RIGIDITY OF MEASURES INVARIANT UNDER SEMISIMPLE GROUPS IN POSITIVE CHARACTERISTIC.

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Abstract. M.Ratner has conjectured a positive characteristic version of her seminal results classifying orbit closures and invariant measures of unipotent flows on homogeneous spaces. In this paper, we provide a partial answer by establishing a positive characteristic version of her classification result for measures invariant under semisimple groups.

1. Introduction

In a series of important papers [29, 30] M.Ratner proved the Raghunathan conjectures for real Lie groups. Ratner's theorems describe orbit closures and invariant measures of actions of unipotent subgroups on homogeneous spaces, thus they have a measure as well as a topological counterpart. Subsequently, these results were extended to products of real and p-adic Lie groups by M.Ratner [31, 32] and independently G.Margulis and G.Tomanov [21, 22] obtained the results for S-arithmetic algebraic groups over local fields of characteristic zero as well as in the more general setting of "almost linear" groups. Ratner's work was preceded by several important results, most notably the proof of the longstanding Oppenheim conjecture due to G.Margulis [17]. Her results have occupied a central place in the theory of homogeneous dynamics and have been extended and applied in various directions and contexts by many authors. The study of dynamics of group actions and applications over ultrametric local fields is very much a subject of current research [8, 11, 16]. Ratner herself has conjectured a positive characteristic version of her seminal work in [32].

In this paper, we will prove a special case of the positive characteristic analogue of Ratner's measure classification theorem. In the interest

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of length, we will omit further historical and motivational remarks about the Raghunathan conjectures, earlier substantial contributions of many mathematicians, especially S.G.Dani, G.Margulis and M.Ratner. We feel the reader will be better served in this regard by the several excellent surveys on these topics, especially [15, 20, 33] as well as the recent monograph [26], and [8] for related research on the action of diagonalizable subgroups.

Our approach is based on an earlier survey paper of the first author [6] where information about finite dimensional representations of $\mathrm{SL}(2,\mathbb{R})$ along with polynomial divergence of unipotent orbits, following ideas from Ratner's work on joinings [27] was used to reprove the classification of $\mathrm{SL}(2,\mathbb{R})$-invariant ergodic probability measures on $\Gamma\backslash G$, where G is a Lie group and Γ is a discrete subgroup of G. We note that following [27] and [28], the polynomial divergence of unipotent orbits the so-called *shearing principle* was further developed in [29], [30] and [38] in the context of general Lie groups.

We will use [6] to our advantage here by referring the reader to that paper for some details, especially when the passage from characteristic zero to p does not pose problems.

1.1. **Notation and statement of result.** Let S be a finite set of powers of primes, which we may refer to as *places*. For each $s \in S$, we denote by $\mathbb{F}_s$ a finite field with s elements and by $k_s \overset{def}{=} \mathbb{F}_s((\pi))$ the field of Laurent series in π with coefficients in $\mathbb{F}_s$. It is classical that a local field of positive characteristic is isomorphic to some k_s [37]. We write $\mathrm{char}(k_s)$ for the characteristic of k_s. For each $s \in S$, we denote by $\mathrm{Mat}_n(k_s)$, the $n \times n$ matrices with entries in k_s, by G_s the k_s-points of a linear algebraic group $\mathbb{G}_s$ defined over k_s embedded into $\mathrm{Mat}_n(k_s)$. And finally, we set $\mathbf{G} = \prod_{s\in S} G_s \subset \prod_{s\in S} \mathrm{Mat}_n(k_s)$.

This notation will be followed throughout. In other words:

- The notation $\mathbb{L}_s$ will be used to denote an algebraic group defined over k_s, and $\mathbb{L}$ will denote $\prod_{s\in S} \mathbb{L}_s$.
- L_s will be used to denote $\mathbb{L}_s(k_s)$.
- $\mathbf{L}$ will be used to denote $\prod_{s\in S} L_s$.
- Occasionally we will drop the subscript s to simplify the notation in proofs where one may consider the places separately.

Let $\mathbb{H}_s$ be a semisimple k_s subgroup of $\mathbb{G}_s$ without anisotropic factors. Set $\mathbf{H} = \prod_{s \in S_0} H_s$ where we allow H_s to be trivial for some of $s \in S$ and denote by $S_0 \subset S$, the set of those s for which $\mathbb{H}_s$ is nontrivial. In other words, H_s consists of the k_s-points of a semisimple algebraic subgroup all of whose k_s-almost simple factors have k_s-rank at least one. We denote by H_s^+, the subgroup of H_s generated by its unipotent one-parameter subgroups, and set $\mathbf{H}^+ = \prod_{s \in S_0} H_s^+$. We refer the reader to Theorem 2.3.1 [19] for more information about the precise relationship between $\mathbf{H}$ and $\mathbf{H}^+$. In general, for a local field k, $\mathbf{H}^+(k)$ is a closed normal subgroup of $\mathbf{H}(k)$ such that $\mathbf{H}(k)/\mathbf{H}^+(k)$ is compact. In certain cases, the two groups are equal, for instance when $\mathbf{H}$ is simply connected, k isotropic and almost k simple.
Let Γ be a discrete subgroup of $\mathbf{G}$, and let μ be an $\mathbf{H}^+$-invariant Borel probability measure on $X \overset{def}{=} \Gamma \backslash \mathbf{G}$. The measure μ is called *homogeneous* if there is a point $x_0 \in X$ and a closed[1] subgroup $L \subseteq \mathbf{G}$ containing $\mathbf{H}^+$ such that μ is L-invariant and $x_0 L$ is closed and supports μ. We note that closed subgroups of algebraic groups, in fact even subgroups of unipotent algebraic groups, appear in positive characteristic in greater abundance.[2] Our theorem is:

Theorem 1.1. *Let μ be an $\mathbf{H}^+$-invariant and ergodic probability measure on X. There is a constant $M > n$ depending only on n such that if $\mathrm{char}(k_s) > M$ for every $s \in S_0$, then μ is homogeneous.*

Remarks.

(1) The restriction on characteristic arises for several reasons and seems indispensable to the structure of our proof. Firstly, we use complete reducibility of linear representations, which in positive characteristic requires some restriction. It turns out that $\mathrm{char}(k_s) > n$ suffices and this is elaborated upon in Section 2.1 as well as in Appendix A. Another, more serious obstruction arises due to the lack of separability of morphisms of algebraic groups in small characteristic. To overcome this, we need bounds arising from algebraic geometry, we provide an elementary but ineffective argument for this bound in Appendix B.

[1] We will be dealing with both the Hausdorff and Zariski topologies and to distinguish between the two, we will explicitly identify the latter. Thus for instance, closed subgroup will imply closed in the Hausdorff topology.

[2] E.g. for any subset $E \subset \mathbb{Z}$ we have the closed subgroup of $\mathbb{F}_s((\pi))$ consisting of all $u \in \mathbb{F}_s((\pi))$ which can be written as $u = \sum_{n \in F} c_n \pi^n$ with $c_n \in \mathbb{F}_s$, and π a uniformizer in $\mathbb{F}_s((\pi))$. In comparison closed subgroups of $\mathbb{R}$ and $\mathbb{Q}_p$ are fewer and allow a very concrete description. However, in the semisimple setting this is somewhat less problematic, see [36].

(2) These restrictions are not optimal. For instance, historically one of the earliest theorems classifying invariant ergodic measures for unipotent actions in the homogeneous setting was proven by Dani in [5] and concerns the so-called horospherical subgroup. The analogous theorem can be proved in positive characteristic without restricting the characteristic, see the recent work of A. Mohammadi [24]. Also in the joint work of M.E. with A. Mohammadi [9] on joinings of maximal horospherical subgroups in almost simple groups there is no need for a restriction on the characteristic. However, all of these situations are rather special and concern cases where the unipotent group is already rather big inside the ambient group. Another instance is the work [25] of A. Mohammadi establishing the positive characteristic analogue of Oppenheim's conjecture. Here the characteristic is assumed to be different from 2, which is a natural restriction for the question. However, somewhat surprisingly the case of characteristic equal to 3 is harder than all other cases.

(3) As will become clear during the proof, our method of proof applies equally well to groups over $\mathbb{Q}_p$ or $\mathbb{R}$. Therefore, the local fields in Theorem 1.1 can be taken to be of arbitrary characteristic, as long as a bound as in the theorem is assumed in case $\mathrm{char}(k_s) > 0$.

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2. Reduction to $\mathrm{SL}(2, k_s)$ resp. $\mathrm{PGL}(2, k_s)$, and finite dimensional representations.

It turns out that the proof of Theorem 1.1 can be carried out using $\mathrm{SL}(2, k_s)$ or $\mathrm{PGL}(2, k_s)$ instead of the larger group H_s. We first record the well known fact that these subgroups exist. Let $\mathbb{T}_s$ denote a maximal k_s-split torus of $\mathbb{H}_s$ and let Φ denote the set of roots of $\mathbb{T}_s$ with respect to $\mathbb{H}_s$. By Theorem 7.2 in [4], there exists a k_s-split subgroup $\mathbb{H}'_s \subseteq \mathbb{H}_s$ containing $\mathbb{T}_s$ such that the roots of $\mathbb{T}_s$ with respect to $\mathbb{H}'_s$ coincide with those with respect to $\mathbb{H}_s$. Let $\mathbb{U}_\alpha$ denote the root subgroup corresponding to $\alpha \in \Phi$. Then $\mathbb{U}_\alpha$ is defined over k_s, as is the opposite root subgroup $\mathbb{U}_{-\alpha}$ and the subgroup $\mathbb{M}_s$ generated by them is defined over k_s and is isomorphic to $\mathbb{SL}_2$ or $\mathbb{PGL}_2$. If $\mathbb{H}_s$ is almost simple we use this subgroup. Otherwise we do the above construction for each almost simple factor and then use a diagonally embedded copy of $\mathbb{SL}_2$ or $\mathbb{PGL}_2$. This way we obtain, for each $s \in S_0$, the subgroup $M_s \subseteq H_s$ isomorphic to $\mathrm{SL}(2, k_s)$ or $\mathrm{PGL}(2, k_s)$ such that M_s does not commute with any almost simple factor of H_s. We set $\mathbf{M} = \prod_{s \in S_0} M_s$ and $\mathbf{M}^+ = \prod_{s \in S_0} M_s^+$ where M_s^+ is the subgroup of M_s generated by its unipotent one-parameter subgroups.

