DUKE'S THEOREM FOR SUBCOLLECTIONS

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Abstract. We combine effective mixing and Duke's Theorem on closed geodesics on the modular surface to show that certain subcollections of the collection of geodesics with a given discriminant still equidistribute. These subcollections are only assumed to have sufficiently large total length without any further restrictions.

1. Introduction

Duke's Theorem, in our context, is concerned with the equidistribution of closed geodesics on the modular surface $Y_0(1) := \mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$ (and its unit tangent bundle). To give the necessary background and its statement, we follow the introduction of [5]. The reader is referred to there for the definitions of the classical notions that we use below.

A non-zero integer d is called a *discriminant* if there exist $a, b, c \in \mathbb{Z}$ such that $d = b^2 - 4ac$. For any non-square positive discriminant d one can associate (see [5, §1.2]) a collection $\mathcal{G}_d$ of $h(d)$ closed geodesics on $X := T^1(Y_0(1)) \cong \mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})$, the unit tangent of the modular surface, where $h(d)$ is the class number of the order $\mathcal{O}_d := \mathbb{Z}[\frac{d+\sqrt{d}}{2}]$ (see [5, §2.1]). Duke's Theorem asserts that the set $\mathcal{G}_d$ becomes equidistributed as $d \to +\infty$ amongst positive non-square discriminants. The aim of this paper is to deduce a similar theorem for subcollections of $\mathcal{G}_d$ of sufficiently large total length without any further restrictions. In order to give a precise formulation and relate this work to previous results we first record the following facts:

Fact 1. *Let d be a positive non-square discriminant and $\mathcal{O}_d := \mathbb{Z}[\frac{d+\sqrt{d}}{2}]$ be the order of discriminant d. We have:*

(1) $|\mathcal{G}_d| = |\mathrm{Pic}(\mathcal{O}_d)|$ where $\mathrm{Pic}(\mathcal{O}_d)$ is the ideal class group of $\mathcal{O}_d$.
(2) The length of any $\phi \in \mathcal{G}_d$ is equal to $\mathrm{Reg}(\mathcal{O}_d)$, the regulator of $\mathcal{O}_d$.
(3) The total length of the collection $\mathcal{G}_d$ is $\mathrm{Reg}(\mathcal{O}_d) \cdot |\mathcal{G}_d| = d^{\frac{1}{2}+o(1)}$.

Proof. See [5, §2]. □

Let $G = \mathrm{SL}_2(\mathbb{R})$, Γ be a finite-index congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and let G act on $X = \Gamma \setminus G$ by $g.\Gamma x = \Gamma x g^{-1}$. Let $C_c^\infty(X)$ denote the space of infinitely differentiable functions with compact support on X and μ_X denote the unique G-invariant probability measure on X. Throughout this paper, given a measure ν on a measurable space X and a ν-measurable function f, we set $\nu(f) := \int_X f d\nu$. For a closed geodesic ϕ, we denote its length by $l(\phi)$ and let $l(I_d) := \sum_{\phi \in I_d} l(\phi)$. By Fact 1, we have $l(I_d) = |I_d|\, Reg(\mathcal{O}_d)$.

We let μ_ϕ denote the normalized arc-length measure along ϕ, that is, for $f \in C_c^\infty(X)$ we set

$$\mu_\phi(f) = \frac{1}{l(\phi)} \int_0^{l(\phi)} f(a_t x)dt, \text{ for some } x \in \phi,$$

where a_t denote the geodesic flow (see §2). For any $I_d \subseteq \mathcal{G}_d$ let $\mu_{I_d} := \frac{1}{|I_d|} \sum_{\phi \in I_d} \mu_\phi$ denote the normalized measure supported on I_d and $\mu_d := \mu_{\mathcal{G}_d}$. Finally, given a sequence of subcollections $\mathcal{I} = \{I_{d_k}\}$ we let $\varphi_{\mathcal{I}}(k) = \frac{l(\mathcal{G}_{d_k})}{l(I_{d_k})}$. In this note, we prove the following:

Theorem 2. *Let $\mathcal{I} = \{I_{d_k}\}$ be a sequence of subcollections such that $\psi(k) := \frac{\varphi_{\mathcal{I}}(k)}{\log(d_k)}$ tends to 0 as $k \to \infty$. Then, for any $f \in C_c^\infty(X)$ we have*

$$\left| \mu_{I_{d_k}}(f) - \mu_X(f) \right| \le C(f)\psi(k)^{\frac{1}{2}}$$

where C is a constant depending only on f. In particular, $\mu_{I_{d_k}}$ equidistribute to μ_X.

