mathrm{in}_{v,\alpha}(g_1),\ldots,\mathrm{in}_{v,\alpha}(g_m)$ *generate* $\mathrm{in}_{v,\alpha}(M)$.

Proof Let $\prec_0$ be an arbitrary term order on the monomials of S^n. Define $\delta\in\{-1,1\}^k$ by letting $\delta(i)=1$ if $v(i)\geq 0$ and $\delta(i)=-1$ if $v(i)<0$. Define the 'absolute vector'

$$|v|=\big(|v(1)|,\ldots,|v(k)|\big)=\big(\delta(1)v(1),\ldots,\delta(k)v(k)\big).$$

For $u,u'\in(\mathbb{Z}^+)^k$ and $1\leq i,i'\leq n$, we define a total order $\prec$ on $\big(\mathbb{R}[x_1^{\delta(1)},\ldots,x_k^{\delta(k)}]\big)^n$. We put $(x_1^{\delta(1)})^{u(1)}\cdots(x_k^{\delta(k)})^{u(k)}e_i\prec(x_1^{\delta(1)})^{u'(1)}\cdots(x_k^{\delta(k)})^{u'(k)}e_{i'}$ if

(1) $\alpha(i)+u\cdot|v|<\alpha(i')+u'\cdot|v|$, or

(2) $\alpha(i)+u\cdot|v|=\alpha(i')+u'\cdot|v|$ and $x^ue_i\prec_0 x^{u'}e_{i'}$.

This defines a term order $\prec$ on the monomials of $(\mathbb{R}[x_1^{\delta(1)},\ldots,x_k^{\delta(k)}])^n$ (treating $x_1^{\delta(1)},\ldots,x_k^{\delta(k)}$ as the independent variables). Let $f\in M$. Find $w\in\mathbb{Z}^k$ such that x^wf lies in the submodule $\langle f_{\delta,1},\ldots,f_{\delta,l(\delta)}\rangle$ of $(\mathbb{R}[x_1^{\delta(1)},\ldots,x_k^{\delta(k)}])^n$. Applying the division algorithm [AL] for Gröbner bases to x^wf and the subset $\{f_{\delta,1},\ldots,f_{\delta,l(\delta)}\}$ of $\{g_1,\ldots,g_m\}$ to find $p_j\in x^{-w}\mathbb{R}[x_1^{\delta(1)},\ldots,x_k^{\delta(k)}]$ such that $f=\sum_{j=1}^m p_jg_j$, we have $p_j=0$ if $g_j\notin\{f_{\delta,1},\ldots,f_{\delta,l(\delta)}\}$, and

$$\mathrm{in}_\prec(x^wf)=\max_\prec\{\mathrm{in}_\prec(x^wp_jg_j):j=1,\ldots,m\}.$$

Using (1) of the definition of $\prec$, it follows that

$$\mathrm{m}_{v,\alpha}(f)=\max\{\mathrm{m}_{v,\alpha}(p_jg_j):j=1,\ldots,m\}.$$

Letting $A=\{1\leq j\leq m:\mathrm{m}_{v,\alpha}(p_jg_j)=\mathrm{m}_{v,\alpha}(f)\}$, we have

$$\mathrm{in}_{v,\alpha}(f)=\sum_{j\in A}\mathrm{in}_{v,\alpha}(p_jg_j)=\sum_{j\in A}\mathrm{in}_v(p_j)\,\mathrm{in}_{v,\alpha}(g_j).\qquad\square$$

3 Directional positivity

The following inductive step will form the core of our proof of (1.2).

(3.1) Proposition *Suppose (1.2) is valid for fewer than k variables, and let M be a submodule of* $(R_k)^n$ *that satisfies (1.2)(b). Then for every rational* $v\in D_k$ *there exists* $f\in M$ *such that* $\mathrm{in}_v(f)\in(R^{++})^n$.

For the proof of (3.1), we may employ a suitable change of variables to assume without loss of generality that $v=-e_k=(0,\ldots,0,-1)$. Write $y=x_k$. For $f\in R^n$, put $\mathrm{m}(f)=\mathrm{m}_{v,\alpha}(f)$ and

$$\mathrm{in}^0_{v,\alpha}(f)(i)=y^{\mathrm{m}(f)-\alpha(i)}\,\mathrm{in}_{v,\alpha}(f)(i).$$

Note that the exponent of y is not always an integer. However, if $\mathrm{in}_{v,\alpha}(f)(i) \neq 0$ then $\max\{N(f(i)) \cdot (-e_k)\} + \alpha(i) = m(f)$ and the exponent is an integer. Furthermore is $\mathrm{in}^0_{v,\alpha}(f)(i) \in R_{k-1} = \mathbb{R}[x_1^\pm, \dots, x_{k-1}^\pm]$. Let $\mathrm{in}^0_{v,\alpha}(M)$ be the R_{k-1}-submodule of $(R_{k-1})^n$ generated by $\{\mathrm{in}^0_{v,\alpha}(f) : f \in M\}$.

(3.2) Lemma *If $\mathrm{in}^0_{v,\alpha}(M) \cap (R^{++}_{k-1})^n \neq \emptyset$ then there exists $f \in M$ such that $\mathrm{in}_v(f) \in (R^{++})^n$.*

Proof By assumption there exist $f_j \in M$ and $p_j \in R_{k-1}$ such that

$$g = \sum_j p_j \ \mathrm{in}^0_{v,\alpha}(f_j) \in (R^{++}_{k-1})^n \, .$$

We will use this expression to build f as in the lemma.

Since multiplying f_j with a power of y does not change $\mathrm{in}^0_{v,\alpha}(f_j)$, we can assume without loss of generality that $m(f_j) \in [0,1)$. Let $m_1 > \cdots > m_d$ be the ordered list of all values $m(f_j)$ which appear. We define $J_e = \{j : m(f_j) = m_e\}$ and

$$I_e = \{i : \mathrm{in}_{v,\alpha}(f_j)(i) \neq 0 \text{ for some } j \in J_e\}.$$

We claim $\{I_1, \dots, I_d\}$ partitions the set of indices $\{1, \dots, n\}$. So suppose $i \in I_e$, and let $j \in J_e$ be as above. By definition of $\mathrm{in}_{v,\alpha}(f_j)$ we get $\max\{N(f_j(i)) \cdot v\} + \alpha(i) = m(f_j) = m_e$. This shows $\alpha(i) \in m_e + \mathbb{Z}$. Since $1 > m_1 > \ldots > m_d \geq 0$ this determines e uniquely. Furthermore the union of the sets I_e must be $\{1, \dots, n\}$ because $g(i) \in R^{++}_{k-1}$ for every i.