Further we let $\mathbb{U}_s, \mathbb{A}_s$ to be respectively the groups of upper unipotent and diagonal subgroups of $\mathbb{M}_s$ for every $s \in S_0$ identified with the image of the respective subgroups by the isomorphism with $\mathbb{SL}_2, \mathbb{PGL}_2$. We denote by $\mathfrak{g}_s$, the Lie algebra of G_s inside $\mathrm{Mat}_n(k_s)$ and set $\mathfrak{G} = \prod_{s \in S} \mathfrak{g}_s$. We will endow each $\mathfrak{g}_s$ and its subspaces with a norm inherited from $\mathrm{Mat}_n(k_s)$.

2.1. **Semisimplicity.**

We now turn our attention to the finite dimensional representation theory of algebraic groups over local fields. It turns out that this depends to a large extent on the characteristic of the base field. Recall that over a field k of characteristic zero, every finite dimensional representation of (the k-points of) a semisimple algebraic k group G is semisimple, i.e every G module splits into a direct sum of simple G modules and for every submodule V' of a G module V, there exists a submodule $V^\perp$ such that

$$V = V' \oplus V^\perp.$$

This is unfortunately no longer true in positive characteristic, and this phenomenon has been studied extensively. Especially determining necessary and sufficient conditions for complete reducibility to hold has been the topic of several recent papers ([13], [23] and [34]). However, counterexamples to complete reducibility require dimension of size similar to the characteristic. We will need the following easier fact. With notation as before,

Proposition 2.1. *Let V be any M_s-module for which all weights are less than* $\mathrm{char}(k_s)$*, then V is semisimple.*

We would like to stress that this theorem is well known in the literature (c.f. [13], [14] and the references therein) in much greater generality for semisimple groups defined over algebraically closed fields. Since we will only require the case of $\mathrm{SL}(2, k_s)$ resp. $\mathrm{PGL}(2, k_s)$ (but actually for the field k_s which is not algebraically closed[3]), we feel that a concrete argument might be helpful to the reader. We provide in the appendix

[3]This is not a big obstacle since we only need to consider the split cases $\mathrm{SL}(2, k_s)$ and $\mathrm{PGL}(2, k_s)$.

a complete proof of the second claim of the above proposition by carefully following one of the standard proofs of complete irreducibility in characteristic zero.

2.2. **Some irreducible representations of** $\mathrm{SL}(2, k_s)$**.** Recall that in characteristic zero one can easily describe all irreducible representations of $\mathrm{SL}(2, k_s)$: they are the symmetric powers of the standard representation on the two-dimensional space k_s^2. Denote by $\mathrm{Sym}^n(k_s^2)$ the n-th symmetric power of the standard representation of $\mathrm{SL}(2, k_s)$ on k_s^2, which we will view as the set of homogeneous polynomials in the variables A, B of degree n where $\mathrm{SL}(2, k_s)$ acts by substitution.

This representation is, unfortunately, no longer guaranteed to be irreducible in positive characteristic. To see this, note that for every positive $r, n \in \mathbb{Z}$, the map $v \to v^{\mathrm{char}(k_s)^r}$ sends $\mathrm{Sym}^n(k_s^2)$ into a proper submodule of $\mathrm{Sym}^{n\,\mathrm{char}(k_s)^r}(k_s^2)$ consisting of all polynomials in $A^{\mathrm{char}(k_s)^r}, B^{\mathrm{char}(k_s)^r}$, and thus $\mathrm{Sym}^{n\,\mathrm{char}(k_s)^r}(k_s^2)$ is not irreducible. This problem has been studied in detail in [14]. For us it will suffice to know that $\mathrm{Sym}^n(k_s^2)$ is irreducible if $\mathrm{char}(k_s) > n$ which follows from the discussion in the appendix.

2.3. **Mautner phenomenon, and replacing H by M.** We wish to replace $\mathbf{H}$ by $\mathbf{M}$ and work with the latter in the rest of this paper. When doing this we need to be sure that we obtain enough information about the adjoint representation of each M_s on $\mathrm{Mat}_n(k_s)$.

Lemma 2.2. *Suppose for every $s \in S_0$ the highest weight of the natural representation of M_s on $\mathrm{Mat}_n(k_s)$ is less than $\mathrm{char}(k_s)$. Then $\mathrm{Mat}_n(k_s)$ is semisimple, and in fact is a direct sum of irreducible representations isomorphic to $\mathrm{Sym}^n(k_s^2)$ for various $n < \mathrm{char}(k_s)$.*

We will show this in the appendix by using techniques similar to the classical characteristic zero argument. We now turn our discussion to invariant measures and the Mautner phenomenon.

Lemma 2.3. *Let $\mathbf{M}, \mathbf{M}^+, \mathbf{H}$ and $\mathbf{H}^+$ be as before. Then any $\mathbf{H}^+$-invariant and ergodic probability measure μ is $\mathbf{M}^+$-ergodic as well as $\mathbf{U}$-ergodic.*

This theorem is a direct consequence of the Howe-Moore theorem on vanishing at infinity of matrix coefficients of unitary representations, (cf. [19], [12]).

3. Coordinates.

We will endow $\mathbf{G}$ with a left invariant metric whose topology is the subspace topology induced from $\mathrm{Mat}_n(k_s)$. This metric can be constructed for example, by taking a compact open subgroup $K < \mathbf{G}$, an invariant metric on K, and the discrete metric on $\mathbf{G}/K$. For convenience the metric on K is assumed to be defined by the matrix norm, i.e. for $g_1, g_2 \in K$ we define

$$d(g_1, g_2) = \|g_1 - g_2\|.$$

Notice that this metric is bi-invariant on K if only K is sufficiently small so that $\|g\| = 1$ for all $g \in K$. This makes X into a metric space. Finally, for any $g \in \mathbf{G}$ and $x \in X$ we define the natural action $R_g(x) = xg^{-1}$. In this section we develop a system of local coordinates in $\mathbf{G}$, i.e. coordinates for elements near the origin in each G_s. Over local fields of characteristic zero, the exponential map provides adequate local coordinates. To get around the lack of exponential map, we will provide an inductive construction of local coordinates which will be built into the proof of our main theorem. We think it would be helpful to provide a concise yet informal description of the structure of the proof before proceeding to actual details. For simplicity, we assume for now that we are working with a single local field (i.e. the set S consists of a single place). A natural candidate for a group which has an orbit which supports μ is the stabilizer of μ i.e.

$$\mathrm{Stab}(\mu) = \{g \in G \ : \ (R_g)_*\mu = \mu\}. \tag{3.1}$$

Note that $\mathrm{Stab}(\mu)$ is a closed subgroup of the locally compact group G but is by no means necessarily an algebraic object. If we suppose for the moment that it is an algebraic subgroup of G (or for instance, has finite index in its Zariski closure), we can consider its Lie algebra $\mathfrak{s}(\mu)$ which, by our weight restriction, will have a “transverse” complement $\mathfrak{s}(\mu)^\perp$ in $\mathrm{Mat}_n(k)$. It is then not very difficult to show (see Lemma 3.1) that an element $g \in G$, close to the identity, can be written as

$$g = (I + v)l$$

where $l \in \mathrm{Stab}(\mu)$ and $v \in \mathfrak{s}(\mu)^\perp$. Using standard techniques as in [27], [29], [30] and [6], we first assume that μ is not homogeneous, start with two suitably generic with respect to the U action, and using a shearing principle for unipotent flows, produce an element and eventually an entire one parameter subgroup which preserves μ but is not contained in $\mathrm{Stab}(\mu)$, thus arriving at a contradiction.

However, we have no information about the structure of $\mathrm{Stab}(\mu)$ and in particular it may make no sense to talk about its Lie algebra, so we will construct an algebraic group by building an inductive procedure into the above outline. Reverting back to our original set-up, we assume inductively that there are unipotent one-parameter subgroups

$\mathbb{F}_1, \ldots, \mathbb{F}_m$ such that $\mathbf{F}_1, \ldots, \mathbf{F}_m$ preserve μ, and together with H^+, generate a group $\tilde{L}$ which is open in the locally compact group $\mathbf{L}$ of points of the group $\mathbb{L}$ generated by $\mathbb{F}_1, \ldots, \mathbb{F}_m$ and $\mathbb{H}$. We can then provide local coordinates as before and proceed with the shearing argument to construct a one-parameter subgroup $\mathbb{F}_{m+1}$, which preserves μ and satisfies the other inductive hypotheses. We therefore find that if μ is not homogeneous, we can use $\mathbb{F}_{m+1}$ and $\mathbb{L}$ to get a new algebraic group $\mathbb{L}'$ which has bigger dimension than $\mathbb{L}$. Dimension considerations then lead to a contradiction, thus completing the proof of Theorem 1.1. The bulk of the complication arises in completing the induction, as we are confronted with difficulties arising from separability issues in positive characteristic which force us to establish a tight connection between the unipotent one parameter subgroup $\mathbb{F}_i$ and the algebraic group $\mathbb{L}$ they generate along with $\mathbb{H}$. We now proceed to precise statements.