Note that the only assumption on I_d is about its total length. Theorem 2 answers a question raised in [11, Remark 6.1.]. In order to discuss a stronger variant of Theorem 2 and to put Theorem 2 in the context of remark [11, Remark 6.1.], one should contrast Theorem 2 with the results in [12, 8]. To explain these results, note that after choosing a base point, $\mathcal{G}_d$ inherits a structure from $\mathrm{Pic}(\mathcal{O}_d)$ (i.e. $\mathcal{G}_d$ is a $\mathrm{Pic}(\mathcal{O}_d)$-torsor, see [5, §2]). In [12, 8, 7], the authors establish the equidistribution of subcollections that correspond to subgroups of $H_d < \mathrm{Pic}(\mathcal{O}_d)$ with $[\mathrm{Pic}(\mathcal{O}_d) : H_d] \gg d^a$ for some $a < \frac{1}{2827}$. In other words, they establish equidistribution of much smaller subcollections which are restricted by some "algebraic" condition. We note that these results, do not imply Theorem 2. First, in the context of Heegner points (which is the framework of the result in [8]), Theorem 2 is clearly false for arbitrary subcollections. Indeed, restricting to points which lie in a certain part of positive measure of the total space, yields subcollections I_d with $|I_d| \ge C|\mathcal{G}_d|$, for some $0 < C < 1$ which do not equidistribute. Moreover, in the context of closed geodesics, arbitrary subcollections with $\frac{l(\mathcal{G}_d)}{l(I_d)} \ll d^a$ for some $a > 0$ do not necessarily equidistribute: following a construction that was outlined to us by Elon Lindenstrauss, for any $a > 0$ we construct in Section 4 subcollections with $\frac{l(\mathcal{G}_d)}{l(I_d)} \ll d^a$ which do not equidistribute, and in fact give positive mass to an arbitrary fixed periodic orbit. This construction uses subcollections of $\mathcal{G}_d$ for which $\mathrm{Reg}(\mathcal{O}_d) = c\log(d)$. While writing this note we found that in an upcoming preprint [1], Bourgain and Kontorovich, construct subcollections with $\frac{l(\mathcal{G}_d)}{l(I_d)} \ll d^a$ that stay uniformly bounded. Moreover, using sieve methods, they manage to construct uniformly bounded subcollections along a sequence that involves only fundamental discriminants.

It is an interesting question to decide whether Theorem 2 holds for smaller subcollections under the assumption $\mathrm{Reg}(\mathcal{O}_d) \gg d^\epsilon$ for some $\epsilon > 0$. (For a stronger conjecture and a related discussion, see also [3, Conjecture 1.9].)

It is important to note that a stronger, but non-effective, equidistribution result on subcollections follows from [5]. Indeed, the fact that $l(\mathcal{G}_d) = d^{\frac{1}{2}+o(1)}$ is the only information that is used in [5] to deduce that the limiting measure has maximal entropy and hence is equal to μ_X. Therefore, the same argument implies that any subcollections with $l(I_d) = d^{\frac{1}{2}+o(1)}$ also equidistribute (see mainly [5, Proposition 3.6]).

