Neural Algebra and Modeling

Erwin Engeler

1. COMBINATORY LOGIC and ALGEBRA

Since Turing, in particular since universal Turing Machines, and also since von Neumann-Machines, inputs and programs are of the same nature; we may simply call them "data". They are perhaps marks on the Turing tape or bits in the computer memory. The basic operation on data is application: Programs may be applied to input data and of course result in data, which may again be programs. Programs may also be applied to programs, again resulting in data, etc. Indeed, we may admit that all combinations of applications on data result again in data, including error messages.

Combining data and programs

x, y	are programs
z	is data
y	yields data
x	yields a program
$(x \cdot z) \cdot (y \cdot z)$	applies the new program to the new data

This combination of x, y, z is regarded as a new program $\mathbf{S}$ with $\mathbf{S} \cdot x \cdot y \cdot z = (x \cdot z) \cdot (y \cdot z)$. Convention: parentheses to the left.

Combinatory Logic

Axiom scheme: For every combination $\phi(x_1, \ldots x_n)$ of data there is a data t_ϕ such that

$t_\phi x_1 \ldots x_n = \phi(x_1, \ldots x_n))$

Such t_ϕ are called "combinators".

Theorem (Schoenfinkel):

Two combinators suffice for expressing all combinators, namely the above $\mathbf{S}$, characterized by its equation $\mathbf{S} \cdot x \cdot y \cdot z = (x \cdot z) \cdot (y \cdot z)$,

together with $\mathbf{K}$, characterized by $\mathbf{K}xy = x$.

Combinatory Logic, was invented by H.B. Curry in his 1929 Göttingen thesis, directed by Paul Bernays, also Doktorvater of the present author as well as of Saunders MacLane and Gerhard

Gentzen. It is the formal theory of equations between combinators as its objects. For Combinatory Logic and its twin, Lambda Calculus, the question of consistency arises. But while Hilbert's program failed for Peano arithmetic, Curry's formal system is consistent, as proved by Church and Rosser using the same finitist proof-theoretic tools that failed in the first case.

Combinatory Algebra

A combinatory algebra $\mathcal{D} = \langle \mathcal{D}, \cdot \rangle$ consist of a set D with an application operation $\cdot$ satisfying the axiom scheme for combinatory logic.

Theorem (EE): Any binary algebraic structure can be isomorphically embedded in a graph model of combinatory logic.

Claim: Combinatory algebras are fundamental mathematical structures, their role is comparable to groups, categories, etc. as organising principles with their collection of tool and templates.

But for forty years the only model was the model consisting of equivalence classes of combinatory expressions which exist non-trivially on basic principles, namely the fact of consistency of the axioms.It was known that no computable model existed. Then in the 1970's Plotkin and Scott invented the set-theoretic and topological models P_ω and then D_∞, constructed to be isomorphic to its function space. Shortly thereafter we ourselves introduced the Graph Model whose transparent and explicit structure lends itself to applications in modelling. Such models are called Combinatory

Algebras.

Neural algebras as presented below, are constructed as a much enriched type of graph models.

2 An Algebraic Model of Interacting Brain Functions

The conceptually simplest model of a brain represents its connectivity, the *connectome* A, as a directed graph whose nodes, called neurons, fire at discrete time instances $t \in Z$. The global activity of the brain, the firing history of these neurons, is represented by the *brain function* f(a,t) which takes the value 1 if the neuron a fires at time t and 0 otherwise. Modeling a brain is accomplished by imposing restrictions on the functions f by a specific *firing law* inherited from abstracting neurological findings. A firing law specifies the condition under which the firing of neurons $a_1,...,a_k$ at times $t_1,...,t_k$ causes the firing of a neuron a_{k+1} at some later time t_{k+1} , assuming the former are connected to it by directed edges.

Cascades and track expressions

A "connectome) a directed graph whose nodes are neurons, edges show forward connections

$f : A \times \mathbb{Z} \to \{0, 1\}$ firing function

track expressions: $x, y, \ldots$

basic track expression: $x(t) = \langle a, t\rangle$ for $f(a, t) = 1$

compose to: $x(t) = \{\langle a, t_1\rangle, \langle b, t_2\rangle \xrightarrow[c]{t} \langle d, t_3\rangle$.

continue by recursion:

$$x_b(t) = \{x_{a_1}(t_1), \ldots x_{a_n}(t_n)\} \xrightarrow[b]{t} x_{a_{n+1}}(t_{n+1}),$$

Each composite track expression is divided by its key neuron into argument track expressions and a value track expression, it is called *causal* if $f(a_i, t_i) = 1, i = 1, \ldots n+1$ and $f(b, t) = 1$, and the firing of $a_1, \ldots a_n$ suffice according to the firing law for the activation of b as well as for all neurons on the paths from $a_1, \ldots a_n$ to b and from b to a_{n+1} at the times given in the expression.

Idea: Define a neural algebra N_A as an algebra of interacting brain functions.

brain functions : temporal patterns of the firings of populations of neurons in a brain A, e.g. visual inputs, recognizing a shape, motor output, etc.

operation: A · B is the brain function which is caused by the activation of B in an active environment A.

Neural Algebras$\mathcal{N}_{\mathcal{A}}$

Given a brain model $\mathcal{A} = (A, f)$ by its directed graph A and the firing function f define *brain functions* $M, N, \ldots$ as arbitrary sets of causal firing tracks.

A brain function M *applied* to a brain function N produces the result of causation as represented in thecausal track expressions in M, on N as follows:

$$M \cdot N = \{x_{n+1}(t_{n+1}) : \exists \{x_1(t_1), \ldots, x_n(t_n)\} \xrightarrow[b]{t} x(t_{n+1}) \in M$$

$$s.t. \quad x_1(t_1), \ldots, x_n(t_n)\} \subseteq N\}.$$

Let $\mathbb{B}_{\mathcal{A}}$ be a set of brain functions closed under composition and union, then

$$\mathcal{N}_{\mathcal{A}} = \langle \mathbb{B}_{\mathcal{A}}, \cdot \cup \rangle$$

is called a *Neural Algebra*.