Inductive assumption. We assume that we are given for each $s \in S_0$ an algebraic subgroup $\mathbb{L}_s \subseteq \mathbb{G}_s$, generated by $\mathbb{H}_s$ and one parameter unipotent subgroups $\mathbb{F}_{s,1}, \ldots, \mathbb{F}_{s,m_s}$ with the following properties:

- For every i, there exists $w_i \in \mathrm{Mat}_n(k_s)$ such that $\mathbb{F}_{s,i}$ is the image variety of the polynomial map $\exp(tw_i)$, which is well defined in large characteristic[4], for $t \in k_s$.
- $\mathbb{F}_{s,i}(k_s) = \exp(k_s w_i)$ stabilizes μ.
- The locally compact group $\tilde{L}_s \subset \mathbb{L}_s(k_s)$ generated by H_s^+ and $F_{s,i}$ for all $i = 1, \ldots, m_s$ is an open subgroup of L_s.
- We have $\mathbb{L}_s = \mathbb{H}_s \mathbb{F}_{s,1} \cdots \mathbb{F}_{s,m_s}$ where the right hand side is understood as the image variety of the product map, and the dimension of $\mathbb{L}_s$ is $3+m_s$. (Recall that we assume $\mathbb{H}_s$ is isomorphic to SL_2 or to PGL_2.)

We note a useful consequence of these hypotheses: There is an analytic map

$$\phi: \text{ a neighborhood of } 0 \in \prod_{s \in S_0} k_s^{\dim \mathbb{L}_s} \to \text{ a neighborhood of } I \in \mathbf{L}. \quad (3.2)$$

We will assume that the ball of radius η around I in $\mathbf{L}$, $B_\eta^{\mathbf{L}}$ is both contained in the image of ϕ as well as in the open subgroup $\tilde{L}$.

Here is a concrete description of the above map for some fixed $s \in S_0$. Since $\mathbb{L}_s$ is smooth at the identity and is defined over k_s, we can find

[4]Note that since w_i is nilpotent and $\mathrm{char}(k_s)$ is assumed large, the power series $\exp(tw_i)$ is actually a polynomial map $t \in k_s \to G_s$ with coefficients in $\mathrm{Mat}_n(k_s)$

polynomials $f_1, \dots, f_{n^2-\dim \mathbb{L}_s}$ vanishing on $\mathbb{L}_s$, generating the ideal of relations for $\mathbb{L}_s$ in the local ring of rational functions, regular at the identity I. If we let $x_1, \dots, x_{\dim \mathbb{L}_s}$ be regular functions vanishing at I which form a basis of m_I/m_I^2, where m_I is the ideal of regular functions vanishing at I. Then we get that

$$g \in \mathrm{Mat}_n(k_s) \to (x_1(g), \dots, x_{\dim \mathbb{L}_s}(g), f_1(g), \dots, f_{n^2-\dim \mathbb{L}_s}(g)) \quad (3.3)$$

is a locally invertible analytic function. We then define ϕ_s to be the inverse of this map restricted to

$$(x_1, \dots, x_{\dim \mathbb{L}_s}, 0, \dots, 0)$$

and define ϕ to be the product of the components ϕ_s for $s \in S_0$.

We now proceed to establish local coordinates, which can be viewed roughly as a weak replacement for the exponential map which is not available to us. Since both L_s and $\mathfrak{l}_s^\perp$ are smooth varieties, the map

$$(v, l) \in \mathfrak{l}_s^\perp \times L_s \to (I + v)l$$

has at I, the natural map

$$(v, w) \in \mathfrak{l}_s^\perp \times \mathfrak{l}_s \to v + w \in \mathrm{Mat}_n(k_s) \quad (3.4)$$

as derivative, since this derivative is an isomorphism, the following lemma follows again from the inverse function theorem.

Lemma 3.1. *Any $g \in G_s$, sufficiently close to I can be uniquely written as*

$$g = (I + v)l \quad (3.5)$$

where $l \in L_s$, $v \in \mathfrak{l}_s^\perp$, and I is the identity matrix in $\mathrm{Mat}_n(k_s)$.

4. Leading up to polynomial divergence.

In this section, we will lay the groundwork which will allow us to deploy the machinery of shearing and polynomial divergence developed in [27], [29] and [30]. The first lemma develops a local characterization of when μ is supported on the orbit of a closed subgroup of $\mathbf{G}$.

Lemma 4.1. *Let $\mathbf{S} \supseteq \mathbf{H}^+$ be a closed subgroup of $\mathbf{G}$ which preserves μ and suppose $\mu(x_0\mathbf{S}) = 1$ for some $x_0 \in X$. Then $x_0\mathbf{S}$ is closed and μ is supported on $x_0\mathbf{S}$.*

This is also one of the steps in [28], see also [6]. The S-arithmetic case of the lemma proceeds along analogous lines and is omitted. The above lemma combined with ergodicity gives us

Lemma 4.2. *With notation as in Lemma 4.1, if x_0 has the property that $\mu(x_0 B_\delta^{\mathbf{S}}) > 0$ for some $\delta > 0$. Then μ is concentrated on $x_0\mathbf{S}$.*

By the Mautner phenomenon in Lemma 2.3 we know that μ is also $\mathbf{U}$-ergodic. Set $B_T(0)$ to be the product of the balls of radius T around 0 over the various $k_s, s \in S_0$ and recall that $x \in X$ is μ-generic for $\mathbf{U}$ if it satisfies

$$\frac{1}{m(B_T(0))} \int_{B_T(0)} f(xu(t))\,\mathrm{d}\lambda_S(t) \to \int f d\mu \text{ for } T \to \infty \tag{4.1}$$

where we fix a parametrization $u(t), t \in \prod_{s\in S_0} k_s$ of $\mathbf{U}$, f is a compactly supported continuous function on X and λ_S is the product Haar measure on $\prod_{s\in S_0} k_s$. Denote by X', the set of μ-generic points for $\mathbf{U}$. By the decreasing Martingale theorem (c.f. e.g. Theorem 5.8 in [10]), $\mu(X') = 1$. We will denote by $C(U_s)$ the centralizer of U_s in G_s and set $C(\mathbf{U}) = \prod_{s\in S_0} C(U_s)$.

The following basic observation (see [28]) clarifies the role of the centralizer.

Lemma 4.3. *Suppose there are two points $x, x' \in X'$ and $c \in C(\mathbf{U})$ such that $x' = xc$. Then c stabilizes μ.*

We remind the reader of the simple argument.

Proof. Let $f : X \to \mathbb{R}$ be a compactly supported continuous function and define $f_c(x) = f(xc)$. We have $u(t)c = cu(t)$ for all $t \in \prod_{s\in S_0} k_s$. Thus,

$$f(x'u(t)) = f(xcu(t)) = f(xu(t)c) = f_c(xu(t)). \tag{4.2}$$

Therefore, the two expressions and their limits

$$\frac{1}{m(B_T(0))} \int_{B_T(0)} f(x'u(t))\,\mathrm{d}\lambda_S(t) \overset{T\to\infty}{\to} \int f d\mu$$

and

$$\frac{1}{m(B_T(0))} \int_{B_T(0)} f_c(xu(t))\,\mathrm{d}\lambda_S(t) \overset{T\to\infty}{\to} \int f_c d\mu = \int f dc_*\mu.$$

are equal for any f, which completes the proof. □

Now take a compact subset K of X' with $\mu(K) > 0.9$. The reason we are doing so will become more transparent in the next section but in a nutshell, is because of the fact that limits of generic points need not be generic. At a later stage, we will be using a limiting argument to try to produce two points in X' and the only way to ensure that this is the case is to start in K.

Recall that λ_S is the product Haar measure on $\prod_{s\in S_0} k_s$. Applying the decreasing Martingale theorem again to the sequence of σ-algebras of $B_T^{\mathbf{U}}$-invariant sets gives us that for μ almost every $y \in X$,

$$\frac{1}{\lambda_S(B_T(0))} \int_{B_T(0)} 1_K(yu(t))d\lambda_S(t) \overset{T\to\infty}{\to} \mu(K). \tag{4.3}$$

By choosing a large T_0, we can arrange that $\mu(X_1) > 0.99$ where

$$X_1 \overset{def}{=} \Big\{ y \in X' \mid \lambda_S(\{t \in B_T(0) \mid yu(t) \in K\}) > 0.8\lambda_S(B_T(0)) \text{ for all } T > T_0 \Big\}. \tag{4.4}$$

And for $\eta > 0$, we set

$$X_2 \overset{def}{=} \left\{ z \in X \mid \frac{1}{m_{\mathbf{L}}(B_\eta^{\mathbf{L}})} \int_{B_\eta^{\mathbf{L}}} 1_{X_1}(zl)dm_{\mathbf{L}}(l) > 0.9 \right\} \tag{4.5}$$

where $m_{\mathbf{L}}$ is Haar measure on the locally compact group $\mathbf{L}$ and $B_\eta^{\mathbf{L}}$ is the ball of radius η around I in $\mathbf{L}$. The definition of X_2 of course depends on the number η (which is chosen sufficiently small such that $B_\eta^{\mathbf{L}} \subset \tilde{L}$ as before). It follows from a standard density argument, (c.f. e.g. Lemma 4.4 in [6]) that

Lemma 4.4. *$\mu(X_2) > 0.9$ for every $\eta > 0$.*

We now record a lemma which will allow us in the next section to produce points in X_2 with certain specified properties.