Apart from the effectivity in the Theorem 2, we also remark that the following variant of Theorem 2 cannot be deduced from [5], i.e. using the above entropy argument. Our method uses as input an effective Duke's Theorem, i.e., the effective equidistribution of $\mathcal{G}_d$ with $d^{-\gamma}$ savings for some $\gamma > 0$ (see §2.2). Note that by [12, 8, 7], a similar effective Theorem exists for collections $\mathcal{H}_d \subset \mathcal{G}_d$ supported on cosets of subgroups $H_d < \mathrm{Pic}(\mathcal{O}_d)$ with $[\mathrm{Pic}(\mathcal{O}_d) : H_d] \gg d^a$ for some $a < \frac{1}{2827}$, with $d^{-\gamma(a)}$ savings for some $\gamma(a) > 0$ (see [7, Corollary 1.4]). Therefore, with the exact same proof, it follows that Theorem 2 holds for any subcollection $\mathcal{I} = \{I_{d_k} \subset \mathcal{H}_{d_k}\}$ with $\tilde{\psi}(k) := \frac{\tilde{\varphi}_{\mathcal{I}}(k)}{\log(d_k)}$ where $\tilde{\varphi}_{\mathcal{I}}(k) = \frac{l(\mathcal{H}_{d_k})}{l(I_{d_k})}$ instead of $\psi(k)$.

This note is organized as follows: Theorem 2 is obtained by a simple application of effective mixing in conjunction with effective version of Duke's Theorem. In hindsight, a similar argument is used in [13]. We review these ingredients in §2 and give the proof of Theorem 2 in §3. Section 4 is devoted for the construction of large but non-equidistributing subcollections as discussed above.

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2. Preliminaries

As above, let $G = \mathrm{SL}_2(\mathbb{R})$, $a_t := \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}$, Γ be a finite-index congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Then, G acts on $X = \Gamma \setminus G$ by $g.\Gamma x = \Gamma x g^{-1}$. Note a left invariant metric d_G on G induces a metric on X which we denote by d_X ([6, §9.3.2]). It is well known (see e.g. [6, §9.4.2]) that under the identification of $\Gamma \setminus G \cong T^1(Y_0(1))$ the action of a_t corresponds to the geodesic flow on X. We denote by $C_c^\infty(X) \oplus \mathbb{C} \subset C^\infty(X)$ the space of compactly supported smooth functions modulo the constants and by $C_0^\infty(X) := \left\{ f \in C_c^\infty(X) \oplus \mathbb{C} : \int_X f d\mu_X = 0 \right\}$. We define f_T by

$$f_T(x) := \frac{1}{T} \int_0^T f(a_t.x)dt.$$

Finally, let $0 \le \theta < \frac{1}{2}$ and assume that the unitary representation of G on $L_0^2(\Gamma \setminus G)$ does not weakly contain any complementary series with parameter $\ge \theta$ (for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ this holds with $\theta = 0$, (the tempered case)).

2.1. **Effective mixing.**

Lemma 3. *For any* $f \in C_0^\infty(X)$, $t \in \mathbb{R}$, $\epsilon > 0$, *we have*

$$|\langle f, a_t \circ f\rangle| \ll_\epsilon S_1(f)^2 e^{t\left(\left(\theta - \frac{1}{2}\right)+\epsilon\right)}$$

where S_1 *is a Sobolev norm.*

Proof. This assertion is proved in [13, Theorem §9.1.2] with an explicit Sobolev norm or in [9, page 216]. In fact, for our argument any bound of the form $\ll S_1(f)^2 e^{t\beta}$ for some $\beta < 0$ will suffice. □

Proposition 4. *For any real valued* $f \in C_0^\infty(X)$ *we have*

$$\mu_X\left(|f_T|^2\right) \ll \frac{S_1(f)^2}{T}$$

where S_1 *is the same as in Lemma 3.*

Proof. We have

$$\mu_X\left(|f_T|^2\right) = \int_X |f_T|^2 \, d\mu_X = \langle f_T, f_T\rangle = \frac{1}{T^2} \int_{0\le t,s\le T} \langle f, f \circ a_{t-s}\rangle ds dt = (*).$$