3. MODELING CONTROL IN NEURAL NETS

A Structure-Function Discipline

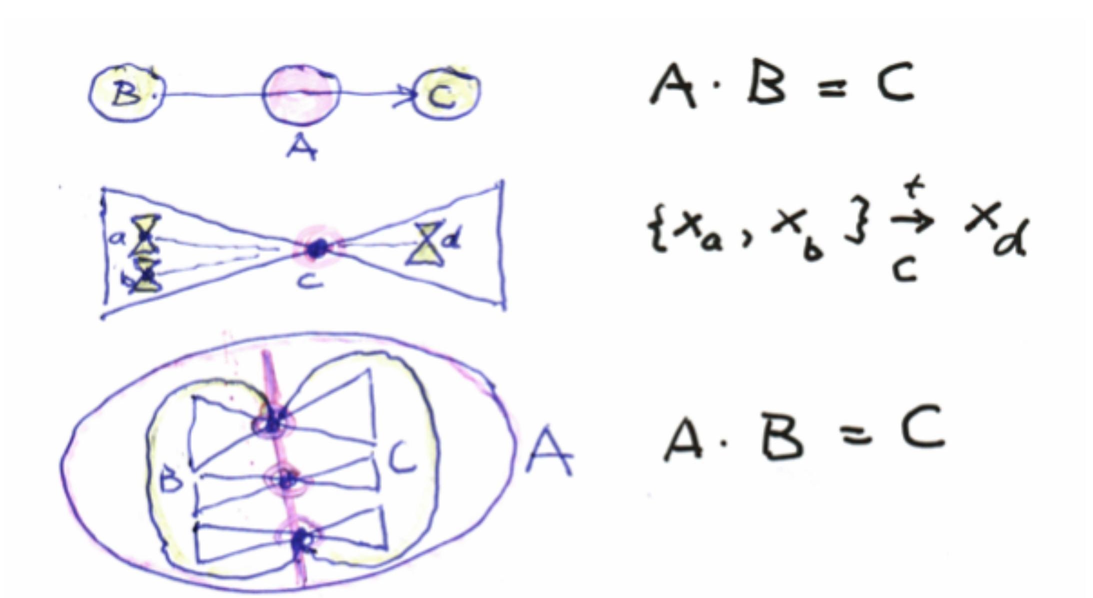
(1) Neurology: Given a connectional structure (”connectome”), analyse it into firing tracks (key neurons!) to reflect a hypothesis of identifiable brain functions.

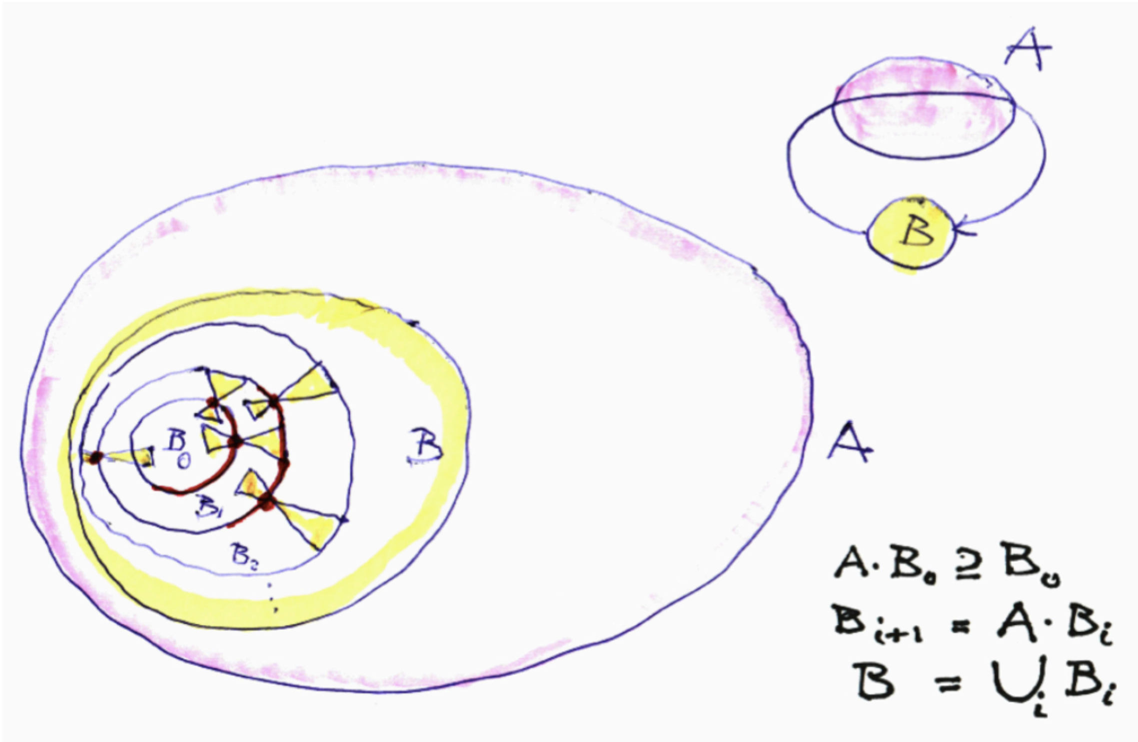
(2) Neural Algebra: Given a diagram of interacting brain functions (and the corresponding equations), solve the equations in N_A and work out underlying sets of track expressions constituting the solutions.

(3) Fundamental Structures: Identify some fundamental neural circuitry by investigating the neural equivalents to basic brain functions on first principles and compare to observa- tional results in neuroscience.