Lemma 4.5. *For every $\epsilon > 0$ there exists $\delta > 0$ such that for $z, z' \in X_2$ with $z' = zg$ and $g \in B_\delta^{\mathbf{G}}$ there exists $l_2 \in B_1^{\mathbf{L}}$ and $l_1 \in B_\epsilon^{\mathbf{L}}(l_2)$, satisfying*

$$zl_1, z'l_2 \in X_1 \tag{4.6}$$

and

$$gl_2 = l_1(I + v) \text{ for some } v \in B_\epsilon^{\mathfrak{l}^\perp}(0). \tag{4.7}$$

Proof. Fix z as in the statement of the lemma and choose $\delta > 0$ so that the map $\pi_{\mathbf{L}}$ which sends $g \in B_\delta^{\mathbf{G}}$ to $l \in \mathbf{L}$ where each component of l for $s \in S_0$ is determined by Lemma 3.1, is well defined. In other words, we have

$$g_s = (I + v_s)(\pi_{\mathbf{L}}(g))_s \tag{4.8}$$

where $v_s \in \mathfrak{l}_s^\perp$ and $\pi_{\mathbf{L}}(g_s) = l_s \in L_s$ for any $s \in S_0$ (and $\pi_{\mathbf{L}}(g_s) = I$, $g_s = I + v_s$ for $s \in S \backslash S_0$).

Fix $g \in B_\delta^{\mathbf{G}}$ and set $z' = zg$. Consider the sets

$$E = \{l \in B_\eta^{\mathbf{L}} \ : \ zl \in X_1\} \tag{4.9}$$

and

$$E' = \{l \in B_\eta^{\mathbf{L}} \ : \ z'l \in X_1\}. \tag{4.10}$$

Note that by construction,

$$m_{\mathbf{L}}(E) > 0.9 m_{\mathbf{L}}(B_\eta^{\mathbf{L}}) \text{ and } m_{\mathbf{L}}(E') > 0.9 m_{\mathbf{L}}(B_\eta^{\mathbf{L}}). \tag{4.11}$$

We remind the reader that it follows from the discussion preceding Lemma 3.1, that there exists $\eta > 0$ and a map ϕ which maps a neighborhood of 0 in $\prod_{s\in S_0} k_s^{\dim \mathbb{L}_s}$ to a neighborhood of size η in $\mathbf{L}$. This determined η and we define ψ on $B_\eta^{\mathbf{L}}$ by the formula

$$\psi(l) = l\pi_{\mathbf{L}}(l^{-1}gl)^{-1} \tag{4.12}$$

and again, we assume that δ is chosen to make $l^{-1}gl$ for all $l \in B_\eta^{\mathbf{L}}$ sufficiently small so that $\pi_{\mathbf{L}}(l^{-1}gl)$ is defined. We note that ψ depends on g implicitly. The conclusion of the lemma is then equivalent to the claim that there exists some $l_1 \in E$ for which $l_2 = \psi(l_1) \in E'$. Because for such l_1, l_2,

$$l_1^{-1}gl_2 = (l_1^{-1}gl_1)l_1^{-1}l_2 = (I+v)\pi_{\mathbf{L}}(l_1^{-1}gl_1)l_1^{-1}\psi(l_1) = (I+v_1) \tag{4.13}$$

where $v_1 \in \mathfrak{l}^\perp$ is as in the definition of $\pi_{\mathbf{L}}(l_1^{-1}g)$. Also note that for $\delta > 0$ sufficiently small, $\pi_{\mathbf{L}}(l_1^{-1}gl)$ (and similar v_1) can be made arbitrarily close to I (resp. $0 \in \mathfrak{l}^\perp$).

To show the existence of $l_1 \in E \cap \psi^{-1}(E')$, we will show that ψ is measure preserving. Since the Haar measure on $\mathbf{L}$ is the product measure and ψ is a product map, we may fix a place $s \in S_0$ and prove that ψ is measure preserving for this place, and we will drop the subscript s here onwards. Notice first that for sufficiently small δ and η, we have $\psi(l) = l\pi_L(l^{-1}gl)^{-1} \in B_\eta^L$ for any $l \in B_\eta^L$ by the ultrametric inequality, so that ψ is indeed a map from $B_\eta^L \to B_\eta^L$. Recall that the metric for sufficiently small elements, say for $g_1, g_2 \in B_\eta^G$, is defined by

$$d(g_1, g_2) = \|g_1 - g_2\|. \tag{4.14}$$

We will show that ψ is an isometry with regard to this metric and that the measure of the ball

$$\bar{B}_\eta^L = \{l \in L \mid d(l, I) \le s^{-n}\}$$

equals s^{-n} as long as $s^{-n} < \eta$, after normalizing the Haar measure on L by a constant as well as the metrics on k by a power (here s is the cardinality of the residue field $\mathbb{F}_s$ defined by $k = k_s$). This implies that ψ is measure preserving on sufficiently small balls thus completing the proof of the theorem.

We first prove the claim regarding the measure of small balls. Notice that $l \in B_\eta^L$ can be written in the form

$$l = I + u + v \text{ with } u \in \mathfrak{l}, v \in \mathfrak{l}^\perp$$

and moreover, that the map $\omega : B_\eta^L \to \mathfrak{l}$ sending l to u is an isometry. To prove this consider the analytic map ϕ preceding Lemma 3.1, but with a more careful choice of the regular functions $x_1, \ldots, x_{\dim \mathbb{L}}$. Let $e_1, \ldots, e_{\dim \mathbb{L}}$ be a basis of $\mathfrak{l}$ and let $e_{\dim \mathbb{L}+1}, \ldots, e_{n^2}$ be a basis of $\mathfrak{l}^\perp$. We now let $x_1, \ldots, x_{\dim \mathbb{L}}$ be the first $\dim \mathbb{L}$ coordinates of $g - I$ with respect to this basis of $\mathrm{Mat}_n(k_s)$, then ϕ has the form

$$\phi(x_1, \ldots, x_{\dim \mathbb{L}}) = I + x_1 e_1 + \cdots + x_{\dim \mathbb{L}} e_{\dim \mathbb{L}} + \text{ higher order terms}$$

and in particular, this shows that ϕ is an isometry in a sufficiently small neighborhood of $0 \in \mathfrak{l}$ by the ultrametric inequality which implies that ω is as well. Moving on to the claim about the volumes of balls, notice that $\bar{B}^L_{s^{-n}}$ is a compact subgroup of L and the index of $\bar{B}^L_{s^{-(n+1)}}$ in $\bar{B}^L_{s^{-n}}$ is equal to the index of $\bar{B}^{\mathfrak{l}}_{s^{-n}}$ in $\bar{B}^{\mathfrak{l}}_{s^{-(n+1)}}$, and in both spaces, this index equals the number of closed balls of radius $s^{-(n+1)}$ needed to cover the ball of radius s^{-n} and because $\hat{\phi}$ is an isometry. The above claim then follows because it is known that the norm on k can be constructed from the Haar measure on k via the module function.

We now move on to the former claim, namely that ψ is an isometry. First of all, note that by construction, $\psi(l)$ satisfies

$$l^{-1} g \psi(l) \in I + \mathfrak{l}^\perp \tag{4.15}$$

and that this property actually characterizes $\psi(l) \in B_\eta^L$. Hence $\psi(l) = \Psi(l, g)$ is the specialization of the analytic map

$$\Psi : \bar{B}_\eta^L \times \bar{B}_\delta^G \to \bar{B}_\eta^L \tag{4.16}$$

which is characterized by the property that

$$l^{-1} g \Psi(l, g) \in I + \mathfrak{l}^\perp \tag{4.17}$$

and in turn is constructed from the function

$$(l, g, l') \in L \times G \times L \to (l, g, u) \in L \times G \times \mathfrak{l}$$

satisfying

$$l^{-1} g l' = I + u + v \text{ with some } v \in \mathfrak{l}^\perp$$

by restricting the local inverse to a neighborhood of $(I, I, 0)$ in $L \times G \times \{0\}$. Since $\Psi(l, e) = l$, in the coordinate system given by $x_1, \ldots, x_{\dim \mathbb{L}}$

via $l = \phi(x_1, \ldots, x_{\dim \mathbb{L}})$, the power series to $\psi(l) = \Psi(l, g)$ equals

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{\dim \mathbb{L}} \end{pmatrix} \to A_g \begin{pmatrix} x_1 \\ \vdots \\ x_{\dim \mathbb{L}} \end{pmatrix} + \text{ higher order terms in } x_1, \ldots, x_{\dim \mathbb{L}} \tag{4.18}$$

where $\|A_g - I_{\dim \mathbb{L}}\| < 1$ as long as δ is sufficiently small. This clearly implies that ψ is an isometry by the ultrametric inequality, which in turn implies that ψ is measure preserving and therefore completes the proof of the lemma.

□

5. Polynomial divergence of unipotent orbits : the induction.

We assume that $\mathbb{L}, \mathbf{L}$ and $\tilde{L}$ are as in the inductive hypothesis. Initially, $\mathbb{L} = \mathbb{H}, \mathbf{L} = \mathbf{H}$ and $\tilde{L} = \mathbf{H}^+$. Let $x_0 \in X_2 \cap \operatorname{supp} \mu|_{X_2}$. If $\mu(x_0.\mathbf{S}) > 0$ for $\mathbf{S} = \operatorname{Stab}(\mu)$, then by Lemma 4.2, the measure μ is homogeneous. So we may assume that

$$\text{for every } \delta > 0, x_0.B_\delta^{\mathbf{G}} \cap X_2 \nsubseteq x_0.B_\delta^{\operatorname{Stab}(\mu)}. \tag{5.1}$$

Fix some $\delta > 0$ and applying Section 4 to our current choice of η to obtain X_2, fix some $z' \in X_2 \cap x_0.B_\delta^{\mathbf{G}} \backslash x_0.B_\delta^{\operatorname{Stab}(\mu)}$. Applying Lemma 4.5 to the points $z = x_0$ and $z' = x_0 g$ with $g \in B_\delta^{\mathbf{G}}$ we find $l_1, l_2 \in \tilde{L}$ such that $x = z l_1, x' = z' l_2 \in X_1 \subseteq X'$. Moreover, by Lemma 4.5, $x' = x(I + v)$ for some $v \in B_\epsilon^{\mathfrak{l}^\perp}$ and $(I + v) = l_1^{-1} g l_2 \notin \operatorname{Stab}(\mu)$. Therefore, by Lemma 4.3, we know that $I + v \notin C(\mathbf{U})$ and we will use this setup to find a one-parameter unipotent subgroup $\mathbb{F}_{s,m+1}$ defined over k_s, not contained in $\mathbb{L}$, but whose k_s-points belong to $\operatorname{Stab}(\mu)$. This will allow us, under our restriction on the characteristic to produce a group $\mathbb{L}'$ containing $\mathbb{L}$ and with bigger dimension, which still satisfies all of the inductive assumptions. Once this has been established, Theorem 1.1 follows, since if the assumption above in (5.1) is true, we can increase the dimension of $\mathbb{L}$ indefinitely. The first step towards proving the above claims will be the following proposition whose proof is completely analogous to Lemma 4.7 and Lemma 4.8 in [6] and previously developed in [27], [29], [30] and therefore is omitted. We remind the reader that by choice we consider the Lie algebra of U to be of positive weight.