Let i be fixed and j, j' be two different indices. Suppose $\mathrm{in}_{v,\alpha}(f_j)(i) \neq 0$, then

$$\begin{aligned} \max\{N(f_j(i)) \cdot v\} &= m(f_j) - \alpha(i) \\ &= m(f_{j'}) - \alpha(i) + (m(f_j) - m(f_{j'})) \\ &\geq \max\{N(f_{j'}(i)) \cdot v\} + (m(f_j) - m(f_{j'})). \end{aligned}$$

From this it is immediate that $m(f_j) = m(f_{j'})$ implies $\max\{N(f_j(i)) \cdot v\} \geq \max\{N(f_{j'}(i)) \cdot v\}$ and $m(f_j) > m(f_{j'})$ implies $\max\{N(f_j(i)) \cdot v\} > \max\{N(f_{j'}(i)) \cdot v\}$. The last possibility is $0 \leq m(f_j) < m(f_{j'}) < 1$, in this case $\max\{N(f_j(i)) \cdot v\} \geq \max\{N(f_{j'}(i)) \cdot v\}$.

For a fixed e we define $h_e = \sum_{j \in J_e} p_j f_j$. Let $i \in I_e$ be fixed, there exists $j \in J_e$ with $\mathrm{in}_{v,\alpha}(f_j)(i) \neq 0$. For every other $j' \in J_e$ we know $\max\{N(f_j(i)) \cdot v\} \geq \max\{N(f_{j'}(i)) \cdot v\}$, and the nonzero terms in the sum $\sum_{j \in J_e} p_j \, \mathrm{in}_{v,\alpha}(f_j)(i)$ have all the same y-degree. If $j' \notin J_e$, then $\mathrm{in}_{v,\alpha}(f_{j'})(i) = 0$. Therefore

$$\sum_{j \in J_e} p_j \, \mathrm{in}^0_{v,\alpha}(f_j)(i) = \sum_j p_j \, \mathrm{in}^0_{v,\alpha}(f_j)(i) = g(i)$$

and

$$\mathrm{in}_{v,\alpha}(h_e)(i) = \sum_{j \in J_e} p_j \, \mathrm{in}_{v,\alpha}(f_j)(i) \in R^{++}.$$

To construct a single element f with $\mathrm{in}_v(f) \in (R^{++})^n$, we need to combine the elements h_e. Consider the combination $f = h_1 + p'_2 h_2 + \cdots p'_d h_d$, where $p'_e \in R_{k-1}$ for

$e = 2, \dots, d$ will be specified later. Let $i \in I_1$ and $j \in J_1$ with $\mathrm{in}_{v,\alpha}(f_j)(i) \neq 0$, then for any $j' \notin J_1$ we know $\max\{N(f_j(i)) \cdot v\} > \max\{N(f_{j'}(i)) \cdot v\}$ from above. This shows $\mathrm{in}_v(f)(i) = \mathrm{in}_v(h_1)(i) \in R^{++}$ independent of the choice of p'_e for $e = 2, \dots, d$. Let now $i \in I_2$ and $j \in J_2$ with $\mathrm{in}_{v,\alpha}(f_j)(i) \neq 0$. If $j' \in J_1$, then $\max\{N(f_j(i)) \cdot v\} \geq \max\{N(f_{j'}(i)) \cdot v\}$. If $j' \notin J_1 \cup J_2$, then $\max N(f_j(i) \cdot v) > \max N(f_{j'}(i) \cdot v)$. Together we see that $\mathrm{in}_v(f)(i) = \mathrm{in}_v(h_1)(i) + p'_2 \,\mathrm{in}_v(h_2)(i)$ (unless the polynomial on the right vanishes). Since $\mathrm{in}_v(h_2)(i) \in R^{++}$ we can choose $p'_2 \in R^{++}_{k-1}$ so that $\mathrm{in}_v(f)(i) \in R^{++}$. We proceed inductively and find $p'_2, \dots, p'_d$ such that $\mathrm{in}_v(f) \in (R^{++})^n$. □

Proof of (3.1) Without loss of generality we assume that $v = -e_k = (0, \dots, 0, -1)$. We continue to write $y = x_k$. Let $\alpha = \alpha_v$ be as in (1.2)(b). Note that condition (1.2)(a) for the module $\mathrm{in}^0_{v,\alpha}(M)$ follows from condition (1.2)(b) for M. In view of (3.2) and the assumption that (1.2) is valid for submodules of $(R_{k-1})^n$, it suffices for the proof of (3.1) to exhibit a continuous function $w \mapsto \gamma_w : D_{k-1} \to \mathbb{R}^n$ such that for every $w \in D_{k-1}$ and $a \in (0,\infty)^{k-1}$ there exists $g \in \mathrm{in}_{w,\gamma}(\mathrm{in}^0_{v,\alpha}(M))$ with $g(a) > 0$.

Let us agree to identify $w \in D_{k-1}$ with $(w, 0) \in D_k$. For small $\epsilon > 0$, put $t = \sqrt{1-\epsilon^2}$, $\widetilde{w} = \epsilon w - t e_k$ and $\widetilde{\alpha} = \alpha_{\widetilde{w}}$. Let U be the neighbourhood of $v = -e_k$ to which (1.2)(b) applies. Pick $\epsilon > 0$ so small that for all $w \in D_{k-1}$, $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n\}$ we have

(1) $\widetilde{w} \in U$,

(2) $\mathrm{in}_{\widetilde{w}}(g_j(i)) = \mathrm{in}_w(\mathrm{in}_v(g_j(i)))$,

(3) if $\max\{N(g_j(i)) \cdot v\} + \alpha_v(i) < \mathrm{m}_{v,\alpha}(g_j)$ then $\max\{N(g_j(i)) \cdot \widetilde{w}\} + \alpha_{\widetilde{w}}(i) < \mathrm{m}_{\widetilde{w},\widetilde{\alpha}}(g_j)$.

Define

$$\gamma(i) = \gamma_w(i) = \left(\alpha_{\widetilde{w}}(i) - t\alpha_v(i)\right)/\epsilon.$$

Observe that continuity of α on U both makes (3) possible and implies the continuity of $w \mapsto \gamma_w : D_{k-1} \to \mathbb{R}^n$.

Now fix $w \in D_{k-1}$.