Let $B = \{(t,s) : 0 \le t \le T, 0 \le s \le t\}$ and note that since f is real-valued and $a_{t-s}\circ$ is a unitary operator, we have $\langle f, f \circ a_{t-s}\rangle = \langle a_{s-t} \circ f, f\rangle = \langle f, f \circ a_{s-t}\rangle$ which implies that

$$(*) = \frac{2}{T^2} \int_B \langle f, f \circ a_{t-s}\rangle ds dt \le \frac{2}{T^2} \int_B |\langle f, f \circ a_{t-s}\rangle| \, ds dt \ll \frac{S_1(f)^2}{T^2} \int_0^T \int_0^t e^{\beta(t-s)} ds dt$$

where $\beta := \frac{\theta - \frac{1}{2}}{2} < 0$. A direct computation yields:

$$\int_0^T \int_0^t e^{\beta(t-s)} ds dt = \frac{1}{\beta} \int_0^T \left(e^{\beta t} - 1\right) dt = \frac{1}{\beta^2}\left(e^{\beta T} - 1 - \beta T\right)$$

and recalling that $\beta < 0$ we have reached the claim above:

$$(*) \ll \frac{S_1(f)^2}{|\beta| \, T}.$$

□

2.2. **Effective Duke's Theorem.** The following theorem is now known as Duke's Theorem [2]:

Theorem 5. *There exist a $\gamma > 0$ and a sobolev norm S_2, such that for any $f \in C_c^\infty(X)$ we have*

$$|\mu_d(f) - \mu_X(f)| \leq S_2(f) d^{-\gamma}$$

where $S_2(f)$ is the Sobolev norm on $C_c^\infty(X)$ (whose further properties are discussed below).

As we want to apply Theorem 5 to $|f_T|^2$, we need to bound the growth rate of $S_2(|f_T|^2)$. To this end, note that

(1) There exists another sobolev norm S_3 such that for any $f_1, f_2 \in C_0^\infty(X) \oplus \mathbb{C}$ we have $S_2(f_1 f_2) \ll S_3(f_1) S_3(f_2)$.
(2) For any $f \in C_0^\infty(X) \oplus \mathbb{C}$ and $g \in G$, $S_3(g.f) \ll \|g\|^\kappa S_3(f)$ for some $\kappa > 0$, where $\|g\|$ denotes the operator norm $Ad(g^{-1}) : \mathrm{Lie(G)} \to \mathrm{Lie(G)}$, $X \mapsto g^{-1} X g$.

For the proof of Theorem 5 and the properties of the above Sobolev norms we refer the reader to [4, Theorem 4.6] and the references therein, in particular to [13, §2.9 and §6].

We thus have:

Lemma 6. *There exists an $\alpha > 0$ such that for any $T > 0$, and any $f \in C_c^\infty(X) \oplus \mathbb{C}$ we have*

$$S_2(|f_T|^2) \ll S_3(f)^2 e^{\alpha T}.$$

Proof. This readily follow from properties (1) and (2). Indeed, first use that that $S_2(|f_T|^2) \ll S_3(f_T) S_3(f_T)$. Further, by the convexity of the norm S_3 and Jensen's inequality, we have

$$S_3(f_T) \leq \frac{1}{T} \int_0^T S_3(f \circ a_t) dt \ll \frac{S_3(f)}{T} \int_0^T e^{\kappa t} dt \ll S_3(f) e^{\kappa T},$$

and the lemma follows. □

3. Proof of Theorem 2

For simplicity, we write $I_k = I_{d_k}$ and $\mu_k = \mu_{I_k}$. Fix $f \in C_c^\infty(X) \oplus \mathbb{C}$ and set $c = \mu_X(f)$ and by abuse of notation, let c also denote the constant function $c \cdot 1_X$. As we aim to estimate $|\mu_k(f) - c|$, note first that

$$\mu_k(f) - c = \mu_k(f - c) = \mu_k\left((f - c)_T\right)$$

where the first equality follows since μ_k is a probability measure and the second since μ_k is supported on closed geodesics.

By the Cauchy-Schwarz inequality, we have

$$\left(\frac{l(I_k)}{l(\mathcal{G}_{d_k})} \mu_k\left((f-c)_T\right)\right)^2 \leq \left(\frac{l(I_k)}{l(\mathcal{G}_{d_k})}\right)^2 \quad \mu_k\left(1_X^2\right) \cdot \mu_k\left(|(f-c)_T|^2\right) \tag{3.1}$$

$$\leq \left(\frac{l(I_k)}{l(\mathcal{G}_{d_k})}\right)^2 \mu_k\left(|(f-c)_T|^2\right) \quad \leq \tfrac{l(I_k)}{l(\mathcal{G}_{d_k})} \mu_{d_k}\left(|(f-c)_T|^2\right) \quad = (*) \tag{3.2}$$

where the last inequality follows since the positivity of $|(f-c)_T|^2$ implies that

$$\frac{l(I_k)}{l(\mathcal{G}_{d_k})}\mu_k\left(|(f-c)_T|^2\right)\leq\mu_{d_k}\left(|(f-c)_T|^2\right).$$