Proposition 5.1. *Suppose for every $\epsilon > 0$, there exist two points $x, x' \in X_1$ and some $v \in \mathfrak{l}^\perp$ with $\|v\| < \epsilon$ such that*

$$x' = x(I + v) \textit{ and } I + v \notin C(\mathbf{U}). \tag{5.2}$$

Then there exists some nonzero $w \in \mathfrak{l}^\perp$ which satisfies $I + w \in \mathrm{Stab}(\mu)$. Moreover, $w = (w_s)$ with $w_s = 0$ for $s \in S \backslash S_0$ and $w_s \neq 0$ for some $s \in S_0$, and for such s, w_s is a sum of positive weight eigenvectors for the adjoint action of A_s which is the k_s-points of the diagonal subgroup $\mathbb{A}_s \subset \mathbb{H}_s$.

5.1. **Producing an invariant one-parameter subgroup in the complement.** We have inductively assumed that μ is invariant under an open subgroup $\tilde{L}$ of $\mathbf{L}$ and have produced (under the assumptions of Proposition 5.1) an element $g = I + w \in \mathrm{Stab}(\mu)$ where $w \in \mathfrak{l}^\perp \backslash \{0\}$. We want to use this element to produce a one parameter subgroup.

Proposition 5.2. *Under the assumptions of Proposition 5.1, and the assumptions on characteristic as in Theorem 1.1*[5]*, there exists some $s \in S_0$ and a nonzero $w' \in \mathrm{Mat}_n(k_s)$ such that*

$$\exp(tw') = I + tw' + \frac{t^2}{2} w'^2 + \cdots + \frac{t^{n-1}}{(n-1)!} (w')^{n-1} \in \mathrm{Stab}(\mu) \quad (5.3)$$

for all $t \in k_s$ and the image variety $\mathbb{F}_{s, m_s+1}$ of this polynomial map does not belong to $\mathbb{L}$.

Proof. Let $g = I + w \in \mathrm{Stab}(\mu)$ be as in Proposition 5.1. As in the proposition, we will fix a place s such that w_s is non-trivial and is a sum of positive weight eigenvectors for the A_s action. In order to prove this proposition, we first prove the existence of $g' \in \mathrm{Stab}(\mu)$ of the form $g' = \exp(w')$ where $w' \in \mathfrak{l}^\perp \backslash \{0\}$ has the following form: w'_s is an eigenvector for A_s, and w' is zero at all other places. We can then produce a one-parameter subgroup using conjugation by elements of A_s, as we show below. It will be useful to be able to measure sizes during the course of the proof and for this we will use the maximum norm (defined on $\mathrm{Mat}_n(k_s)$) on matrix entries, denoting it by $\|\cdot\|$. We now choose $a \in \mathbf{A} \cap \mathbf{H}^+ \subseteq \mathrm{Stab}(\mu)$ such that $a_{s'} = I$ for every $s' \neq s$ and a_s satisfies $\|a_s w_s a_s^{-1}\| < 1$. Replacing g by aga^{-1} if necessary we may assume that $\kappa \overset{def}{=} \|w_s\|_\infty < 1$ and then choose $a \in \mathbf{A}$ such that $\|a_s w_s a_s^{-1}\| \leq \kappa^2$, and again $a_{s'} = I$ for $s' \neq s$. With this choice, $(ag^{-1}a^{-1}g)'_s = I$ for $s' \in S \backslash \{s\}$ and

$$\begin{aligned}(ag^{-1}a^{-1}g)_s &= (I + aw_s a^{-1})^{-1}(I + w_s) \\ &= (I - aw_s a^{-1} + aw_s^2 a^{-1} - \cdots \pm aw_s^n a^{-1})(I + w_s) \\ &= I + w_s + \ldots \end{aligned} \quad (5.4)$$

[5]This is overkill for the purposes of this proposition. We only need $\mathrm{char}(k_s) > n$ for all $s \in S_0$.

where the dots indicate various terms of norms smaller than κ^2. Clearly, $g'' \overset{def}{=} ag^{-1}a^{-1}g \in \mathrm{Stab}(\mu)$. Finally, using the standard power series again which is a polynomial due to the fact that g'' is unipotent, we define the element $w'' = \log(g'')$ which satisfies $(w'')_{s'} = 0$ for $s' \in S\backslash\{s\}$ and $(w'')_s = w_s + \dots$ where the dots again refer to terms of norm less than κ^2. The next step will be to ensure that w'' belongs to $\mathfrak{l}_s^\perp$. It turns out that this is already almost true. Note that we have the weight space decomposition for the A_s action:

$$\mathrm{Mat}_n(k_s) = \bigoplus_i V_i \tag{5.5}$$

and so we can write

$$w_s = \sum_{i=i_0}^{\infty} w_i \tag{5.6}$$

with $i_0 \geq 1, w_{i_0} \neq 0$. Conjugating by elements in A_s as before, we can contract all the w_i's to ensure that from the beginning $\kappa = \|w_s\| = \|w_{i_0}\|$. What we have therefore shown is that if w_s'' is written as $w_s'' = \sum_{i \geq i_0} w_i''$, then

$$w_{i_0}'' = (1+\lambda) w_{i_0} \text{ for some } \lambda \in k_s. \tag{5.7}$$

Since $w \in \mathfrak{l}_s^\perp$ and $\mathfrak{l}_s^\perp$ is A_s-invariant, this shows that $w_{i_0}'' \in \mathfrak{l}_s^\perp$, and as it turns out, this is good enough for our purposes. Write $a_s(t) \in A_s$ for the element corresponding corresponding to the diagonal matrix with entries t, t^{-1}. Recall that

$$a_s(t) g_s'' a_s(t)^{-1} = \exp(\mathrm{Ad}_{a_s(t)}(w_s'')) = \exp\left(\sum_{i=i_0}^{\infty} t^i w_i''\right) \tag{5.8}$$

Now let i_1 be the smallest integer greater than i_0 such that $w_{i_1} \neq 0$, if no such i_1 exists, we are done. Then, for $t' \in \mathbb{Z}$ and any $t \in k_s$ we have

$$\begin{aligned} \exp(t'w'') \exp\left(\sum_{i=i_0}^{\infty} t^i w_i''\right) = \\ \exp\left((t' + t^{i_0}) w_{i_0} + (t' + t^{i_1}) w_{i_1} + \dots\right) \in \mathrm{Stab}(\mu)(\mu) \end{aligned} \tag{5.9}$$

where the dots indicate terms of higher weight. We now proceed to systematically rid ourselves of higher weight terms. First, we choose $t \in \mathbb{Z}$ such that t^{i_0} is not congruent to t^{i_1} modulo $\mathrm{char}(k_s)$ and then choose $t' \in \mathbb{Z}$ such that $t' + t^{i_1} = 0$, which will get rid of w_{i_1}. Repeating this process, we can cancel all terms of bigger weight than i_0, at the small cost of replacing w_{i_0} by some non-trivial integer multiple. Since $w_{i_0} \in \mathfrak{l}_s^\perp\backslash\{0\}$, we have shown the existence of $w' \in \mathfrak{l}_s^\perp \cap V_{i_0}$

(the aforementioned multiple of w_{i_0}) such that $g' = \exp(w') \in \mathrm{Stab}(\mu)$. Moreover, $a_s(t)g'a_s(t)^{-1} = \exp(t^{i_0}w') \in \mathrm{Stab}(\mu)$ for all $t \in k_s$. Since all these elements belong to the one parameter subgroup $\mathbb{F}_{s,m+1}$ which is defined to be the image variety of the polynomial map $\exp(tw')$, we only have to prove the following lemma in order to complete the proof of the proposition. $\square$

Lemma 5.3. *If $i < \mathrm{char}(k_s)$ then the expressions of the form t^i for $t \in k_s$ span a dense subgroup P of k_s.*

Proof. We first note that since P as above is invariant under multiplication by $\pi^{\pm i}$ where π is the uniformizer of k_s, it is enough to prove that every $t \in k_s, \|t\| < 1$ can be approximated by elements of P. For such an t, we can build a sequence of elements in P approximating it to an arbitrary degree as follows. Note that,

$$(1 + t/i)^i - 1 = (1 + t + \cdots) - 1 = t + \cdots = t + t_1 \in P \tag{5.10}$$

where the dots stand for terms with norm smaller than $\|t\|$, represented by t_1. We can now repeat the procedure for t_1, to get that

$$(1 + t_1/i)^i - 1 = t_1 + \cdots = t_1 + t_2 \tag{5.11}$$

where t_2 is even smaller in norm. This procedure can be continued till we make the error as small as we please, hence completing the proof of the lemma. $\square$