(3.3) Lemma *For all j, i we have*

$$(*) \qquad \mathrm{in}_{\widetilde{w},\widetilde{\alpha}}(g_j)(i) = y^{\alpha(i) - \mathrm{m}(g_j)} \,\mathrm{in}_{w,\gamma}(\mathrm{in}^0_{v,\alpha}(g_j))(i).$$

Proof Observe that (3) of our choice of ϵ means that we have $\mathrm{in}_{\widetilde{w},\widetilde{\alpha}}(g_j)(i) = 0$ whenever $\mathrm{in}^0_{v,\alpha}(g_j)(i) = 0$, ensuring that the lemma holds whenever $\mathrm{in}^0_{v,\alpha}(g_j)(i) = 0$. Fix j and let i, i' be such that $\mathrm{in}^0_{v,\alpha}(g_j)(i)$, $\mathrm{in}^0_{v,\alpha}(g_j)(i') \neq 0$. From (2) and the definition of $\widetilde{w}$ we have

$$\begin{aligned} & \max\{N(g_j(i)) \cdot \widetilde{w}\} + \widetilde{\alpha}(i) \\ = \; & \max\{N(y^{\alpha(i) - \mathrm{m}(g_j)} \,\mathrm{in}_w(\mathrm{in}^0_{v,\alpha}(g_j)(i))) \cdot \widetilde{w}\} + \widetilde{\alpha}(i) \\ = \; & \epsilon \max\{N(\mathrm{in}_{v,\alpha}(g_j)(i)) \cdot w\} - t(\alpha(i) - \mathrm{m}(g_j)) + \widetilde{\alpha}(i). \end{aligned}$$

Using the analogous equalities for i', we see that the equality

$$\max\{N(g_j(i)) \cdot \widetilde{w}\} + \widetilde{\alpha}(i) = \max\{N(g_j(i')) \cdot \widetilde{w}\} + \widetilde{\alpha}(i')$$

holds if and only if

$$\epsilon \max\{N(\mathrm{in}_{v,\alpha}(g_j)(i)) \cdot w\} + \widetilde{\alpha}(i) - t\alpha(i) = \epsilon \max\{N(\mathrm{in}_{v,\alpha}(g_j)(i')) \cdot w\} + \widetilde{\alpha}(i') - t\alpha(i'),$$

which happens if and only if

$$\max\{N(\mathrm{in}_{v,\alpha}(g_j)(i)) \cdot w\} + \gamma(i) = \max\{N(\mathrm{in}_{v,\alpha}(g_j)(i')) \cdot w\} + \gamma(i').$$

This means that the left-hand side of $(*)$ is nonzero if and only if the right-hand side is nonzero. Recalling that, by (2), $\mathrm{in}_{\widetilde{w}}(g_j(i)) = y^{\alpha(i)-\mathrm{m}(g_j)}\,\mathrm{in}_w(\mathrm{in}^0_{v,\alpha}(g_j)(i))$ whenever $\mathrm{in}^0_{v,\alpha}(g_j)(i)$ is nonzero, the lemma is proved. □

Returning to the proof of (3.1), consider $a \in (0,\infty)^{k-1}$. Put $\widetilde{a} = (a, 1)$. Since M satisfies (1.2)(b), there exists $f \in \mathrm{in}_{\widetilde{w},\widetilde{\alpha}}(M)$ with $f(\widetilde{a}) > 0$. Use (2.1) to find $p_j \in R$ such that

$$f = \sum_{j=1}^{m} p_j \ \mathrm{in}_{\widetilde{w},\widetilde{\alpha}}(g_j)\,.$$

By (3.3),

$$f(i) = \sum_j p_j\, y^{\mathrm{m}(g_j)-\alpha(i)}\ \mathrm{in}_{w,\gamma}(\mathrm{in}^0_{v,\alpha}(g_j))(i).$$

Define $q_j \in R_{k-1}$ by letting $q_j(x_1,\ldots,x_{k-1}) = p_j(x_1,\ldots,x_{k-1},1)$ and evaluate the last equation at $\widetilde{a}$:

$$\sum_j q_j(a)\ \mathrm{in}_{w,\gamma}(\mathrm{in}^0_{v,\alpha}(g_j))(i)(a) = f(i)(\widetilde{a}) > 0.$$

Hence, $g = \sum\limits_j q_j\ \mathrm{in}_{w,\gamma}(\mathrm{in}^0_{v,\alpha}(g_j))$ is an element of $\mathrm{in}_{w,\gamma}(\mathrm{in}^0_{v,\alpha}(M))$ with $g(a) > 0$. This completes the proof of (3.1). □

We are now in a position to deduce (1.2) from (1.3).

Proof of (1.2) We have already observed that (a) and (b) are necessary. To deduce their sufficiency from (1.3), we verify that (1.2)(b) implies (1.3)(b): When $k = 1$, for any $v \in D_1 = \{-1, 1\}$ and $f \in R^n$, each entry $\mathrm{in}_v(f)(i)$ consists of a single term and the existence of $f \in M$ with $\mathrm{in}_v(f) \in (R^{++})^n$ is immediate from (1.2)(b). Induction on k, with (3.1) as the inductive step, does the rest. □

We end the section with an example showing that the requirement that α vary continuously with v cannot be dropped from (1.2)(b).

(3.4) Example Consider the case $k = 3$ and write x, y, z for the three variables. Let $p_1 = 1 + x^3 + y^3 - 3xy$ and $p_2 = 1 - 2x + x^2 + y$. Since p_1 vanishes when $x = y = 1$ and $\mathrm{in}_{(0,0,-1)}(p_2) = (1-x)^2$ vanishes when $x = 1$, no multiple of either of p_1, p_2 can belong to R^{++}. Let

$$f_1 = (zp_1 + 1, zp_1 + 1),\ \ f_2 = (zp_2, 1)\,,$$

and let M be the submodule of R^2 generated by f_1 and f_2. Put $w = (0, 0, 1)$. To see that M does not intersect $(R^{++})^2$, suppose $q_1, q_2 \in R$ are such that $g = q_1 f_1 + q_2 f_2 \in (R^{++})^2$. Then $\text{in}_w(g(2)) \in R^{++}$. Since no multiple of p_1 belongs to R^{++}, this means that the z-degree of q_2 must exceed that of q_1; that is,

$$\max\{N(q_2) \cdot w\} > \max\{N(q_1) \cdot w\} .$$

On the other hand, since $\text{in}_w(g(1)) \in R^{++}$ and no multiple of p_2 belongs to R^{++},

$$\max\{N(q_2) \cdot w\} \leq \max\{N(q_1) \cdot w\} .$$

These contradictory inequalities reveal that $M \cap (R^{++})^2 = \emptyset$.

Observe that M satisfies (1.2)(a) since $f_2(a) > 0$ for all $a \in (0, \infty)^3$. Also note that $\text{in}_v(f_1) \in (R^{++})^2$ for all $v \in D_3 \setminus \{w\}$. Define $v \mapsto \alpha_v : D_3 \to \mathbb{R}^2$ by putting $\alpha_w = (0, 1)$ and letting $\alpha_v = (0, 0)$ for all $v \in D_3 \setminus \{w\}$. Then $\text{in}_{w,\alpha_w}(f_2) = f_2$, and $\text{in}_{v,\alpha_v}(f_1) = \text{in}_v(f_1) \in (R^{++})^2$ for all $v \in D_3 \setminus \{w\}$. Hence (1.2)(b) is satisfied, except for the fact that $v \mapsto \alpha_v$ has a discontinuity at w.

4 Irrational directions

In preparation for the proof of (1.3), we next show that positivity in irrational directions follows from that in rational directions. We continue to work with the super Gröbner basis $g_1, \ldots, g_m$ constructed in section 2.