The functional diagrams below are based on the combinatory logic idea that rules and their arguments are one and the same type of objects, and that any two may be applied to each other: A . B results again in such an object C. In Neural Algebra A, B, etc are sets of formal representations of cascades (each represented as a sort of butterfly with the key neuron in the middle), and their relation is sketched in this figure. The set of key neurons activated in the execution of the brain function A is characteristic for A; it is sometimes is identified locally by fMRI with the brain function itself.

In a functional diagram the application operation is represented as an arrow going through the circle named A, starting at the periphery of B and ending at the periphery of C. Thus “rules may be applied to rules” etc.



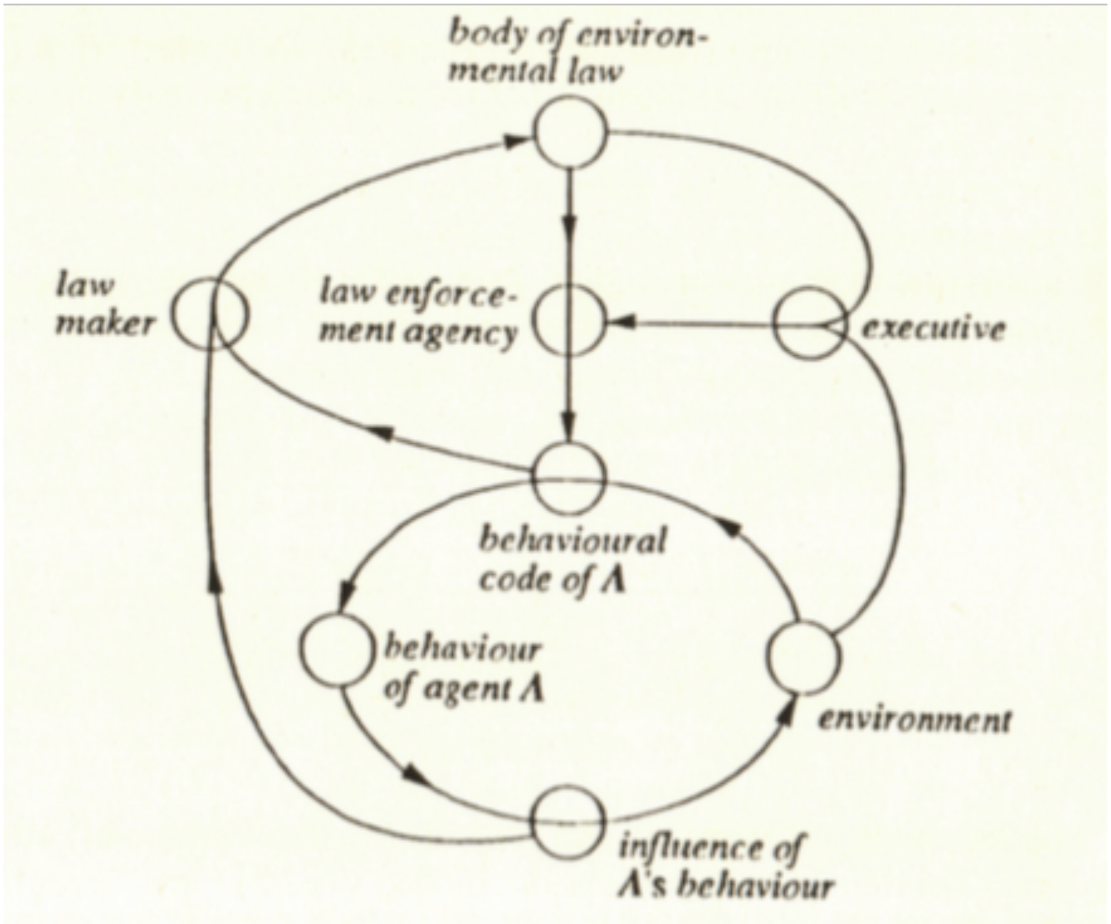


The above picture contains the functional diagram of a simple feedback circuit. Its equation is simply A . B = B, a fixpoint equation, and the solution algorithm is given both mathematically and by its realization as a neural net.

The subsequent examples are much of the same nature.

Of course, functional diagram have a much wider range of use: Below is an old example of mine, depicting the functional diagram of the social functions operating in the context of environmental problems. Note that the diagram depicts a dictatorial regime; you may want to change that. The underlying social net is not depicted; that would be the subject of a “Social Algebra”, which I, and others, have clearly not mastered.

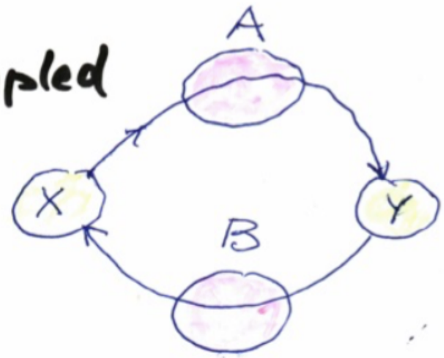
Also, I have not written out the equations...



Neural nets are the subject of Neural Algebras, and the functional diagrams concern brain functions. The examples further below are simpler, their corresponding equations are obvious, and the solutions (for the controlled objects, usually denoted B or X, Y, C as the case may be, are quite simple cases of the fixpoint theorem of Neural Algebras (and Combinatory Algebras). We have also developed algorithms solving for other components in the equations, which is sometimes a bit harder.

The basic idea of the approach is of course to clarify algorithmically the relation been structure and function in neural nets.