Now we apply Theorem 5. Note that $|(f-c)_T|^2$ does not have compact support, but it is eventually constant since $|(f-c)_T|^2-c^2$ has compact support. Noting that μ_d and μ_X are probability measures, we can apply Theorem 5 to estimate $\mu_{d_k}\left(|(f-c)_T|^2\right)$ and get that

$$(*)\leq\frac{l(I_k)}{l(\mathcal{G}_{d_k})}\left(\mu_X\left(|(f-c)_T|^2\right)+d_k^{-\gamma}S_2\left(|(f-c)_T|^2-c^2\right)\right)=(**)\,. \tag{3.3}$$

Note that $S_2\left(|(f-c)_T|^2-c^2\right)\ll S_2\left(|(f-c)_T|^2\right)+\|f\|_\infty^2$. Now, as $f-c$ has mean zero we can apply Proposition 4 to estimate $\mu_X\left(|(f-c)_T|^2\right)$ and Lemma 6 to bound $S_2\left(|(f-c)_T|^2\right)$, in order to get

$$(**)\ll_f\frac{l(I_k)}{l(\mathcal{G}_{d_k})}\left(S_1(f-c)^2T^{-1}+S_3(f-c)^2d_k^{-\gamma}e^{\alpha T}+d_k^{-\gamma}\|f\|_\infty^2\right).$$

Putting all of the above together and choosing $T=\eta\log(d_k)$, we have

$$\frac{l(I_k)}{l(\mathcal{G}_{d_k})}\left(\mu_k(f)-\mu_X(f)\right)^2\ll_f S_1(f-c)^2\eta^{-1}\log(d_k)^{-1}+S_3(f-c)^2d_k^{\eta\alpha-\gamma}+d_k^{-\gamma}\|f\|_\infty^2\,. \tag{3.4}$$

Choosing $\eta<\frac{\gamma}{\alpha}$ and multiplying both sides by $\varphi_{\mathcal{I}}(k)=\frac{l(\mathcal{G}_{d_k})}{l(I_k)}$, we get with $\psi(k)=\frac{\varphi_{\mathcal{I}}(k)}{\log(d_k)}$ that

$$\left(\mu_k(f)-\mu_X(f)\right)^2\ll_f\psi(k)$$

as claimed.

4. Large but non-equidistributing subcollections

Recall that $\mathcal{O}_d:=\mathbb{Z}[\frac{d+\sqrt{d}}{2}]$, the unique order of discriminant d. The following construction was outlined to us by E. Lindenstrauss:

Theorem 7. *Let $\{d_k\}_{k\in K}\nearrow\infty$ be any sequence with $\operatorname{Reg}(\mathcal{O}_{d_k})\ll\log(d_k)$. Given $a>0$ and a fixed periodic orbit P, there exist subcollections $I_{d_k}\subset\mathcal{G}_{d_k}$ with $l(I_{d_k})\gg d_k^{\frac{1}{2}-a}$ such that any weak-* limit of a subsequence of $\mu_{I_{d_k}}$ gives a positive mass to P and in particular, the sequence $\left\{\mu_{I_{d_k}}\right\}_{k\in\mathbb{N}}$ does not equidistribute.*

Remark 8. Such sequences of discriminants do exist and even exist in any given fixed real quadratic field (see e.g. [10, §6]).

Let P be a periodic orbit and note that since P is compact, it has a uniform injectivity radius which we denote by $\operatorname{inj}(P)$. For any $r<\operatorname{inj}(P)$ we let $U_r=\{x\in X:d_X(x,P)<r\}$.