5.2. **Finishing the induction.** We devote this section to generating a bigger subgroup $\mathbb{L}' \supseteq \mathbb{L}$ satisfying the inductive hypotheses. In Proposition 5.1, we have found some $s \in S_0$ and a one parameter unipotent subgroup $\mathbb{F}_{s,m_s+1}$ defined over k_s for some $s \in S$ with Lie algebra contained in $\mathfrak{l}_s^{\perp}$ which satisfies $\mathbb{F}_{s,m_s+1}(k_s) \subseteq \mathrm{Stab}(\mu)$. We first recall the simple procedure of generating an algebraic group from given algebraic varieties. By the inductive assumption $\mathbb{H}_s, \mathbb{F}_{s,1}, \ldots, \mathbb{F}_{s,m_s}$ are subgroups of $\mathbb{G}_s$ such that

$$\mathbb{L}_s = \mathbb{H}_s\mathbb{F}_{s,1} \ldots \mathbb{F}_{s,m_s} \tag{5.12}$$

is also a subgroup of $\mathbb{G}_s$ of dimension $3+m_s$. We define $\mathbb{V}_1 = \mathbb{L}_s\mathbb{F}_{s,m_s+1}$, $\mathbb{V}_2 = \mathbb{V}_1\mathbb{F}_{s,m_s+2}, \ldots, \mathbb{V}_\ell = \mathbb{V}_{\ell-1}\mathbb{F}_{s,m_s+\ell}$ and we will stop when $\mathbb{V}_\ell = \mathbb{L}'$ is again an algebraic group, see Prop. 2.2.7 in [35]. Here $\mathbb{F}_{s,m_s+2}$ is either $\mathbb{F}_{s,j}$ for some $j = \{1, \ldots, m_s\}$ or $\mathbb{U}^-$ (which is the lower triangular unipotent subgroup of $\mathbb{H}_s$) or $\mathbb{F}_{s,m_s+2} = \mathbb{U}$, depending on which of these choices make the dimension of $\mathbb{V}_2$ bigger than the dimension of $\mathbb{V}_1$. The varieties $\mathbb{V}_i$ are chosen similarly. If, at some stage, none of the available choices increase the dimension of the resulting variety, we would have found our group $\mathbb{L}'_s$. The group $\mathbb{L}$ is then defined as the product of $\mathbb{L}_{s'}$ for $s' \neq s$ and and $\mathbb{L}'_s$. We are now ready for:

Proposition 5.4. *There is a constant M depending on n such that if char$(k_s) > M$ for all $s \in S_0$, then the above map*

$$\mathbb{H}_s \times \mathbb{F}_{s,1} \times \cdots \times \mathbb{F}_{s,m_s+\ell} \to \mathbb{L}'_s \subseteq \mathbb{G} \tag{5.13}$$

is separable. In particular, the group $\tilde{L}'_s$ generated by $\tilde{L}_s$ and $\mathbb{F}_{s,m+1}(k_s)$ is open in the group L'_s of points of $\mathbb{L}'_s$.

Proof. Recall that $\mathbb{H}_s$ can also be generated by unipotent one parameter subgroups, in fact $\mathbb{H}_s = \mathbb{U}\mathbb{U}^-\mathbb{U}$ as varieties and so we define $\mathbb{U}_1 = \mathbb{U}$, $\mathbb{U}_2 = \mathbb{U}^-$, and $\mathbb{U}_3 = \mathbb{U}$. Hence it suffices to bound the degree of the following finite morphism. Let $\phi_i(t)$ for $i = 1, 2, 3$ be parameterizations of $\mathbb{U}_i$. Let $\psi_j(t) = \exp(tw_j)$ for $j = 1, \ldots, m + \ell$ be parameterizations of $\mathbb{F}_{s,j}$. Define

$$\Theta : \mathbb{A}^{3+m_s+\ell} \to \mathbb{L}'_s \tag{5.14}$$

by

$$\Theta(t_1, t_2, t_3, r_1, \ldots, r_{m_s+\ell}) = \phi_1(t_1) \cdots \psi_1(r_1) \ldots \psi_{m_s+\ell}(r_{m_s+\ell}). \tag{5.15}$$

By definition of $\mathbb{L}'_s$, this map is both dominant, i.e. $\Theta(\mathbb{A}^{3+m+\ell})$ is Zariski dense in $\mathbb{L}'_s$, and the field extension induced by

$$\Theta^* : k_s(\mathbb{L}'_s) \to k_s(\mathbb{A}^{l+m_s+\ell}) \tag{5.16}$$

is algebraic. This follows from the construction of $\mathbb{L}'_s$: In the inductive choice of the subgroups $\mathbb{F}_{s,j}$ we always choose it such that the dimension of the image $\mathbb{H}_s\mathbb{F}_{s,1} \cdots \mathbb{F}_{s,j}$ is equal to $3 + j$. (If the dimension of the product does not increase when $\mathbb{F}_{s,j}$ is multiplied on the right then a different subgroup is chosen to be the next one and if for no choice the dimension goes up, we have found the subgroup $\mathbb{L}'_s$ as needed). Therefore, the dimension of $\mathbb{L}'_s$ equals $3+m_s+\ell$ so the transcendence degrees of the fields are the same. We remind the reader that the groups $\mathbb{F}_{s,j}$ constructed so far in the proof depend on the properties of the measure. However, the structure of the maps $\psi_1, \ldots, \psi_{m_s+\ell}$ is such that the degree of these polynomials in bounded by the nilpotency degree of the matrices $w_j \in \mathrm{Mat}_n(k_s)$ and hence by n, i.e. we may view all of the maps $\psi_1, \ldots, \psi_{m+\ell}$ as particular elements of a finite dimensional family of polynomial maps. By Bezout's theorem (but see the Appendix B for a more concrete argument), this implies that there exists a universal bound M depending only on the degree of the polynomials ϕ_i, ψ_j which is bounded by n, and their number $3 + m_s + \ell$ (bounded by $n^2 = \dim \mathrm{Mat}_n(k_s)$) so that for any choice of parameters used to define $\psi_1, \ldots, \psi_{m_s+\ell}$ and hence Θ, the extension induced by (5.14) is either not algebraic or has degree bounded by M. Our assumption that

$\mathrm{char}(k_s) > M$ for all $s \in S_0$ ensures the separability of the field extension[6]. By Theorem 4.3.6 (ii) in [35] there exists a nonempty Zariski open subset of simple points in $\mathbb{A}^{3+m_s+\ell}$ where the derivative of Θ is an isomorphism. Let $(t_1, t_2, t_3, r_1, \ldots, r_{m_s+\ell})$ be such a point, at which Θ has a local analytic inverse. Then there is an open neighborhood in L'_s of $\Theta(t_1, t_2, t_3, r_1, \ldots, r_{m_s+\ell})$ which entirely belongs to the subgroup $\tilde{L}'_s$ generated by $\tilde{L}_s$ and $\mathbb{F}_{s,m+1}(k_s)$ which in return implies that $\tilde{L}'_s$ is open in $\mathbf{L}'_s$. ☐

Appendix A. Semisimplicity of finite dimensional representations

In this appendix, we provide a self contained proof of the semisimplicity of representations of $\mathrm{SL}(2, k)$ under the assumption that the weights are small in comparison to the characteristic. Though this result seems to be well known in the literature, we have not been able to identify a source which treats the case when k is not algebraically closed. Our strategy will be as in the characteristic zero case.

Let V be an algebraic $\mathrm{SL}(2, k)$-module and $\phi : \mathrm{SL}(2, k) \to \mathrm{GL}(V)$ be the natural homomorphism. By Theorem 2.4.8 (2) in [35], $\phi(a)$ is a semisimple endomorphism of V for every $a \in A$ where A denotes the diagonal subgroup of $\mathrm{SL}(2, k)$.

The characters of $A \cong \mathbb{G}_m$ are in one-one correspondence with the integers. Therefore there exists $n \in \mathbb{Z}$ such that

$$V = \bigoplus_{i=-n}^{n} V_i, \tag{A.1}$$

here the diagonal matrix $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ for $a \in k^*$ acts in V_i through multiplication by a^i. To simplify notation, we will write $\phi(a)$ for $\phi\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right)$. Thus $\phi(a)|_{V_i}$ is multiplication by a^i on V_i. Also i will be referred to as the weight of V_i and V_i will be referred to as the weight space of i. As ϕ is defined over k, each V_i is k-rational. Recall that we denote by π, the uniformizer of k. Then $\phi(\pi)$ is k-linear and V_i is the eigenspace of $\phi(\pi)$ corresponding to π^i.

Let $u(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ be the standard parametrization of the upper triangular unipotent subgroup U of $\mathrm{SL}(2, k)$ and let $w \in \mathrm{End}(V)$ be a

[6]In fact, we only need $\mathrm{char}(k_s)$ not to divide the degree of the extension, which is clearly satisfied.

non-trivial element of the one-dimensional Lie algebra of the algebraic group $\phi(V)$. We now prove,

Lemma A.1. *Assuming that* $\mathrm{char}(k) \neq 2$*, the adjoint representation of* $\phi(a)$ *satisfies*[7]

$$\mathrm{Ad}_{\phi(a)}(w) = a^2 w. \tag{A.2}$$

Proof. First we notice that since ϕ is a non-trivial rational representation of $\mathrm{SL}(2,k)$, $\phi(\mathrm{SL}(2,k))$ is isomorphic to $\mathrm{SL}(2,k)$ or $\mathrm{PGL}(2,k)$. For $\mathrm{SL}(2,k)$, the lemma holds by a simple matrix calculation. Hence the lemma follows if $d\phi$ is invertible which in turn follows if ϕ is invertible. Otherwise $\phi(\mathrm{SL}(2,k))$ is isomorphic to $\mathrm{PGL}(2,k_s)$ and $d\phi$ is invertible as long as $\mathrm{char}(k) \neq 2$ by [35, 4.4.11]. □

Take $v \in V_i$ and consider $w(v)$. Clearly,

$$\phi(a)(w(v)) = \mathrm{Ad}_{\phi(a)}(w)(\phi(a)(v)) = a^2 w(a^i v) = a^{i+2}(w(v)) \tag{A.3}$$

and so $w(v_i) \subset V_{i+2}$. In particular, $w(V_n) = 0$. We now assume that the highest weight n appearing in V satisfies

$$n < \mathrm{char}(k) = p. \tag{A.4}$$

We thus get

$$w^n(V) = w^n(V_{-n}) \subset V_n \tag{A.5}$$

and

$$w^{n+1}(V) = 0. \tag{A.6}$$

So that the polynomial map exp defined on the Lie algebra of $\phi(U)$ by

$$\exp(w) = I + w + \frac{1}{2!}w^2 + \cdots + \frac{1}{n!}w^n \tag{A.7}$$

is well defined.