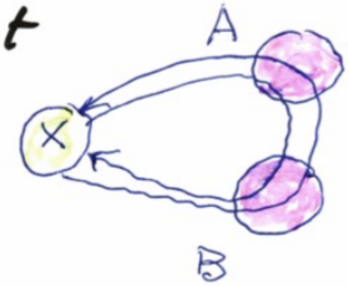
coupled



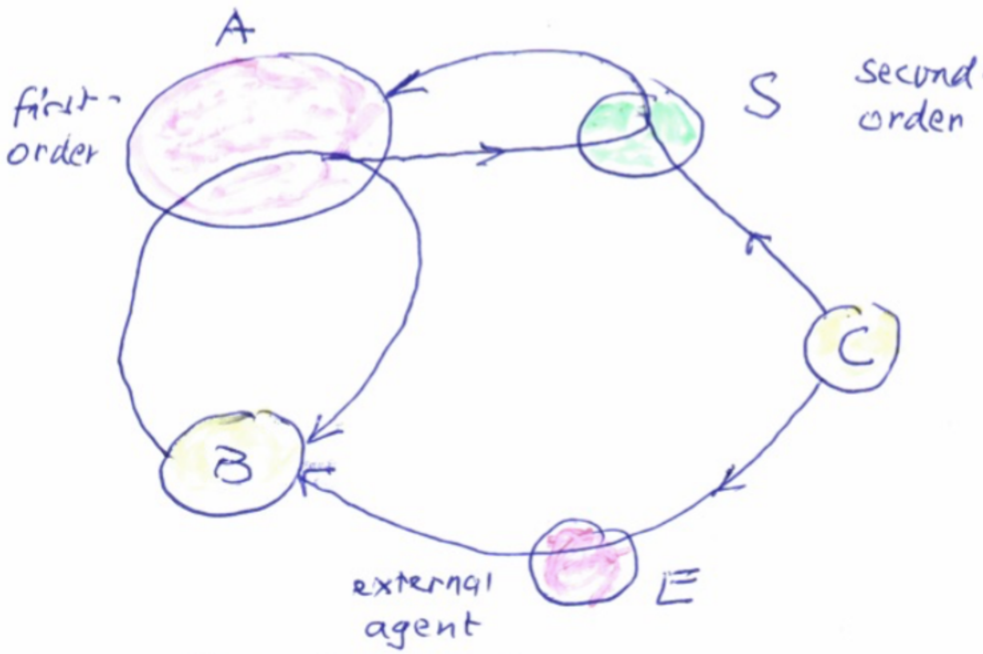
$$A \cdot X = Y$$

$$B \cdot Y = X$$

joint

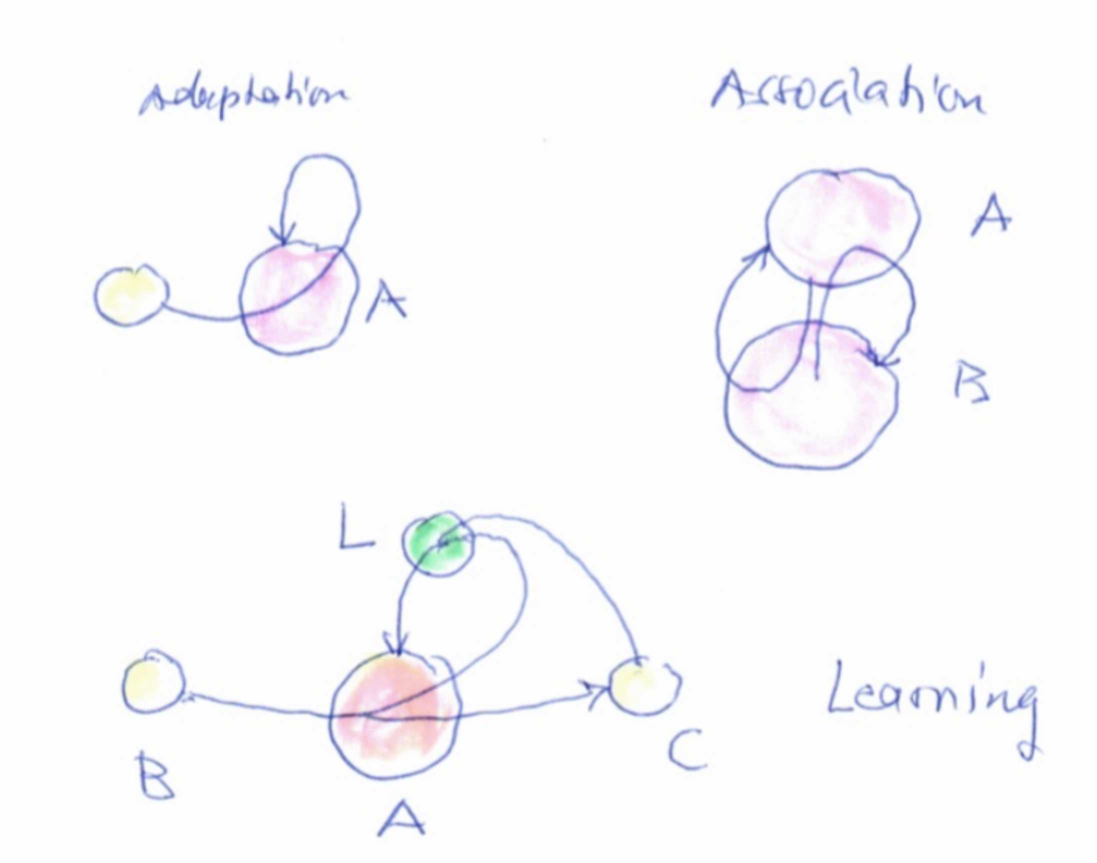


$$X = A \cdot (B \cdot X) \cup B \cdot (A \cdot X)$$



$$A = S \cdot (A \cup C)$$

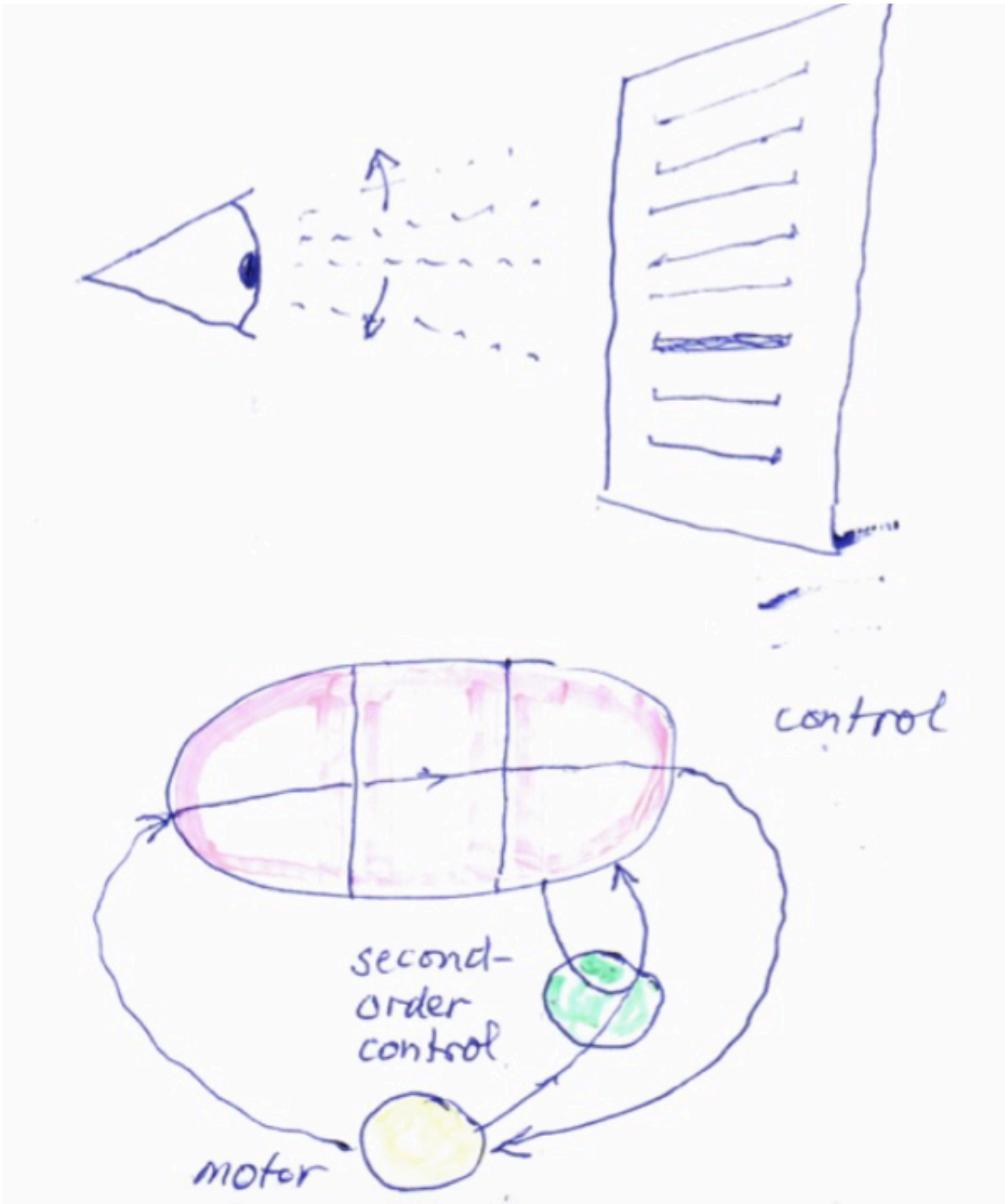
$$B = A \cdot B \cup E \cdot C$$



4. EXAMPLE: EYE MOVEMENT