(4.1) Lemma *Let $\widetilde{v} \in D_k$ and $\epsilon > 0$. There exists a rational direction $v \in D_k$ such that $\|v - \widetilde{v}\| < \epsilon$ and $\text{in}_v(g_j(i)) = \text{in}_{\widetilde{v}}(g_j(i))$ for all $j \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, n\}$.*

Proof For each i, j such that $g_j(i) \neq 0$, pick $u_{i,j} \in \text{Log}(\text{in}_{\widetilde{v}}(g_j(i)))$. The condition $\text{in}_v(g_j(i)) = \text{in}_{\widetilde{v}}(g_j(i))$ amounts to the requirements that $u \cdot v = u_{i,j} \cdot v$ for all $u \in \text{Log}(\text{in}_{\widetilde{v}}(g_j(i)))$ and that $u \cdot v < u_{i,j} \cdot v$ for all $u \in \text{Log}(g_j(i)) \setminus \text{Log}(\text{in}_{\widetilde{v}}(g_j(i)))$. Treating $v(1), \ldots, v(k)$ as variables and running through all i, j with $g_j(i) \neq 0$, we obtain a set of homogeneous linear equations and a set of strict linear inequalities in $v(1), \ldots, v(k)$. These equations and inequalities have integral coefficients. Reduce the equations to echelon form. The reduced equations will have rational coefficients. Since the equations have a nontrivial solution, namely $\widetilde{v}$, the echelon form contains fewer than k equations, leaving $l \geq 1$ of the variables $v(1), \ldots, v(k)$ free. For the l free variables choose rational values close to the corresponding entries of $\widetilde{v}$, and use the reduced equations to determine the values of the remaining $k - l$ variables. The resulting rational direction $\frac{v}{\|v\|}$ may be made arbitrarily close to $\widetilde{v}$ by choosing the values of the free variables sufficiently close the the corresponding entries of $\widetilde{v}$, and $\frac{v}{\|v\|}$ will satisfy the set of inequalities because $\widetilde{v}$ does. □

(4.2) Proposition *Suppose M is a submodule of R^n and for every rational $v \in D_k$ we have $f \in M$ with $\text{in}_v(f) \in (R^{++})^n$. Then for every $v \in D_k$ there exists $g \in M$ with $\text{in}_v(g) \in (R^{++})^n$.*

Proof Let $\widetilde{v} \in D_k$ and use (4.1) to find rational $v \in D_k$ such that $\mathrm{in}_v(g_j(i)) = \mathrm{in}_{\widetilde{v}}(g_j(i))$ for all j, i and $v(i)\widetilde{v}(i) > 0$ whenever $\widetilde{v}(i) \neq 0$. By assumption there is $f \in M$ such that, writing $\beta(i) = -\max\{N(f(i)) \cdot v\}$, we have $\mathrm{in}_{v,\beta}(f) \in (R^{++})^n$. Put

$$\widetilde{\beta}(i) = -\max\{N(\mathrm{in}_v(f(i))) \cdot \widetilde{v}\},$$

so that $\mathrm{m}_{\widetilde{v},\widetilde{\beta}}(\mathrm{in}_{v,\beta}(f)) = 0$ and $\mathrm{in}_{\widetilde{v},\widetilde{\beta}}(\mathrm{in}_{v,\beta}(f)) \in (R^{++})^n$. Let $\prec_0$ be an arbitrary term order on the monomials of S^n. Define $\delta \in \{-1, 1\}^k$ by letting $\delta(i) = 1$ if $v(i) \geq 0$ and $\delta(i) = -1$ if $v(i) < 0$. Let $|v|$ and $|\widetilde{v}|$ denote the elements of $\mathbb{R}^k$ with $|v|(i) = |v(i)|$ and $|\widetilde{v}|(i) = |\widetilde{v}(i)|$. For $u, u' \in (\mathbb{Z}^+)^k$ and $i, i' \in \{1, \ldots, n\}$, put $(x_1^{\delta(1)})^{u(1)} \cdots (x_k^{\delta(k)})^{u(k)} e_i \prec (x_1^{\delta(1)})^{u'(1)} \cdots (x_k^{\delta(k)})^{u'(k)} e_{i'}$ if

(1) $\beta(i) + u \cdot |v| < \beta(i') + u' \cdot |v|$, or

(2) $\beta(i) + u \cdot |v| = \beta(i') + u' \cdot |v|$ and $\widetilde{\beta}(i) + u \cdot |\widetilde{v}| < \widetilde{\beta}(i') + u' \cdot |\widetilde{v}|$, or

(3) $\beta(i) + u \cdot |v| = \beta(i') + u' \cdot |v|$, $\widetilde{\beta}(i) + u \cdot |\widetilde{v}| = \widetilde{\beta}(i') + u' \cdot |\widetilde{v}|$ and $x^u e_i \prec_0 x^{u'} e_{i'}$.

This defines a term order on the monomials of $(\mathbb{R}[x_1^{\delta(1)}, \ldots, x_k^{\delta(k)}])^n$. As in the proof of (2.1), we have $w \in \mathbb{Z}^k$ and $p_j \in R$ such that $x^w f, x^w p_j g_j \in (\mathbb{R}[x_1^{\delta(1)}, \ldots, x_k^{\delta(k)}])^n$, $f = \sum_{j=1}^{m} p_j g_j$ and

$$\mathrm{in}_\prec(x^w f) = \max_\prec\{\mathrm{in}_\prec(x^w p_j g_j) : j = 1, \ldots, m\}.$$

Considering (1) of the definition of $\prec$ and letting

$$A = \{1 \leq j \leq m : \mathrm{m}_{v,\beta}(f) = \mathrm{m}_{v,\beta}(p_j g_j)\},$$

we obtain

$$\mathrm{in}_{v,\beta}(f) = \sum_{j \in A} \mathrm{in}_{v,\beta}(p_j g_j) \in (R^{++})^n.$$

Furthermore, considering (2) of the definition of $\prec$ and letting

$$B = \{j \in A : \mathrm{m}_{\widetilde{v},\widetilde{\beta}}(\mathrm{in}_{v,\beta}(p_j g_j)) = 0\},$$

we have

$$(*) \qquad \mathrm{in}_{\widetilde{v},\widetilde{\beta}}(\mathrm{in}_{v,\beta}(f)) = \sum_{j \in B} \mathrm{in}_{\widetilde{v},\widetilde{\beta}}(\mathrm{in}_{v,\beta}(p_j g_j)) \in (R^{++})^n.$$

Let $i \in \{1, \ldots, n\}$. Put $B_i = \{j \in B : \mathrm{in}_{\widetilde{v},\widetilde{\beta}}(\mathrm{in}_{v,\beta}(p_j g_j))(i) \neq 0\}$, so that $(*)$ and the fact that $\mathrm{in}_{\widetilde{v}}(g_j(i)) = \mathrm{in}_v(g_j(i))$ imply

$$(**) \qquad \mathrm{in}_{\widetilde{v}}(\mathrm{in}_v(f(i))) = \sum_{j \in B_i} \mathrm{in}_{\widetilde{v}}(\mathrm{in}_v(p_j))\, \mathrm{in}_{\widetilde{v}}(g_j(i)) \in R^{++}.$$

