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Hausdorff Dimension and Fractal Geometry

Bachelor Thesis

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Abstract

In this thesis we are interested in giving a rigorous Hausdorff dimension of a subset of $\mathbb{R}^n$.
We provide a method to compute the Hausdorff dimension for a self-similar and fractal sets. We then compute explicitly the Hausdorff dimension of some well known fractals (Cantor set, Koch curve and others).

Contents

Chapter 1

Introduction

Historically, the sets now described as fractals where always regarded as "pathological" or "monstrous" purely mathematical structures without any application to the real world.
This perception changed thanks to the work made by Benoit Mandelbrot (1924-2010), who introduced the term **fractal** to the world (fractal comes from the Latin adjective *fractus*, which translates approximatively into "broken") and who applied this mathematical theory to the physical world.
His efforts in this field culminated into publishing the book *"Fractal geometry of nature"*[1] in 1982 (a revisited version of the previously published book *"Fractals: Form, Chance and Dimension"* published in 1977). This book is considered a milestone for the development of mathematics and science and it appears in the 100 books of the 20th century by the magazine "American scientist".[2]
In his original essay, Mandelbrot defines a fractal as a set with Hausdorff dimension (which he calls fractal dimension, see definition in chapter 2) different from its topological dimension (the topological dimension, also known as covering dimension or Lebesgue covering dimension is defined inductively: a set has topological dimension 0 if and only if it is totally disconnected, a set has topological dimension n if every point has arbitrary small neighbourhoods with boundary of dimension (n-1)). Note that the topological dimension of a set is always an integer; therefore (by Mandelbrot's definition) every set with non-integer Hausdorff dimension is considered a fractal.
Instead of using this classical definition, we decided to go with the more vague definition of fractal given by the book *"Fractal Geometry, Mathematical Foundations and Applications"* by Kenneth Falconer[3], since it allows us give the tag of fractal to more sets than Mandelbrot's definition. Therefore, we say that a set F is fractal if it is satisfies some of the following properties:

fine structure Which means that F has details on arbitrary small scales.

irregular i.e it cannot be described by traditional geometry.

self-similar F is composed by smaller copies of itself.

simple F must be easily describable by some rule.

The objective of the first chapter of this thesis is to give a rigorous definition of Hausdorff dimension and prove the invariance of this quantity under application of a bi-Lipschitz map. To do this we will define the Hausdorff measure and work our way up to the main definition, with an excursus in measure theory, where we will show the equivalence of Hausdorff measure and Lebesgue measure in $\mathbb{R}^n$.
In the third chapter of this thesis we will focus on calculating the Hausdorff dimension of some self-similar fractals in a rigorous mathematical way, but before that, we would like to show some examples of fractal structures that can be found in nature, architecture and some other fields of mathematics (in particular statistics).
We would like to emphasise the fact that fractals (just like circles and spheres) do not exist in the real world, but only in our mathematical abstraction. Therefore, we refer to the **fractal representative** of a real object when talking about the mathematical abstract fractal similar to the original object (in the same way that we refer to a sphere as the geometrical representative of an orange).
Mandelbrot himself in the above-mentioned book proposed the first classic example of fractal in nature: the coast of Great Britain. The usual method for measuring the length of a coast is to fit a polynomial over a map of the coast and to add the length of the segments of this polygon to find an approximation of the wanted quantity. If we use a rough approximation (for example if the sides of the polynomial are all of length 125 km) we find a certain value. If we refine this approximation (length of the sides equal to 62.5 km), we will find a much higher, and intuitively much better, approximation of the length of the coastline. Repeating this procedure, we find out that the value of the approximation tends to infinity whenever the length of the side tends to zero.

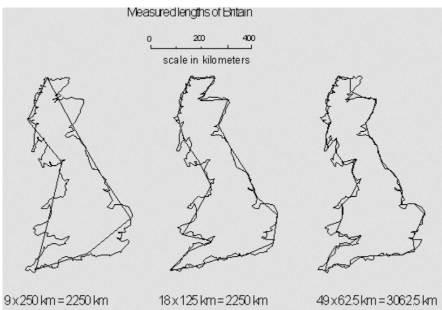

Figure 1.1: Coastline of Britain. [4]

The motivation for this behaviour is the fact that the coastline has a fractal-like structure, and its fractal representative would have Hausdorff dimension bigger than one. Therefore its "length" cannot be measured with the same instruments that we would use for a set of dimension one (in a certain way it is as if we were trying to measure the volume of a sphere using ruler).
The second and more interesting example is the Roman cauliflower. The characteristic in which we are interested is the fact that if we zoom into a portion of this cauliflower, we find the same structure as the original cauliflower itself. We can repeat this procedure a couple of times and continue finding the same result. Of course, in the real world we cannot zoom in infinitely many times, because sooner or later we would soon reach the molecular level, which is not self-similar to the initial cauliflower. Nevertheless, we can say that the fractal representative of the cauliflower has a self-similar structure.