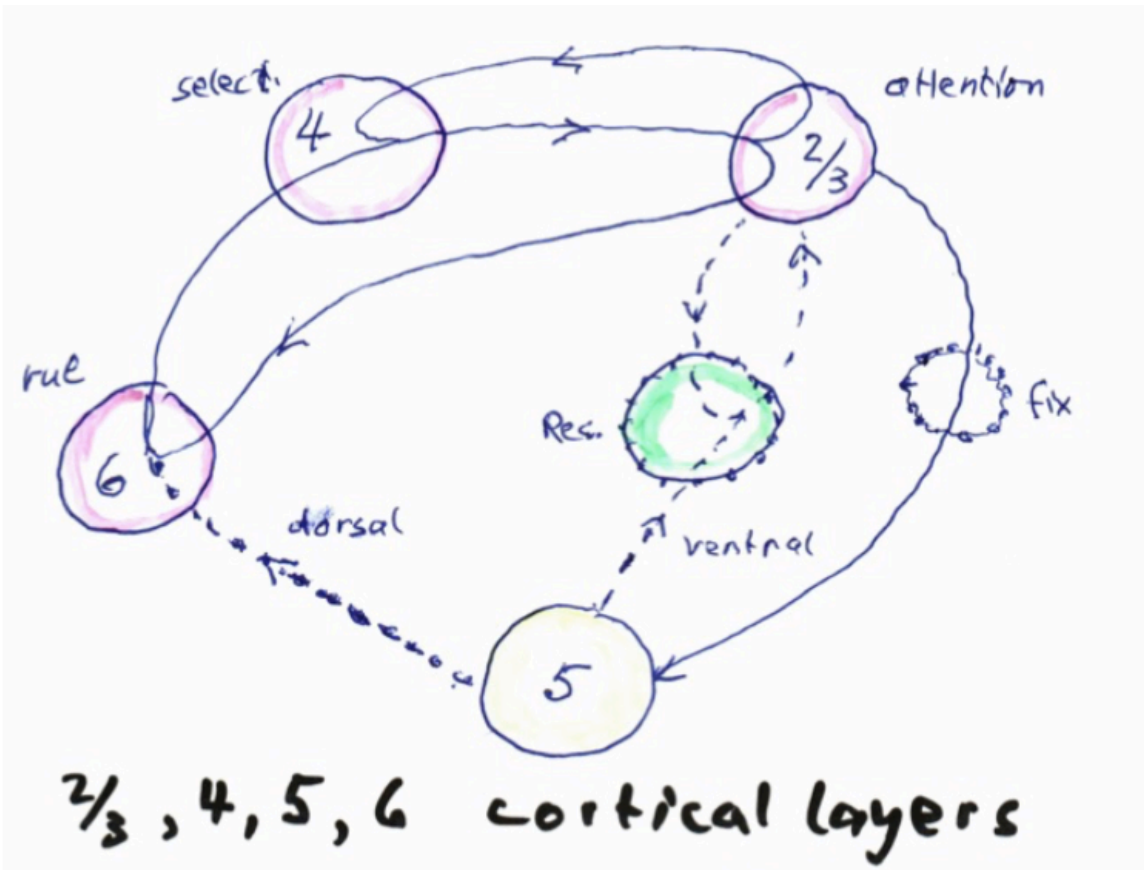
To illustrate the neural algebra of control in the context of neuroscience we choose an example that has the advantage of being both easy to explain and having interesting ramifications. We proceed by starting with a simple feedback and progressively include analyses of higher forms of control, input, output and additional operands.