Consider $g = \sum_{j \in B} \mathrm{in}_v(p_j)\, g_j \in M$. The polynomials $h_j = \mathrm{in}_v(p_j)$ satisfy $h_j = \mathrm{in}_v(h_j)$. Note that

$$\mathrm{in}_{\widetilde{v}}(h_j g_j(i)) = \mathrm{in}_{\widetilde{v}}(h_j)\, \mathrm{in}_{\widetilde{v}}(g_j(i)) = \mathrm{in}_{\widetilde{v}}(h_j)\, \mathrm{in}_{\widetilde{v}}(\mathrm{in}_v(g_j(i))) = \mathrm{in}_{\widetilde{v}}(\mathrm{in}_v(h_j g_j(i))).$$

Since B consists of those $j \in A$ where the maximum $\mathrm{m}_{\widetilde{v},\widetilde{\beta}}(\mathrm{in}_{v,\beta}(p_j g_j)) = \mathrm{m}_{\widetilde{v},\widetilde{\beta}}(h_j \, \mathrm{in}_{v,\beta}(g_j))$ is attained, we also have

$$\begin{aligned}
& \max_{j \in B} \max\{N(h_j g_j(i)) \cdot \widetilde{v}\} \\
= \; & \max_{j \in B} \max\{N(\mathrm{in}_v(h_j g_j(i))) \cdot \widetilde{v}\} \\
= \; & \max_{j \in A} \max\{N(\mathrm{in}_{v,\beta}(h_j g_j)(i)) \cdot \widetilde{v}\} \\
= \; & \max\{N(\mathrm{in}_{v,\beta}(f)(i)) \cdot \widetilde{v}\} = -\widetilde{\beta}(i).
\end{aligned}$$

As $B_i = \{j \in B : N(\mathrm{in}_{\widetilde{v}}(\mathrm{in}_v(p_j) g_j(i))) \cdot \widetilde{v} = -\widetilde{\beta}(i)\}$, it then follows from $(**)$ and the definition of g that

$$\mathrm{in}_{\widetilde{v}}(g)(i) = \mathrm{in}_{\widetilde{v},\widetilde{\beta}}(g)(i) = \sum_{j \in B_i} \mathrm{in}_{\widetilde{v}}(\mathrm{in}_v(p_j) g_j(i)) \in R^{++}. \qquad \square$$

5 Gluing

We will prove (1.3) by gluing together various elements of M to come up with an element whose components satisfy (ii) of the following theorem of Handelman[H].

(5.1) Theorem *[H] For $p \in R$ the following are equivalent.*

(i) There exists $q \in R^{++}$ such that $qp \in R^{++}$.

(ii) $\mathrm{in}_v(p)(a) > 0$ for all $v \in \mathbb{R}^k$ and $a \in (0, \infty)^k$.

Handelman's theorem may be viewed as dealing with principal ideals of R; it was also used in proving the result of [ET] for arbitrary ideals. A short self-contained proof of (5.1) may be found in [DT].

We continue to work with a submodule $M \subset R^n$.

(5.2) Lemma *Suppose that for every $v \in D_k$ we have $g \in M$ with $\mathrm{in}_v(g) \in (R^{++})^n$. Then there exists $f \in M$ such that $\mathrm{in}_v(f) \in (R^{++})^n$ for every $v \in D_k$.*

Proof For $v \in D_k$, let $f_v \in M$ be such that $\mathrm{in}_v(f_v) \in (R^{++})^n$. Note that if $v' \in D_k$ is close enough to v we have $\mathrm{in}_{v'}(f_v) = \mathrm{in}_{v'}(\mathrm{in}_v(f_v)) \in (R^{++})^n$. Hence, there exists $\epsilon_v > 0$ such that $\mathrm{in}_{v'}(f_v) \in (R^{++})^n$ for all v' in the ϵ_v-ball $B(v, \epsilon_v)$ around v. The set of all such balls, $\mathcal{B} = \{B(v, \epsilon_v) : v \in D_k\}$, forms an open cover of the compact set D_k and hence has a Lebesgue number, say 2λ with $\lambda < 1$.

Take a finite collection of balls of radius λ which cover D_k, and label their centers $v_1, \ldots, v_m$. Note that each ball $B(v_j, \lambda)$, $j \in \{1, \ldots, m\}$, is contained in some $B(v'_j, \epsilon_{v'_j}) \in \mathcal{B}$ and let $f_j = f_{v'_j}$, so that $\mathrm{in}_v(f_j) \in (R^{++})^n$ for all $v \in B(v_j, \lambda)$. Let 2κ be a Lebesgue number for the cover $\{B(v_j, \lambda) : j = 1, \ldots, m\}$ of D_k. Then for any $v \in D_k$ there exists $j \in \{1, \ldots, m\}$ such that $B(v, \kappa) \subset B(v_j, \lambda)$ and, in particular, $\|v_j - v\| < \lambda - \kappa$.

Let δ be the infimum of

$$\{v \cdot v_j - v \cdot v_{j'} : v \in D_k\,, \|v_j - v\| < \lambda - \kappa\,, \|v_{j'} - v\| \geq \lambda\,, \, j, j' \in \{1, \dots, m\}\}\,.$$

Note that $\delta \geq \kappa(\lambda - \frac{\kappa}{2}) > 0$ since for all $v, w, w' \in D_k$ we have

$$v \cdot w - v \cdot w' = \frac{1}{2}\left(\|w' - v\|^2 - \|w - v\|^2\right).$$

Choose r large enough that $N(f_j(i)) \subset B(0, \frac{\delta}{2}r - \frac{\sqrt{k}}{2})$ for all $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n\}$.

For $j = 1, \dots, m$ pick an integral vector w_j such that $\|w_j - rv_j\| \leq \frac{\sqrt{k}}{2}$. Let

$$f = \sum_{j=1}^{m} x^{w_j} f_j\,.$$

Consider any $v \in D_k$ and $i \in \{1, \dots, n\}$. Let $j \in \{1, \dots, m\}$ be such that $\|v - v_j\| < \lambda - \kappa$. For all $j' \in \{1, \dots, m\}$ with $\|v - v_{j'}\| \geq \lambda$ we have

$$\begin{aligned} \max\left\{N\left(x^{w_{j'}} f_{j'}(i)\right) \cdot v\right\} &< r\, v \cdot v_{j'} + \frac{\delta}{2} r \\ &\leq r\, v \cdot v_j - \frac{\delta}{2} r \\ &< \min\left\{N\left(x^{w_j} f_j(i)\right) \cdot v\right\}. \end{aligned}$$

For the remaining indices j' with $\|v - v_{j'}\| < \lambda$ we know already that $\mathrm{in}_v(f_{j'}) \in (R^{++})^n$. Since there can be no cancellation with those initial parts, we get

$$\mathrm{in}_v(f) = \mathrm{in}_v\left(\sum_{j' : \|v - v_{j'}\| < \lambda} x^{w_{j'}} f_{j'}\right) \in (R^{++})^n. \qquad \square$$

Let $p \in R$. Put $|p| = \sum_{u \in \mathrm{Log}(p)} |p_u| x^u$ and, considering the boundary $\partial N(p)$ of the Newton polyhedron of p, let $p_\partial = \sum_{u \in \partial N(p)} p_u x^u$ and $p_c = p - p_\partial$. For $v \in \mathbb{R}^k$, write $\mathrm{m}_v(p) = \max\{N(p) \cdot v\}$ and $e^v = (e^{v(1)}, \dots, e^{v(k)})$. Let $\underline{1} = (1, \dots, 1) \in \mathbb{R}^k$.