Figure 1.2: Roman Cauliflower. [5]

The more interesting point of this (and the following) example is the reason why the Roman cauliflower has this structure. Evolution has created this vegetable in a way that allows it to have a big surface in a confined volume. This way the cauliflower is very efficient in absorbing oxygen from the air, without needing a big space. This is the beauty and the key for technical applications of fractal geometry: simple rules, which lead to efficiency (in this case in the absorption of oxygen).
The same concept can be applied to the third example: the lungs. The structure of the lung is very similar to the structure of a tree: there are two main bronchi, which divide into secondary bronchi, which divide further into tertiary bronchi. The tertiary bronchioles divide into bronchioles, which divide into terminal bronchioles. All of those parts of the lung are so similar to each other that they almost have the same name. Therefore, we can say that the fractal representative of the lung has a self-similar, fractal structure. Following the same reasoning as in the case of the Roman cauliflower, evolution

has modelled the lungs efficiently, creating a structure with a big area inside a confined volume (due to the ribcage) for the better transfer of oxygen form the air to the blood.
Even before the publication of Mandelbrot's book, fractal structures have appeared into the work of many artist because of their beauty. Already in the Early Middle Ages, we can find an example of Sierpinski triangle structure in a mosaic on the floor of the Anagni Cathedral in Lazio.

Figure 1.3: Floor of the Anagni Cathedral in Lazio. [6]

A modern example of the application of fractal theory to architecture is the work made by Wolfgang E. Lorenz [7], where he uses the idea behind the concept of Hausdorff dimension to create an index, with which he can then compare some architectural structures over the aspect of self-similarity.
The last example diverges from the solid world and starts dealing with statistical self-similarity. The argument in question is Brownian motion, which was based on the study of the movement of a small particle of dust in a fluid. Mandelbrot himself said that the work done by Jean Perrin on the subject was a great motivation for the development of fractal geometry. The reason why this motion might be considered fractal might not be immediately clear, since the trajectory of one particle is not exactly self-similar to itself. Let it be noted that cases like these are the reason why we have decided to go with a loose definition of fractal. The first point to be made is that the trajectory the this particle is considered to be nowhere differentiable, therefore it is **irregular**. Secondly, the probability distribution of the trajectory of a particle is the same at any time. This creates a sort of **self similar** structure of the trajectory.
We hope that this small introduction helped in tickling your interest for fractal structures and that it has given motivation to read forward.

Chapter 2

Hausdorff dimension in $\mathbb{R}^n$

This chapter has the goal of introducing the basic notions of set theory and measure theory that build up to the definition of Hausdorff dimension. This chapter will also contain the proof that the n-dimensional Hausdorff measure and the Lebesgue measure agree on $\mathbb{R}^n$.

2.1 Set and Measure theoretical concepts

This section will seem a bit disjointed, but its purpose is to introduce some results that will be used later on in the thesis.

Definition 2.1.1 *Given a set $C \subset \mathbb{R}^n$, we define the **Diameter** of C to be:*

$$diam(C) = \sup\{|x-y|; x, y \in C\}$$

Where $|\cdot|$ represents the euclidean norm.

Lemma 2.1.2 *Given a set $C \subset \mathbb{R}^n$, we have that*

$$diam(C) = diam(\overline{C})$$

Proof $'' \leq ''$ This inequality is clear, since $C \subset \overline{C}$
$'' \geq ''$ Let $p, q \in \overline{C}$ and let $\varepsilon > 0$.
By the definition of closure in a metric space, we have that $\exists \tilde{p}, \tilde{q} \in C$ s.t. $|p - \tilde{p}| < \varepsilon, |q - \tilde{q}| < \varepsilon$.
By the triangular inequality it follows that
$|p-q| \leq |p-\tilde{p}| + |\tilde{p}-\tilde{q}| + |\tilde{q}-q| \leq 2\varepsilon + diam(C)$
Since we took ε arbitrary, we can let it tend to 0 and we get that
$|p-q| < diam(C), \forall p, q \in \overline{C} \Rightarrow sup\{|p-q|; p, q \in \overline{C}\} = diam(\overline{C}) \leq diam(C)$ □

After this small preliminary lemma, which will be used later in the thesis, we introduce some measure theoretical concepts. It should be noted that when we talk about the Lebesgue measure $\mathcal{L}^n$ we are referring to the **Hahn-Carathéodory** extension of the pre-measure defined on the set of cubes in $\mathbb{R}^n$. We first introduce a useful constant, which will appear later in the definition of Hausdorff measure as a normalizing factor, and then calculate the Lebesgue measure of a ball of radius r in $\mathbb{R}^n$.