Consider the movement of the eye as it scans a text for a particular passage.



The following diagram analyses to some extent the successive stages and components of control:

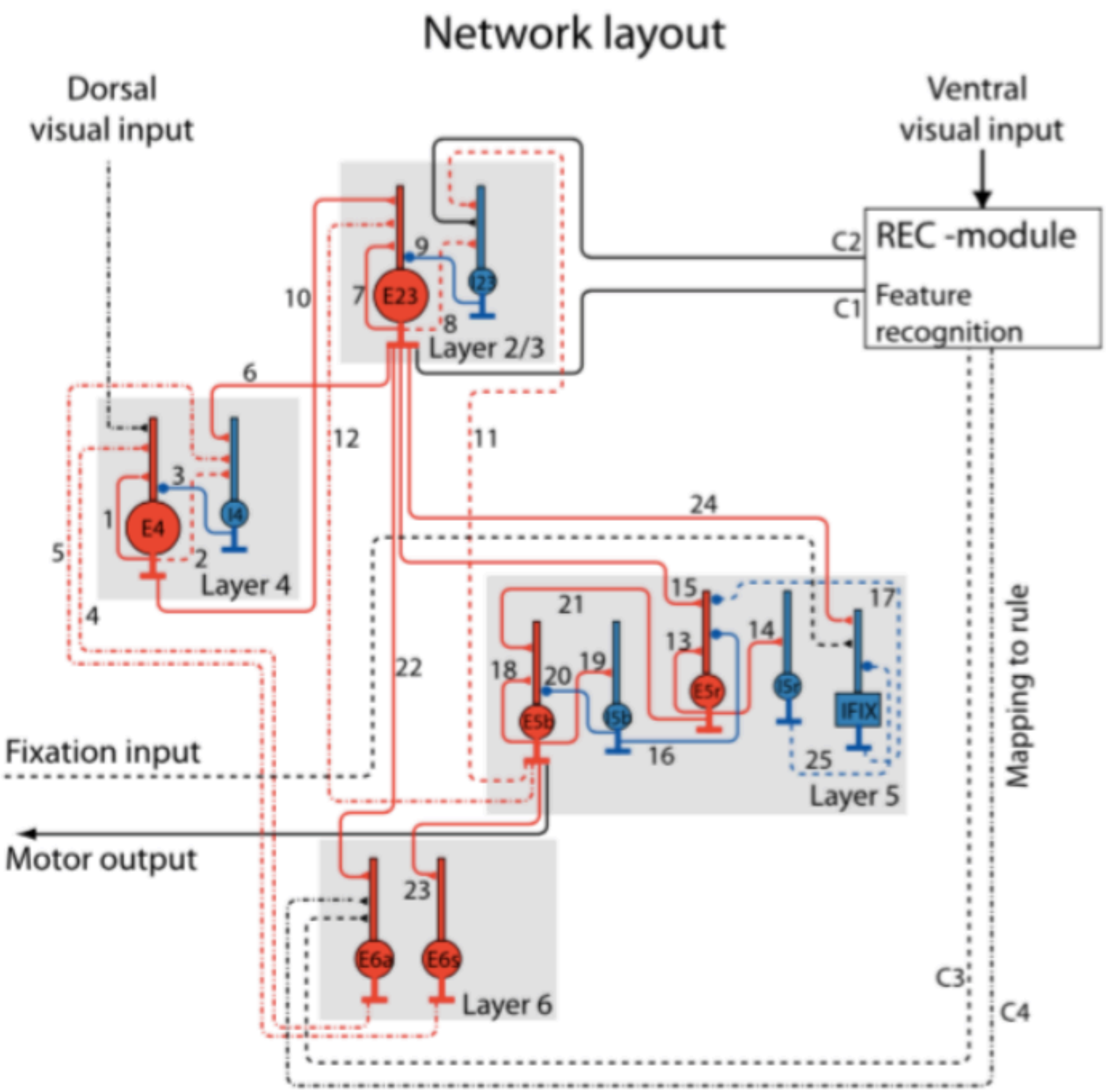
Observe the *second-order control* character of the operation ”Rec” of recognising a text passage.