(5.3) Lemma *Let $p \in R$ be such that $\mathrm{in}_v(p) \in R^{++}$ for every $v \in D_k$. There exists $d > 0$ such that*

$$p(e^{tv}) \geq e^{\mathrm{m}_v(p)t}\left(\mathrm{in}_v(p)(\underline{1}) - e^{-dt}|p_c|(\underline{1})\right)$$

for all $v \in D_k$ and $t \geq 0$. In particular, there is a compact set $K \subset (0, \infty)^k$ such that $p(a) > 0$ for all $a \in (0, \infty)^k \backslash K$.

Proof Note that $\mathrm{Log}(p_c) = \mathrm{Log}(p)\backslash\partial N(p)$. Use the compactness of D_k to find $d > 0$ so that, putting $m_v = \mathrm{m}_v(p)$, we have

$$m_v - \mathrm{m}_v(p_c) = \max\{\mathrm{Log}(p)\cdot v\} - \max\{\mathrm{Log}(p_c)\cdot v\} \geq d$$

for all $v \in D_k$. For $t \geq 0$ and $v \in D_k$, put $a = e^{tv}$. Observe that, for $u \in \mathbb{Z}^k$,

$$a^u = \prod_i a(i)^{u(i)} = \prod_i e^{tv(i)u(i)} = e^{t(u\cdot v)}\,.$$

Also using the fact that $p_\partial \in R^{++}$, we have

$$\begin{aligned} p(e^{tv}) &= p(a) = p_\partial(a) + p_c(a) \\ &\geq \mathrm{in}_v(p)(a) + p_c(a) \\ &= e^{m_v t}\,\mathrm{in}_v(p)(\underline{1}) + \sum_{u\in\mathrm{Log}(p_c)} p_u\, e^{t(u\cdot v)} \\ &\geq e^{m_v t}\,\mathrm{in}_v(p)(\underline{1}) - e^{(m_v-d)t} \sum_{u\in\mathrm{Log}(p_c)} |p_u| \\ &= e^{m_v t}\left(\mathrm{in}_v(p)(\underline{1}) - e^{-dt}|p_c|(\underline{1})\right). \end{aligned}$$

□

Proof of (1.3) Clearly (a) and (b) are necessary. For the converse, use (4.2) and (5.2) to find $f \in M$ such that $\mathrm{in}_v(f) \in (R^{++})^n$ for every $v \in D_k$. Applying (5.3) to the entries of f, pick $C > 1$ so that we have $f(a) > 0$ whenever $a \in (0,\infty)^k\backslash[C^{-1}, C]^k$. Put $K = [C^{-1}, C]^k$ and $\widetilde{K} = [(3kC)^{-1}, 3kC]^k$. Use (1.3)(a) to find $h_1, \ldots, h_l \in M$, $a_1, \ldots, a_l \in \widetilde{K}$ and $r_1, \ldots, r_l > 0$ such that the open balls $B(a_j, r_j)$ cover $\widetilde{K}$ and $h_j > 0$ on $B(a_j, 2r_j)$. For small $\delta > 0$, let $q_j \in R$ be such that $|q_j - 1| < \delta$ on $B(a_j, r_j)$ and $|q_j| < \delta$ on $\widetilde{K}\backslash B(a_j, \frac{3}{2}r_j)$. Pick $\delta > 0$ small enough for $g = \sum_{j=1}^{l} q_j h_j$ to have $g(a) > 0$ for all $a \in \widetilde{K}$. Letting

$$q(x_1, \ldots, x_k) = \frac{1}{2k}\sum_{i=1}^{k} \frac{1}{C}(x_i + x_i^{-1}) \in R\,,$$

fix $\epsilon > 0$ small enough to have $\epsilon f(a) + g(a) > 0$ for all $a \in K$. We will complete the proof by showing that, for sufficiently large $N \in \mathbb{N}$, every entry of $h_N = \epsilon\, q^N f + g \in M$ satisfies (5.1)(ii).

First note that for some $N_0 \in \mathbb{N}$ the Newton polytope of g will be in the interior of that of $\epsilon\, q^{N_0} f$ and we will have

$$\mathrm{in}_v(h_N) = \mathrm{in}_v(\epsilon\, q^N f) = \epsilon\ \mathrm{in}_v(q)^N\, \mathrm{in}_v(f) \in (R^{++})^n$$

for all $v \neq 0$ and $N \geq N_0$. Considering the case $v = 0$, we need to make sure that $h_N(a) > 0$ for all $a \in (0,\infty)^k$.

Since $0 < q \leq 1$ on K, our choice of ϵ guarantees that $h(a) > 0$ for all $a \in K$. In fact, $h(a) > 0$ for all $a \in \widetilde{K}$ since both $f, g > 0$ on $\widetilde{K}\backslash K$. By Lemma 5.3, $h_{N_0}(a) > 0$ for

$a \in (0,\infty)^k \setminus L$ where $L \supset \widetilde{K}$ is a compact set. Let $N \geq N_0$ and $a \in (0,\infty)^k \setminus \widetilde{K}$. Since $0 < a(i) \notin ((3kC)^{-1}, 3kC)$ for some i, we have $\frac{1}{2C}(a(i) + a(i)^{-1}) \geq \frac{3k}{2}$ and $q(a) \geq 3/2$. Since $f(a) > 0$ we see that $h_N(a) > 0$ implies $h_{N+1}(a) > 0$. Therefore $h_N(a) > 0$ for $a \in (0,\infty)^k \setminus L$ and $N \geq N_0$. Finally consider $a \in L \setminus \widetilde{K}$. In this case we have

$$h_N(a) \geq \epsilon\,(3/2)^N f(a) - g(a),$$

and for each a the last quantity will be positive for sufficiently large N. By compactness, for large N we will also have $h_N(a) > 0$ for all $a \in L \setminus \widetilde{K}$. □

6 A finite set of directions

In this section we use the super Gröbner basis $g_1, \ldots, g_m$ to show that it is enough to verify (1.3)(b) for finitely many $v \in D_k$. We then describe a procedure for checking whether a given module $M \subset R^n$ contains a positive element and for finding such an element. The procedure will be based on (1.3); it will use recursion on the number of variables, as in the proof of (1.2).