Lemma A.2. *If the heighest weight n in the representation V of* $\mathrm{SL}(2,k)$ *satisfies* $n < \mathrm{char}(k)$*, then* exp *is an isomorphism of algebraic groups between the Lie algebra of $\phi(U)$ and $\phi(U)$.*

Similarly, we get $w'(V_i) \subset V_{i-2}$ and may define the isomorphism $\exp(w') \in \phi(U')$ *for elements w' of the Lie algebra of $\phi(U')$ where* $U' = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$.

A corollary to the above is that a subspace which is invariant under the Lie algebras of U is actually invariant under U.

[7] The variable a always stands for the diagonal matrix in $\mathrm{SL}(2,k)$ with entries a, a^{-1} even in the case where $\phi(\mathrm{SL}(2,k))$ is isomorphic to $\mathrm{PGL}(2,k)$.

Proof. Notice first that because $w^{n+1} = 0$ it is easily checked that exp is a homomorphism. Also, the terms $w, w^2, \ldots$ are in different weight spaces for $\mathrm{Ad}_{\phi(a)}$ which shows that exp is an isomorphism between the Lie algebra of $\phi(U)$ and a group with the same Lie algebra (which does not yet imply that the image equals $\phi(U)$).

To see that the image indeed equals $\phi(U)$ we analyze the map

$$\psi(t) = \phi(u(t)) = I + w_1 t + w_2 t^2 + \cdots w_\ell t^\ell$$

where $\ell \in \mathbb{N}$ and $w_i \in \mathrm{End}(V)$ for $i = 1, \ldots, \ell$. Because $au(t)a^{-1} = u(a^2 t)$ we get $\phi(a)\psi(t)\phi(a)^{-1} = \psi(a^2 t)$ which implies that w_i has weight $2i$ under conjugation by $\phi(a)$. However, since the biggest weight of conjugation on $\mathrm{End}(V)$ is $2n$ we get that the degree ℓ of ψ satisfies $\ell \le n$. Also we know that $\psi(t+s) = \psi(t)\psi(s)$ which we may expand and then compare the terms corresponding to the monomial ts^{i-1}. This gives $iw_i ts^{i-1} = w_1 t w_{i-1} s^{i-1}$ for all $i = 1, \ldots, \ell$ and so $i! w_i = w_1$. In other words, $\psi(t) = \exp(tw_1)$, which gives the lemma. □

We are now ready for

Lemma A.3. *Let V be an $\mathrm{SL}(2,k)$-module defined over k, and let $n < \mathrm{char}(k)$ be the highest weight appearing in V. Let $v_{\mathrm{highest}} \in V$ be a highest weight vector, i.e. a nonzero vector $v_{\mathrm{highest}} \in V_r$ (for some $r \le n$) which satisfies $w(v_{\mathrm{highest}}) = 0$ for every $w \in \phi(U)$. Then v_{highest} generates a submodule V_1 isomorphic to $\mathrm{Sym}^r(k^2)$ and the latter is irreducible.*

Proof. Let $v_r = v_{\mathrm{highest}}$ with r as above. We let V_1 be the vector space spanned by v_r and the vectors $w_2^i(v)$ where $w_2 = D\phi\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$ belongs to the Lie algebra of $\phi(U')$. Note that V_1 is by construction, invariant under w_2. We claim that V_1 is also invariant under $w_1 = D\phi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$. Let $w_a = D\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right) = [w_1, w_2]$. Then it follows that $w_a(v_r) = rv_r$ since v_r has weight r. Consequently,

$$\begin{aligned} w_1(w_2(v_r)) = w_1(v_{r-2}) = w_1(w_2(v_r)) \\ = [w_1, w_2](v_r) + w_2(w_1(v_r)) = rv_r. \end{aligned}$$

It follows by induction, that for $l \in \mathbb{Z}_+$,

$$w_1(w_2^l(v_r)) = l(r - l + 1)w_2^{l-1}(v_r). \tag{A.8}$$

And this shows that w_1 leaves V_1 invariant. We call the coefficients $l(r-l+1)$ in this formula the structure constant of the representation. Note that for $l, r < n$, the structure constants are non-zero modulo char(k) by assumption on the highest weight. Thus and by Lemma A.2 V_1 is invariant under $\phi(U)$ as well as $\phi(U')$, and consequently, under $\phi(\mathrm{SL}(2,k))$. The lemma now follows: Any nonzero vector $v \in V_1$ splits a sum of terms of the form $t_i w_2^i v_r$, let i be the maximal index with $t_i \neq 0$. Applying w_1 to v replaces the term $t_i w_2^i v_r$ in v by $t_i i(r-i+1) w_2^{i-1} v_r$, which gives that $w_1^i v$ is a nonzero multiple of the vector v_r we started with. Hence applying powers of w_2 all of V_1 is generated by a single nonzero vector v. It is now easy to give a concrete isomorphism between the irreducible representation V_1 and $\mathrm{Sym}^r(k^2)$. □

It only remains to prove:

Lemma A.4. *Let V be an $\mathrm{SL}(2,k)$-module defined over k and let $n <$ char(k) be the highest weight appearing in V. Then V is semisimple. In other words, any invariant submodule of V_1 of V has a complementary submodule V_1' and V splits into irreducible submodules isomorphic to $\mathrm{Sym}^r(k^2)$ for various $r \leq n$.*

Proof. Let V_1 be an invariant submodule of V. We will prove the lemma by induction on the codimension of V_1 in V. Clearly, the claim is trivial if $V_1 = V$.

Suppose now V_1 is a proper subspace. Then there exists an $r \leq n$ and a nonzero $v \in V_r \backslash V_1$ such that $w_1(v) \in V_1$. By the arguments in the proof of Lemma A.3 it follows easily that $w_1(v) = w_1(v')$ for some $v' \in V_1$. Replacing v by $v - v'$, we assume that $w_1(v) = 0$. By Lemma A.3, there is an invariant submodule V_2 containing v which is isomorphic to $\mathrm{Sym}^r(k^2)$ and irreducible. Therefore, V_1 intersects V_2 trivially. Thus $V_1 \oplus V_2$ is a bigger submodule of V, which by the inductive hypothesis has an invariant complement V' as in the lemma. Clearly $V_2 \oplus V'$ is then an invariant complement to V_1 which also splits into irreducible submodules isomorphic to various symmetric powers. □

Appendix B. A direct argument for the bound on the degree of Φ

In this appendix we give a direct proof for the bound M of the degree of Φ that we needed to finish the induction in Section 5. In the purely algebraic discussion below we will use capital letters for variables and lower case letters for elements of a field.

Proposition B.1. *Let $m, n, r \geq 1$ be integers and let*

$$\Phi = (\phi_1, \ldots, \phi_n)$$

be an n-tuple of polynomials

$$\phi_i \in \mathbb{Z}[X_1, \ldots, X_m, T_1, \ldots, T_r].$$

Then there exists some M (depending on Φ) with the following property for all fields k and all choices of parameters $\mathbf{t} = (t_1, \ldots, t_r) \in k^r$. Let

$$\Phi_{\mathbf{t}} : \mathbb{A}^m \to \mathbb{A}^n$$

be the morphism defined by $\Phi_{\mathbf{t}}(\mathbf{x}) = \Phi(\mathbf{x}, \mathbf{t})$, and let $\overline{\Phi_{\mathbf{t}}(\mathbb{A}^m)} \subseteq \mathbb{A}^n$ be the image variety. Then, depending on $\mathbf{t}$, either $\Phi_{\mathbf{t}}$ has infinite degree or has degree $\leq M$, i.e. either the embedding

$$\Phi_{\mathbf{t}}^* : k\big(\Phi_{\mathbf{t}}(\mathbb{A}^m)\big) \to k(\mathbb{A}^m)$$

does not give rise to an algebraic field extension or gives rise to an algebraic field extension of degree bounded by M.

Proof. We start by proving the existence of the bound M for a given field k. Consider the prime ideal

$$\mathcal{P}_0 = \big\langle Y_1 - \phi_1(\mathbf{X}, \mathbf{T}), \ldots, Y_n - \phi_n(\mathbf{X}, \mathbf{T})\big\rangle \subset k[\mathbf{X}, \mathbf{Y}, \mathbf{T}],$$

where we used bold capital letters as abbreviations for tuples of variables. We now construct inductively a finite list of ideals and associated numbers, where we may view the ideals according to the construction as labels on a finite tree and $\mathcal{P}_0$ is the label of the root. We set initially $\mathcal{P} = \mathcal{P}_0$, but later in the construction we will consider other prime ideals $\mathcal{P} \supset \mathcal{P}_0$.

So let $\mathcal{P} \supseteq \mathcal{P}_0$ be a prime ideal. If for every variable X_i there is a polynomial

$$f_{d(i)}(\mathbf{Y}, \mathbf{T}) X_i^{d(i)} + \cdots f_0(\mathbf{Y}, \mathbf{T}) \in \mathcal{P} \cap k[X_i, \mathbf{Y}, \mathbf{T}] \tag{B.1}$$

with $f_{d(i)} \notin \mathcal{P}$ then we record the product $d(1) \cdots d(m)$ of the exponents (and later will make sure that $M \geq d(1) \cdots d(m)$). Let us refer to this case by saying that $\mathcal{P}$ has finite degree. If for some i there is no such polynomial, then let us call $\mathcal{P}$ of infinite degree and we do not associate a number to this ideal.