Lemma 9. *For any small enough $r>0$ and $y\in U_r$ there exists an interval I of length $\asymp -\log(r)$ such that for any $t\in I$, we have $a_t.y\in U_{r^{\frac{1}{2}}}$.*

Proof. Let $x\in P$ such that $d_X(x,y)<r$ for some r that will be determined momentarily. Denote $x=\Gamma g_1, y=\Gamma g_2$ and $h=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in G$ with $\Gamma g_1 h=\Gamma g_2$. Fix a norm on $M_{2\times 2}(\mathbb{R})$, say $\left\|\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right\|=\max\left(|a|,|b|,|c|,|d|\right)$ and restrict it to G. We know that the resulting metric $d_{\|\|}$ on G is bi-Lipschitz equivalent to d_G (say $\kappa d_{\|\|}\leq d_G\leq K d_{\|\|}$), and for any $z=\Gamma g_0\in P$ and $r<\min\left(\mathrm{inj}(P),1\right)$ the projection $B_r^G(e)\to B_r^X(x),\ g\mapsto \Gamma g_0 g$ is an isometry between d_G and d_X. Thus by assumption $\|h\|\leq\frac{r}{\kappa}$ and since $a_t.y=\Gamma g_1 a_t^{-1}a_t h a_t^{-1}=a_t.x\left(a_t h a_t^{-1}\right)$ we have $d_X(a_t.x,a_t.y)\leq K\left\|a_t h a_t^{-1}\right\|$. As $a_t.x\in P$ we have to show that there is an interval I of length $\asymp-\log(r)$ such that $t\in I$ implies

$$\left\|a_t h a_t^{-1}\right\|\leq\frac{r^{\frac{1}{2}}}{K}. \tag{4.1}$$

Since $a_t h a_t^{-1}=\begin{pmatrix} a & be^t\\ e^{-t}c & d\end{pmatrix}$, and by assumption $|a-1|,|d-1|,|b|,|c|\leq\frac{r}{\kappa}\ll\frac{r^{\frac{1}{2}}}{K}$ (where the last inequality holds for small enough r), (4.1) amounts to $e^t|b|\leq e^t\frac{r}{\kappa}\leq\frac{r^{\frac{1}{2}}}{K}$ and $e^{-t}|c|\leq e^{-t}\frac{r}{\kappa}\leq\frac{r^{\frac{1}{2}}}{K}$. Thus, for small enough r, we have $d_X(a_t.x,a_t.y)\leq r^{\frac{1}{2}}$ if and only if

$$e^t\leq\frac{\kappa r^{-\frac{1}{2}}}{K}\text{ and }e^{-t}\leq\frac{\kappa r^{-\frac{1}{2}}}{K}$$

if and only if $t\leq\log(\frac{\kappa}{K})-\frac{1}{2}\log(r)$ and $-t\leq\log(\frac{\kappa}{K})-\frac{1}{2}\log(r)$. As for small enough r, $I=[-\log(\frac{\kappa}{K})+\frac{1}{2}\log(r),\log(\frac{\kappa}{K})-\frac{1}{2}\log(r)]$ has length $\asymp-\log(r)$ we are done. □

Lemma 10. *Let P be a fixed closed geodesic and $0\leq r\leq\min(1,\mathrm{inj}(P))$. There exists a function $f_r=f(r,P)$ such that*

1. $\forall x\in X$, $0\leq f_r(x)\leq 1$,
2. $\mathrm{supp}(f_r)\subset U_r$, $f_r|_P\equiv 1$.
3. $S_2(f_r)\ll r^{-b}$ *for some* $b>0$ *(where* S_2 *is as in Theorem 5),*
4. $\mu_X(f_r):=\int f_r d\mu_X\asymp r^c$ *for some* $c>0$.

Proof. As $0\leq r\leq\mathrm{inj}(P)$, this construction takes place in the compact region of X. Therefore, to estimate the Sobolev norm any standard Sobolev norm on $\mathbb{R}^3$ will do. The most obvious construction works. Namely, note that the set $\tilde{U}_r=P\cdot\{u^+(s)u^-(s):|s|<r\}$ is a subset of U_r, and we can use the standard bump function $\Theta(x)=\exp(\frac{-1}{(x-r)^2})$ on $(-r,r)$ to define $f_r:\tilde{U}_r\to\mathbb{R}$ by $f_r(\Gamma g_P a_t u^+(s_1)u^+(s_2))=\Theta(s_1)\Theta(s_2)$ where $P=\{\Gamma g_P a_t\}_{0\leq t\leq l(P)}$ and $|s_2|,|s_2|<r$. One easily checks that f_r has the desired properties. □

Proof of Theorem 7. For simplicity we restrain from mentioning injectivity radius issues any further as these may always be resolved by taking some variables to be large/small enough.