For every polynomial $p \in R_k$ there exists a finite partition of D_k such that two directions of the same partition element give you the same initial part of p; in the terminology of polyhedral geometry this partition is the intersection of the normal fan to $N(p)$ with D_k (see Chapter 2 in [S]). The next lemma can be considered as a generalization to a module, but first we need some notation.

Let $v \in D_k, \alpha \in \mathbb{R}^n$ and let $x^u e_i$ be a monomial of R^n. We introduce the new variables $t_1, \ldots, t_n$ and define an R-module homomorphism

$$\phi : M \to R[t_1, \ldots, t_n]$$

by letting $\phi(x^u e_i) = x^u t_i = x^u t^{e_i}$. Then

$$\mathrm{m}_{v,\alpha}(x^u e_i) = u \cdot v + \alpha(i) = (u, e_i) \cdot (v, \alpha)$$

and therefore we have

$$(\dagger) \qquad \phi(\mathrm{in}_{v,\alpha}(f)) = \mathrm{in}_{(v,\alpha)}(\phi(f))$$

for every $f \in M$.

Since $\phi(g_j)$ is a polynomial there exists a partition of D_{k+n} such that for any two directions in the same partition element the initial parts are the same. Let $\mathcal{Q}$ be the common refinement of the partitions associated to the polynomials $\phi(g_1), \ldots, \phi(g_m)$.

Using the map $\rho : D_k \times \mathbb{R}^n \to D_{k+n} : (v,\alpha) \mapsto \frac{(v,\alpha)}{\|(v,\alpha)\|}$, we can consider $D_k \times \mathbb{R}^n$ as a subset of D_{k+n}. We also have the scaled projection $\pi : \mathrm{Im}(\rho) \subseteq D_{k+n} \to D_k$ with

$$\pi((v(1), \ldots, v(k+n))) = (v(1), \ldots, v(k))/\|(v(1), \ldots, v(k))\|\,.$$

For each $Q \in \mathcal{Q}$ we consider the two-element partition $\{\pi(Q), D_k \setminus \pi(Q)\}$ and define

$$\mathcal{P} = \bigvee_{Q \in \mathcal{Q}} \{\pi(Q), D_k \setminus \pi(Q)\}.$$

The following lemma clarifies the connection between the finite partitions $\mathcal{P}, \mathcal{Q}$ and the initial modules $\mathrm{in}_{v,\alpha}(M)$.

(6.1) Lemma *For a submodule M of R^n there are finitely many initial modules $\mathrm{in}_{v,\alpha}(M)$. In fact, we have $\mathrm{in}_{v_1,\alpha_1}(g_j) = \mathrm{in}_{v_2,\alpha_2}(g_j)$ and $\mathrm{in}_{v_1,\alpha_1}(M) = \mathrm{in}_{v_2,\alpha_2}(M)$ whenever $\rho((v_1,\alpha_1))$ and $\rho((v_2,\alpha_2))$ belong to the same element of $\mathcal{Q}$. In addition, if $v_1, v_2 \in D_k$ belong to the same element of $\mathcal{P}$ and $\alpha_1 \in \mathbb{R}^n$, then there exists $\alpha_2 \in \mathbb{R}^n$ such that $\mathrm{in}_{v_1,\alpha_1}(g_j) = \mathrm{in}_{v_2,\alpha_2}(g_j)$ and $\mathrm{in}_{v_1,\alpha_1}(M) = \mathrm{in}_{v_2,\alpha_2}(M)$.*

Proof Assume that $\rho((v_1,\alpha_1))$ and $\rho((v_2,\alpha_2))$ lie in the same element of $\mathcal{Q}$. Since $\mathcal{Q}$ is defined as the refinement of the partitions associated to $\phi(g_j)$ we have

$$\mathrm{in}_{(v_1,\alpha_1)}(\phi(g_j)) = \mathrm{in}_{(v_2,\alpha_2)}(\phi(g_j)),$$

and it follows from (†) that $\mathrm{in}_{v_1,\alpha_1}(g_j) = \mathrm{in}_{v_2,\alpha_2}(g_j)$ for $j = 1 \dots m$. We obtain $\mathrm{in}_{v_1,\alpha_1}(M) = \mathrm{in}_{v_2,\alpha_2}(M)$ by (2.1).

For the final assertion, suppose v_1 and v_2 belong to the same element P of $\mathcal{P}$ and $\alpha_1 \in \mathbb{R}^n$. Let Q be the element of $\mathcal{Q}$ such that $\rho((v_1,\alpha_1)) \in Q$. Since v_1 belongs to both P and $\pi(Q)$, the set P is contained in $\pi(Q)$. Hence, for $v_2 \in P$ there exists $\alpha_2 \in \mathbb{R}^n$ with $\rho((v_2,\alpha_2)) \in Q$. Since $\mathrm{in}_{v_1,\alpha_1}(M) = \mathrm{in}_{tv_1,t\alpha_1}(M)$ for any $t > 0$, we then have $\mathrm{in}_{v_1,\alpha_1}(M) = \mathrm{in}_{v_2,\alpha_2}(M)$ by the first part of the lemma. □

We can now construct an element of M that is positive for all directions in P.

(6.2) Lemma *Let $P \in \mathcal{P}$ and assume that after a change of coordinates P is an open subset of $D_k \cap \langle e_1, \dots, e_d\rangle^\perp$. Let $v \in P$ and $\alpha \in \mathbb{R}^n$. There exist $b_i \in \mathbb{Z}^k$ and $c_j \in \mathbb{Z}^k$ such that $x^{b_i} x^{c_j} \mathrm{in}_{v,\alpha}(g_j)(i) \in R_d$ for $i = 1, \dots, n$ and $j = 1, \dots, m$. If the module $\mathrm{in}_{v,\alpha}(M)$ contains a positive element, then there are polynomials $p_j \in x^{c_j} R_d$ such that*

(i) $\sum p_j \,\mathrm{in}_{v,\alpha}(g_j) \in (R_k^{++})^n$, and

(ii) $f_P = \sum p_j g_j$ has $\mathrm{in}_v(f_P) \in (R_k^{++})^n$ for every $v \in P$.