If $\mathcal{P}$ has finite degree we continue inductively in an algorithmic manner and consider next all ideals of the form

$$\mathcal{I}_i = \langle \mathcal{P}_0, f_{d(i)}(\mathbf{Y}, \mathbf{T})\rangle$$

for $i = 1, \ldots, m$ but ignore those i for which $\mathcal{I}_i = k[\mathbf{X}, \mathbf{Y}, \mathbf{T}]$. Roughly speaking, in the case where $\mathcal{P}$ has finite degree, we will have that $\Phi_{\mathbf{t}}$ has finite degree for a *generic* choice of the parameter $\mathbf{t}$ and the ideals $\mathcal{I}_i$ correspond to the *non-generic* choices.

If $\mathcal{P}$ has infinite degree, then we will see later that $\Phi_{\mathbf{t}}$ has infinite degree except possibly for some *non-generic* choices of the parameters. Here we need to define the notion of non-generic using some polynomials obtained from a Gröbner basis. We refer to [1] for this notion. Let

$\ll$ be a linear ordering of the monomials in the variables $\mathbf{X}, \mathbf{Y}, \mathbf{T}$ such that any power of X_i is less than X_j for $j \neq i$, any monomial in $\mathbf{Y}$ is less than X_i, and any monomial in $\mathbf{T}$ is less than any of the variables in $\mathbf{Y}$ (and satisfying the usual properties in the theory of Gröbner basis). Now let $G_1, \dots, G_\ell$ be a reduced Gröbner basis of $\mathcal{P}$ with respect to this order $\ll$, and we write $G_j = g_j(\mathbf{T})\chi_j + \cdots$ where $g_j(\mathbf{T})$ is a polynomial, χ_j is the $\ll$-biggest monomial in the variables $\mathbf{X}$ and $\mathbf{Y}$ appearing in G_j, and the dots indicate all other monomials in $\mathbf{X}, \mathbf{Y}, \mathbf{T}$ which are smaller than χ_j. In particular, the above list would contain a Gröbner basis (and hence generators) of the ideal $\mathcal{P} \cap k(\mathbf{T})$ if this ideal is nonempty. Similarly, it would contain a polynomial as in (B.1) if there were such a polynomial in $\mathcal{P}$ (but we assumed there is no such polynomial), see [1].

Also recall the test whether $G_1, \dots, G_\ell$ is a Gröbner basis: It needs to generate the ideal and for every two elements G_{j_1}, G_{j_2} with $j_1 \neq j_2$ one needs to be able to reduce the polynomial $F_0 = S(G_{j_1}, G_{j_2})$ (which is the difference of minimal multiples of the polynomials G_{j_1}, G_{j_2} chosen such that their leading terms cancel) to zero. Here a single reduction step is given by subtracting from F_e a multiple of any G_j such that the leading term of the original polynomials cancel in their difference F_{e+1}. Here we may choose the G_j as we wish, and we always use a polynomial $G_j \in k[\mathbf{T}]$ if there is such an element of the Gröbner basis that can be used at this stage. Using the same notation as before we may write $F_e = f_e(\mathbf{t})\psi_e + \cdots$.

Our notion of genericity now involves all of the polynomials g_j and f_e obtained above. We may assume that $g_j(\mathbf{T}) \notin \mathcal{P}$ except when $\chi_j = 1$ (for otherwise the Gröbner basis is not reduced). Again we will continue the construction of the tree by considering the ideals

$$\mathcal{I} = \langle \mathcal{P}, g_j(\mathbf{T}) \rangle$$

for any $j = 1, \dots, \ell$, and the ideals

$$\mathcal{I} = \langle \mathcal{P}, f_e(\mathbf{T}) \rangle$$

obtained with any choice of $j_1 \neq j_2$ and $e = 1, 2, \dots$ — unless $\mathcal{I} = \mathcal{P}$ or $1 \in \mathcal{I}$.

So in both cases we obtain finitely many strictly bigger ideals $\mathcal{I} \supset \mathcal{P}$. Applying the primary decomposition [1] we get $\mathcal{I} = \tilde{\mathcal{Q}}_1 \cap \ldots \cap \tilde{\mathcal{Q}}_\ell$ where each $\tilde{\mathcal{Q}}_j$ is a $\mathcal{Q}_j$-primary ideal and $\mathcal{Q}_j \supset \mathcal{P}_0$ is a prime ideal in $k[\mathbf{X}, \mathbf{Y}, \mathbf{T}]$. Each of these prime ideals obtained are now the labels to new vertices of the tree.

Now we go through the above procedure for each $\mathcal{Q} = \mathcal{Q}_j$: We look for each $i = 1, \dots, m$ for a polynomial as in (B.1). If all of them exist, we record the product of the exponents, and put all of the nontrivial ideals of the form $\mathcal{I}_i = \langle \mathcal{Q}, f_{d(i)}(\mathbf{Y}, \mathcal{T})\rangle$ into our stack of ideals to be considered in the future. If, however, for some i there is no polynomial as in (B.1) that belongs to $\mathcal{Q}$, then we call $\mathcal{Q}$ of infinite degree and proceed as above using a Gröbner basis of $\mathcal{Q}$ which again will put a finite list of bigger ideals into our stack.

Note that by Noetherianness of $k[\mathbf{X}, \mathbf{Y}, \mathbf{T}]$ this procedure stops. In other words in any of the transitions from one prime ideal $\mathcal{P}$ to another prime ideal $\mathcal{Q}$ in the primary decomposition of $\mathcal{I} \supset \mathcal{P}$ the dimension of the corresponding variety goes down, and so the resulting tree has only finite depth.

Suppose now the above construction is finished giving us a finite tree, labeled with ideals and finitely numbers. Let $M = M(k)$ be the maximum of the numbers appearing in the tree.

We now show why the above M has the desired property. Fix some $t_1, \ldots, t_r \in k$ (or even in some field extension of k). We will now go through the constructed tree starting with $\mathcal{P}_0$. We define

$$\mathcal{R} = \left\langle Y_1 - \phi_1(\mathbf{X}, \mathbf{t}), \ldots, Y_n - \phi_n(\mathbf{X}, \mathbf{t}) \right\rangle \subset k[\mathbf{X}, \mathbf{Y}],$$

which is the ideal obtained from $\mathcal{P}_0$ by evaluating the variables in $\mathbf{T}$ to the elements in $\mathbf{t}$. We assume inductively to have reached a prime ideal $\mathcal{P}$ which when evaluated as above gives $\mathcal{R}$.

Suppose first $\mathcal{P}$ has finite degree, in which case there exists for every i a polynomial as in (B.1). If in addition, also for all i, $f_{d(i)}(\mathbf{Y}, \mathbf{t}) \notin \mathcal{R}$, then we get that $X_i + \mathcal{R}$ is algebraic of degree $\leq d(i)$ over $k[\mathbf{Y}]/\mathcal{R} \cap k[\mathbf{Y}]$, so that $\Phi_{\mathbf{t}}$ has degree $\leq M$. If, for some i, $f_{d(i)}(\mathbf{Y}, \mathbf{t}) \in \mathcal{R}$, then $\mathcal{I}_i$ gets evaluated into $\mathcal{R}$. We will finish the inductive step after our discussion of the case of infinite degree.

Now suppose $\mathcal{P}$ has infinite degree, i.e. there is some i for which there is no polynomial of the form (B.1) in $\mathcal{P}$. Here again we have to consider two cases depending on how the elements of the Gröbner basis and their leading terms are evaluated into $\mathcal{R}$: If, for some j, $g_j(\mathbf{t}) = 0$, or for some choice $j_1 \neq j_2$ and e, $f_e(\mathbf{t}) = 0$, then the corresponding ideal $\mathcal{I}$ gets evaluated into $\mathcal{R}$ and we may proceed to the next paragraph. If on the other hand we have $g_j(\mathbf{t}) \neq 0$ and $f_e(\mathbf{t}) \neq 0$ for all possible choices, then we claim that the Gröbner basis $G_1, \ldots, G_\ell$ of $\mathcal{P}$ gets evaluated into a Gröbner basis of $\mathcal{R}$ (after removing the zeroes in the list if there are any). In fact, let $\ll$ denote the restriction of the previous order to the remaining variables $\mathbf{X}, \mathbf{Y}$. Note that we have ensured that the leading coefficients of the evaluated polynomials are precisely the evaluations of the leading polynomial (obtained by collecting all terms with the same $(\mathbf{X}, \mathbf{Y})$-monomial as the leading term), both for the polynomials G_j and for the polynomials F_e in the calculation that shows that we have a Gröbner basis. Hence we avoid any coincidences and get a Gröbner basis for the evaluated ideal $\mathcal{R}$. In particular, this shows that there does not exist a polynomial of the form $f_{d(i)}X_i^{d(i)} + \cdots$ in $\mathcal{R}$ — $\Phi_{\mathbf{t}} : \mathbf{A}^m \to \mathbf{A}^n$ has infinite degree.

So in both cases we are reduced to the case of an ideal $\mathcal{I}$ that is evaluated into $\mathcal{R}$. As $\mathcal{R}$ is a prime ideal the same must be true for one of the primary ideals $\tilde{\mathcal{Q}}_j$ and then also for the prime ideal $\mathcal{Q}_j$. This finishes the inductive step.

We now argue why M can be chosen independent of the field k, and in particular of the characteristic of k. Apply the argument above to $k = \mathbb{Q}$ to get $M_{\mathbb{Q}}$. Then there are only finitely many rational numbers used in the whole argument and from this one sees that the number $M_{\mathbb{Q}}$ has the desired property for a general field k if only the characteristic of k is different than all primes appearing in any of these rational numbers. To get the proposition for all characteristics one now has to apply the above argument to all remaining finite fields. $\square$

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