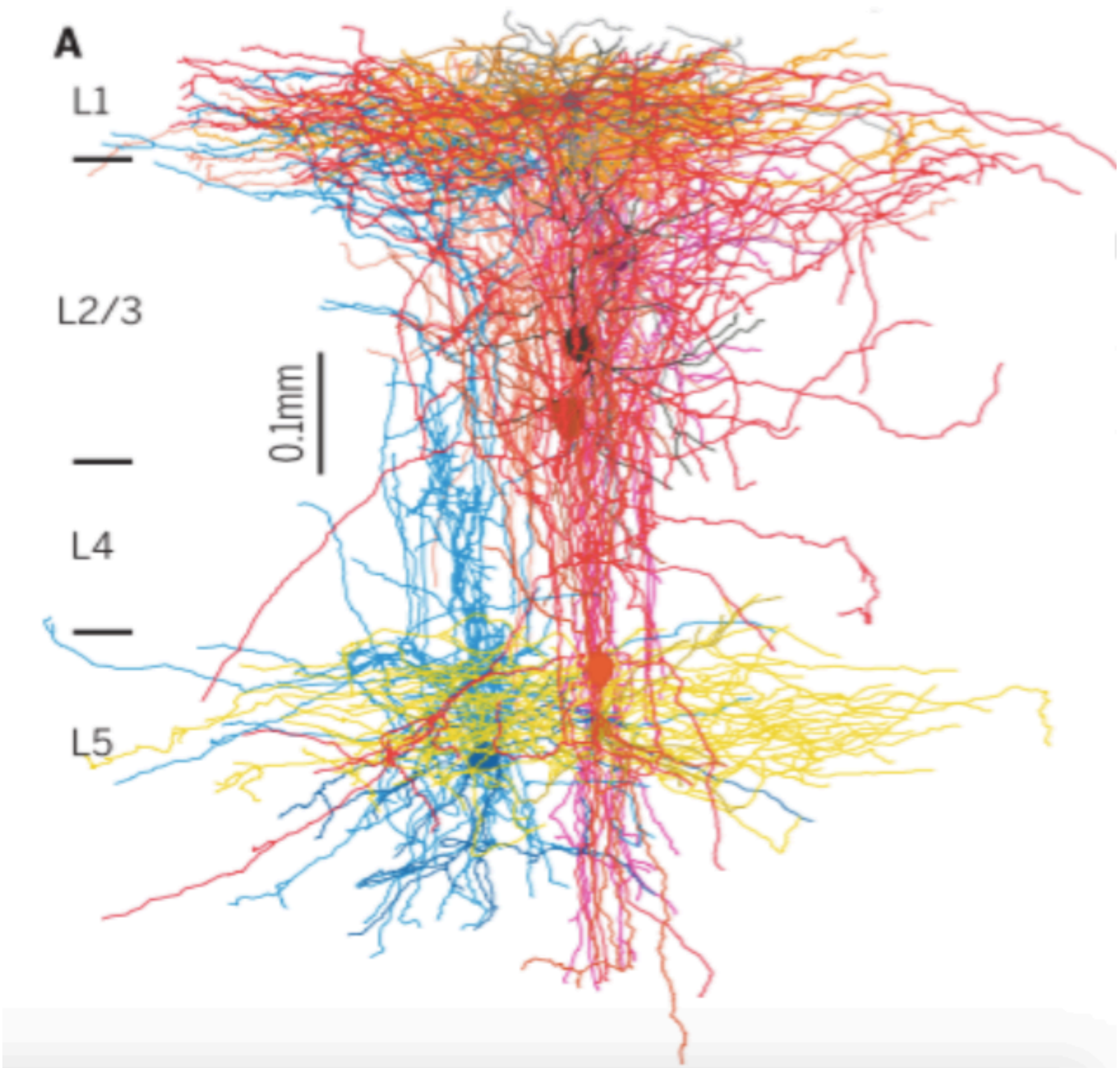
Here is what our neurologists determined by investigating the network of ”integrate and fire” neurons of the relevant area in the cat brain: Microcircuits of the frontal eyefield (Martin, Hepp and Heinzle, J.Neuroscience 29 (2007) 9341 ff).

The diagram is based on a quantitative study of the connection matrix. The research included the construction of a model to simulate the electrophysiological and behaviour findings that captured the functionality. It shows some of the connections. Note that the REC module is not detailed. The symbols used in this drawing are the standard symbols for indicating the firing law: In the figure the distinction between incoming and outgoing signals and between excitatory and inhibitory synapses are coded graphically; these codes in fact represent the firing law of this brain model.

Observe the sequence of operands; they correspond to a layered arrangement of individual cascades, named by the given layer-numbers in the diagram.



The interest in this neural circuit is that its structure is omnipresent in the neocortex; it is called the *canonical circuit* of dominant interactions. Observe that the feedback connections go across layers. It has been conjectured (Chklovskii et al. 2002) that cortical lamination is providing a general scaffold and the canonical circuits may allow neurons to connect with each other with a minimum of wires.



Jiang, X and al., Science 350 (2015), 1055 ff-

5. REFLEXIVE CONTROL

A quote from McCarthy (Makig Robots Conscious of their Mental States, retrieved from formal.stanford.edu/jmc ca.2005):

> 2. Already [Turing, 1950] disposes of "the claim that a machine cannot be the subject of its own thought". Turing further remarks
>
> > By observing the results of its own behavior it can modify its own programs so as to achieve some purpose more effectively. These are possibilities of the near future rather than Utopian dreams.
>
> We want more than than Turing explicitly asked for. The machine should oberve its processes in action and not just the results.
>
> 3. The preceding sections are not to be taken as a theory of human consciousness. We do not claim that the human brain uses sentences as its primary way of representing information.
>
> Of course, logical AI involves using actual sentences in the memory of the machine.

Reflexive control C reflects on the controlling process itself. It is the ability of the brain B to observe itself as it is planning, acting and reacting. This definition, at first sight, appears circular. Interpreted in N_A it is simply self - referential:

$C \cdot C$: the controlling function observes itself

$C \cdot (B \cdot C)$: it observes the influence of the brain on the control activity

$C \cdot (C \cdot B)$: it observes the result of the controlling action on the brain

$C \approx C \cdot C \cup C \cdot (B \cdot C) \cup C \cdot (C \cdot B).$

A set of track expressions R makes sense as a mental activity if its firing is *sustained* for a time interval $[t_0, t_1]$ with $t_1 - t_0 > \nu$ for some arbitrarily fixed number ν, say 10^5. Write $\{R\}_{t_0}^{t_1}$. The composition of sustained brain functions may not be sustained.