Proof Let $h_j = \mathrm{in}_{v,\alpha}(g_j)$, and let Q be the element of $\mathcal{Q}$ to which $\rho((v,\alpha))$ belongs. Then $h_j = \mathrm{in}_{v',\alpha'}(g_j)$ whenever $\rho((v',\alpha')) \in Q$. Suppose $i \in \{1, \dots, n\}$ is such that $h_j(i) \neq 0$. Then $h_j(i) = \mathrm{in}_{v'}(h_j(i))$ for every $v' \in P$. Since P is assumed to be an open subset of $D_k \cap \langle e_1, \dots, e_d\rangle^\perp$ this shows that $x^{a_{ij}} h_j(i) \in R_d$ for some $a_{ij} \in \{0\}^d \times \mathbb{Z}^{k-d}$. Letting $F = \{(i,j) : \mathrm{in}_{v,\alpha}(g_j)(i) \neq 0\}$, this defines a_{ij} for all $(i,j) \in F$. Note that for $(i,j) \in F$ we have

$$\mathrm{m}_{v',\alpha'}(g_j) = -a_{ij} \cdot v' + \alpha'(i).$$

Considering a sequence

$$(*) \qquad (i_0, j_0), (i_1, j_0), (i_1, j_1), \dots, (i_l, j_{l-1}), (i_l, j_l), (i_0, j_l)$$

in F, and writing $i_{l+1} = i_0$, we find that

$$0 = \sum_{s=0}^{l} \mathrm{m}_{v',\alpha'}(g_{j_s}) - \mathrm{m}_{v',\alpha'}(g_{j_s}) = \sum_{s=0}^{l} a_{i_{s+1} j_s} \cdot v' - a_{i_s j_s} \cdot v'$$

for every $v' \in P$. By the assumption on P we get

$$(**) \qquad \sum_{s=0}^{l} a_{i_s j_s} - a_{i_{s+1} j_s} = 0$$

for every allowed sequence $(*)$ in F.

We now extend a_{ij} and $(**)$ to all pairs $(i,j) \in \{1,\dots,n\} \times \{1,\dots,m\}$. Assume a_{ij} is already defined on a set $E \supset F$ and $(**)$ is valid on E. Pick $(i,j) \notin E$. If there exists a sequence

$$(i_1, j), (i_1, j_1), \dots, (i_l, j_{l-1}), (i_l, j_l), (i, j_l) \in E\,,$$

we put $i_{l+1} = i$ and define

$$a_{ij} = a_{i_1 j} - \sum_{s=1}^{l} a_{i_s j_s} - a_{i_{s+1} j_s}\,.$$

One easily verifies that $(**)$ then holds for every allowed sequence $(*)$ in $E \cup \{(i,j)\}$.

If there is no sequence

$$(i_1, j_0), (i_1, j_1), \dots, (i_l, j_{l-1}), (i_l, j_l), (i_0, j_l)$$

in E, we can take a_{ij} to be any element of $\{0\}^d \times \mathbb{Z}^{k-d}$ and have $(**)$ hold for all sequences $(*)$ in $E \cup \{(i,j)\}$.

Having thus extended a_{ij} to all pairs $(i,j) \in \{1,\dots,n\} \times \{1,\dots,m\}$, we define

$$b_i = a_{i1}$$

and

$$c_j = a_{ij} - a_{i1}\,,$$

which is independent of i by $(**)$ (using the allowed sequence $(i,j), (i',j), (i',1), (i,1)$). Then $a_{ij} = b_i + c_j$ and the first part of the lemma follows.

Multiplying every i-th coordinate of every element of M with x^{b_i} we get a conjugated module, and multiplying g_j with x^{c_j} we get a different set of generators. So, we can assume that $b_i = c_j = 0$. This means that $\mathrm{in}_{v,\alpha}(g_j) \in R_d^n$ for every j. If $\mathrm{in}_{v,\alpha}(M)$ contains a positive element, we can choose $p_j \in R_d$ such that

$$g = \sum p_j \, \mathrm{in}_{v,\alpha}(g_j) \in (R_d^{++})^n.$$

Define $f = \sum_j p_j g_j$ and fix $v' \in P$ and α' with $\rho((v', \alpha')) \in Q$. As in (3.2), we have $\mathrm{in}_{v',\alpha'}(f) = \sum_j p_j \, \mathrm{in}_{v',\alpha'}(g_j) = g \in (R_d^{++})^n$. Going back to the original module and the original generators we see that $p_j \in x^{c_j} R_d$ and $\mathrm{in}_{v',\alpha'}(f) \in (R_k^{++})^n$ only. □

We now describe a procedure for deciding whether M contains a positive element and for finding such an element.

(6.3) Procedure *1. Construct a super Gröbner basis $g_1, \dots, g_m \in M$.*

2. *Using the Newton polytopes of* $\phi(g_j)$ *calculate the partition* $\mathcal{Q}$ *of* D_{k+n} *and project its sets to define* $\mathcal{P}$.
3. *Pick for every* $P \in \mathcal{P}$ *a rational direction* $v_P \in P$ *and, for every* $Q \in \mathcal{Q}$ *with* $P \subset \pi(Q)$, *pick a vector* $\alpha_{P,Q} \in \mathbb{R}^n$ *with* $\rho((v_P, \alpha_{P,Q})) \in \mathcal{Q}$.
4. *Fix a partition element* $P \in \mathcal{P}$.
   - *Make a coordinate change in the variables* $x_1, \dots, x_k$ *using a matrix* $A \in \mathrm{Gl}(k, \mathbb{Z})$ *such that after the change* $v_P = e_k$.
   - *For every* $Q \in \mathcal{Q}$ *with* $P \subset \pi(Q)$, *consider the module*

$$\mathrm{in}^0_{v_P,\alpha_{P,Q}}(M) = \langle \mathrm{in}^0_{v_P,\alpha_{P,Q}}(g_1), \dots, \mathrm{in}^0_{v_P,\alpha_{P,Q}}(g_m) \rangle \subset R^n_{k-1}$$

     and determine whether this R_{k-1} *module contains a positive element* h_P.
   - *If there is a positive element* h_P *in one of the above modules, use* (6.2) *to construct an element* $f_P \in M$ *with* $\mathrm{in}_v(f_P) \in (R_k^{++})^n$ *for every* $v \in P$.
   - *If there is no positive element in any of the above modules, then* M *does not contain a positive element either.*
5. *Having completed the last step for every* $P \in \mathcal{P}$, *glue the vectors* f_P *together to get an element* $f \in M$ *with* $\mathrm{in}_v(f) \in (R_k^{++})^n$ *for every* $v \in D_k$ *(see* (5.2)*).*
6. *Use* (5.3) *to find a compact set* $K \subset (0, \infty)^k$ *such that* $f(a) > 0$ *for* $a \notin K$.
7. *Check condition* (1.3)(a) *for* $a \in K$. *If the condition fails for some* $a \in K$, *the module* M *does not contain a positive element. If the condition holds for every* $a \in K$ *find, as in the proof of* (1.3), *an element* $h \in M$ *which satisfies* (5.1)(ii) *in every coordinate.*
8. *Let* $q = \prod_i \left(\sum_{u \in \mathrm{Log}(h(i))} x^u \right)$, *and find* $l \in \mathbb{N}$ *such that* $q^l h \in (R_k^{++})^n$.

The above procedure might be called an algorithm except for two questions. The first is whether (1.3)(a) can be checked algorithmically; as we have seen above, it is sufficient to have an algorithm for checking this condition on a compact set K. In particular, when $k = 1$ we have polynomials and we require an algorithm for checking a compact set for zeros. The existence of l as in step 8 is a consequence of the proof of Handelman's theorem (see [DT]); the second question is whether there is a computable bound on l.

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