Let γ be as in Theorem 5 and fix $a > 0$ and a periodic orbit P. Let $\eta = \eta(a) > 0$ that will be determined later. Applying Theorem 5 to the functions $f_k := f(d_k^{-\eta}, P)$, which are provided by Lemma 10, we get that there exist $C_1, C_2 > 0$ such that

$$\mu_{d_k}(f_k) \geq \mu_X(f_k) - d_k^{-\gamma} S_2(f_k) \geq C_1 d_k^{-c\eta} - C_2 d_k^{-(\gamma - b\eta)}.$$

Setting η so small such that $\gamma - b\eta \geq c\eta$, i.e. $\eta \geq \frac{\gamma}{b+c}$, we have

$$\mu_{d_k}(f_k) \gg d_k^{-c\eta}.$$

For a closed geodesic ϕ and $f \in C_c^\infty(X)$ we set $\phi(f)$ to be the line integral of f along ϕ. Recall that $\mu_{d_k}(f) = l(\mathcal{G}_{d_k})^{-1} \sum_{\phi \in \mathcal{G}_{d_k}} \phi(f)$, and note that since $0 \leq f_k \leq 1$, for any $\phi \in \mathcal{G}_{d_k}$ and any $\epsilon > 0$, we have $\phi(f) \leq \mathrm{Reg}(\mathcal{O}_{d_k}) \ll d_k^\epsilon$ since by assumption $\mathrm{Reg}(\mathcal{O}_{d_k}) \ll \log(d_k)$. Therefore, if we let I_{d_k} denote the subcollection of all the elements of $\mathcal{G}_{d_k}$ that intersect the support of f_k, for any $\epsilon > 0$ we have

$$d_k^{-c\eta} \ll \mu_{d_k}(f_{d_k}) \ll l(\mathcal{G}_{d_k})^{-1} \left|I_{d_k}\right| d_k^\epsilon = d_k^{-\left(\frac{1}{2}+o(1)-\epsilon\right)} \left|I_{d_k}\right|.$$

Thus, by choosing $\epsilon(a)$ and $\eta(a)$ accordingly, we have $\left|I_{d_k}\right| \gg d_k^{\left(\frac{1}{2}+o(1)-\epsilon\right)-c\eta} \gg d_k^{\frac{1}{2}-a}$.

Let ν be a weak-* limit of $\mu_{I_{d_k}}$ and we claim that $\nu(P) > 0$. It is enough to show that there exists a $C > 0$ such that for any large enough k, and any small enough r we have $\mu_{I_{d_k}}(U_r) > C$. Let $U^k = U_{d_k^{-\frac{\eta}{2}}}$ and as $d_k \nearrow \infty$ it is clear that $\cap_k U^k = P$. Thus it is enough to verify that for any k_0, $\mu_{I_{d_k}}(U^{k_0}) > C$ for $k \gg 0$. Fix $k_0 \in \mathbb{N}$; for any $k \geq k_0$ any element of $\phi \in I_{d_k}$ intersects $\mathrm{supp}(f_k)$ by definition, and therefore there is a point $x \in \phi$ with $d(x, P) \leq d_k^{-\eta}$. By Lemma 9 there exists an interval I_k of length $\eta \log(d_k)$ such that for any $t \in I_k$ we have $a_t.x \in \phi \cap U^k \subset \phi \cap U^{k_0}$. Since the length of any element of I_{d_k} is $\mathrm{Reg}(\mathcal{O}_{d_k}) \leq c_1 \log(d_k)$ for some constant c_1, any element of I_{d_k} spends at least $\frac{\eta}{c_1}$ of its length in U_{k_0}. It follows that $\nu(U_{k_0}) > \frac{\eta}{c_1} > 0$ and so that $\nu(P) > 0$ and the claim follows. □

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