We write $X \approx Y$ if the sustained brain functions X and Y overlap for a subinterval of length at least ν.

A causal cycle is a sequence $\{y_{c_0}(t_0), y_{c_1}(t_1), y_{c_2}(t_2), \dots\}$ of causal track expressions of the form $\alpha_i \xrightarrow[c_i]{t_i} x_{c_{i+1}}$ with $x_{c_{i-1}} \in \alpha_i$ for $i = 0, 1, \dots$, which is cyclic in the indices i modulo some period n.

Theorem: A neural algebra admits nontrivial reflexive control if

and only if it contains at least one sustained causal cycle.

Speculation: Reflexive control in a brain may be identified with "Consciousness.

The cycles of consciousness are all within a general consciousness area. This in turn is divided into subareas that would be active as "states of consciousness". One might suspect that these areas all intersect in a central area of fundamental cycles, a "core consciousness". And one might conjecture with Crick and Koch that in the human brain this is located in the claustrum.

At this point, many more theoretical perspectives open up, for example with respect to logic and language. But that is another chapter. Let me just add, as a sort of suggestion the consciousness of animals: of course what we believe to experience as consciousness may, from a purely connectional point of view also be experienced by animals with a sufficiently structured brain. So the question is: how does it feel to be a worm? (I once constructed the neural net of a worm that can act as an universal Turing machine…)

There is of course also the possibility that super organisms such as the Portuguese Man of War, or more to the point: that ant colonies have consciousness. Indeed, one may even invent something to be called ”political consciousness”, that is the shared consciousness of some human societies, a kind of network of shared awareness. (Remember the example of environmental policy above.)

6. INTELLIGENCE

Let me finally share some speculations about intelligence, a notion that has aquired various qualifications such as ”artificial-”, ”machine-”, and even ”emotional” intelligence. Let me add another: *Combinatory Intelligence*.

To start, remember one person who claimed superior intelligence of his grey cells and then went on to demonstrate it: Hercule Poirot.

In the final scene of the Agatha Christie’s classic detective story “Five Little

Pigs,” Poirot presents his parsing of the events that led to the trial and conviction of Caroline (C) for the murder of her husband, Amythas (Am), who in fact had been killed by his mistress, Elsa (E). But Caroline confessed (C_1) to the murder because she had observed (C_3) her sister Angela (A) in some activity at the time, which convinced her (C_2) that the sister was the murderess. But Caroline had felt bad all her life for having accidentally mutilated (C_4) Angela as a child, and she therefore decided to take advantage of some facts (F_1) that pointed to herself, confessed (C_1) and was convicted, while in fact many clues (F_2) pointed Poirot (P_2) to the murder of Amythas by Elsa. Poirot, detecting (P_3) the background of Caroline's confession, correctly reconstructs the circumstances (P_1) and accuses Elsa, who departs to places unknown.

Formalized as mental activities of the people involved, symbolically indicated above, the story parses as:

$(P_1 \cdot(P_3 \cdot(((C_1 \cdot F_1)\cdot((C_2 \cdot A)\cdot(C_4 \cdot A)))\cdot(C_3 \cdot(A\cdot Am))))\cdot((P_2 \cdot F_2)\cdot(E\cdot Am))$

I submit that Poirot's intelligence manifests itself as the complexity of the bracketing of this combinatory expression.

To measure the combinatory complexity of the tale of the five little pigs, consider its six components P, C, Am, E, A, F .

In combinatory logic there is an object T which represents the tale.

$$T \cdot P \cdot C \cdot Am \cdot E \cdot A \cdot F =$$

$$(P_1 \cdot (P_3 \cdot (((C_1 \cdot F_1) \cdot ((C_2 \cdot A) \cdot (C_4 \cdot A))) \cdot (C_3 \cdot (A \cdot Am)))) \cdot ((P_2 \cdot F_2) \cdot (E \cdot Am))$$

The proof of the existence of T in a neural algebra consists of a construction of the corresponding connection diagram. There is an algorithm for that. But here we take a simple example, already encountered at the beginning of this talk, the diagram for the S-combinator.

An example of transforming an equational definition of a functional diagram into a connection diagram, the basic combinator S:

$\mathbf{S} \cdot X \cdot Y \cdot Z = (X \cdot Z) \cdot (Y \cdot Z)$

compiles to:

$$\mathbf{S} = \{\{\tau \xrightarrow[c_1]{\cdot} (\{r_1, \ldots, r_n\} \xrightarrow[c_2]{\cdot} s)\} \xrightarrow[c_3]{\cdot} (\{\sigma_1 \xrightarrow[d_1]{\cdot} r_1, \ldots, \sigma_n \xrightarrow[d_n]{\cdot} r_n\} \xrightarrow[e]{\cdot} (\sigma \xrightarrow[f]{\cdot} s))\},$$

where $n \geqq 0, \tau \cup \bigcup_i \sigma_i = \sigma \subseteq X$,
$\{\sigma_1 \xrightarrow[d_1]{\cdot} r_1, \ldots, \sigma_n \xrightarrow[d_n]{\cdot} r_n\} \subseteq Y$,
and $\{\tau \xrightarrow[c_1]{\cdot} (\{r_1, \ldots, r_n\} \xrightarrow[c_2]{\cdot} s) \in Z$.

Indeed, I believe that the ability to think using highly complex combinators such as other people's thoughts, in fact quite generally of internal modelling is a defining quality of human intelligence, as combinatory intelligence.