Definition 2.1.3 $\alpha(s) = \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}$, *where* $\Gamma(\cdot)$ *is the usual gamma function.*

Lemma 2.1.4 *Let* $B_R(x)$ *be a ball with centre in* x *and radius* R. *It holds that:*

$$\mathcal{L}^n(B_R(x)) = \alpha(n)R^n$$

Proof For this proof we use the system of spherical coordinates in $\mathbb{R}^n$, which is described by a radius r, n-2 angles $\varphi_1, \dots, \varphi_{n-2} \in [0, \pi[$ and one angle $\varphi_{n-1} \in [0, 2\pi[$.
The Cartesian coordinates are given by:

$$\begin{cases} x_1 & = r\cos(\varphi_1) \\ x_2 & = r\sin(\varphi_1)\cos(\varphi_2) \\ x_3 & = r\sin(\varphi_1)\sin(\varphi_2)\cos(\varphi_3) \\ \vdots & \\ x_{n-1} & = r\sin(\varphi_1)\dots\sin(\varphi_{n-2})\cos(\varphi_{n-1}) \\ x_n & = r\sin(\varphi_1)\dots\sin(\varphi_{n-2})\sin(\varphi_{n-1}) \end{cases}$$

By calculating the determinant of the Jacobian of the transformation above, we can conclude that

$$dV = r^{n-1}sin^{n-2}(\varphi_1)sin^{n-3}(\varphi_2)\dots sin(\varphi_{n-2})d\varphi_{n-1}d\varphi_{n-2}\dots d\varphi_1 dr$$

And the volume becomes:

$$
\begin{aligned}
&\mathcal{L}^n(B_R(x)) = \int_{B_R(x)} dV = \\
&= \int_0^R \int_0^\pi \dots \int_0^\pi \int_0^{2\pi} r^{n-1} sin^{n-2}(\varphi_1) sin^{n-3}(\varphi_2) \dots sin(\varphi_{n-2}) d\varphi_{n-1} d\varphi_{n-2} \dots d\varphi_1 dr \\
&= \left(\int_0^R r^{n-1} dr\right)\left(\int_0^\pi sin^{n-2}(\varphi_1) d\varphi_1\right) \dots \left(\int_0^{2\pi} d\varphi_{n-1}\right) \\
&\overset{(1)}{=} \frac{R^n}{n}\left(2\int_0^{\frac{\pi}{2}} sin^{n-2}(\varphi_1) d\varphi_1\right) \dots \left(4\int_0^{\frac{\pi}{2}} d\varphi_{n-1}\right) \\
&\overset{(2)}{=} \frac{R^n}{n} B\left(\frac{n-1}{2}, \frac{1}{2}\right) B\left(\frac{n-2}{2}, \frac{1}{2}\right) \dots B\left(1, \frac{1}{2}\right) \cdot 2B\left(\frac{1}{2}, \frac{1}{2}\right) \\
&\overset{(3)}{=} \frac{R^n}{n} \frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2})} \dots \frac{\Gamma(1)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} \cdot 2\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} \\
&\overset{(4)}{=} \frac{2\pi^{\frac{n}{2}} R^n}{n\Gamma(\frac{n}{2})} = \frac{\pi^{\frac{n}{2}} R^n}{\Gamma(\frac{n}{2}+1)} = \alpha(n) R^n
\end{aligned}
$$

Some explanations:

(1) We can change the integral boundaries since *sin* is symmetric with respect to $\frac{\pi}{2}$.

(2) $B(.,.)$ is the **Euler beta function**, which has an alternative definition that coincides with the formula from the line above.

(3) $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

(4) Here we use telescopic multiplication and the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

□

In the following we will give a list of properties of measures and in the next section we will show that the Hausdorff measure has some of those properties (it will not be Radon in every case).

Definition 2.1.5 *Let μ be a measure on $\mathbb{R}^n$. We say that μ is:*

(Borel) *If every Borel set is μ measurable.*

(Borel-Regular) *If μ is **Borel** and if :*

$$\forall C \subset \mathbb{R}^n, \exists B \text{ such that } C \subset B \text{ and } \mu(C) = \mu(B).$$

(Radon) *If μ is **Borel-Regular** and if $\mu(K) < \infty$, $\forall K \subset \mathbb{R}^n$ compact.*

(Metric) *If $\forall A, B \subset \mathbb{R}^n$ such that $dist(A,B) > 0$ it follows that*

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

Remark 2.1.6 *In the definition of metric measure, the function dist is defined by* $dist(A, B) = inf\{|x - y|; x \in A, y \in B\}$ *and should not be confused with the Hausdorff distance, which will be introduced in a later chapter.*

The last definition is useful, because in most cases it is easier to prove that a measure is metric rather than prove directly the Borel property of a measure. The following theorem links the two concepts together.

Theorem 2.1.7 (Carathéodory's criterion) *Let μ be a measure on $\mathbb{R}^n$.*

$$\mu \textit{ is metric} \Rightarrow \mu \textit{ is Borel.}$$

We will make use of this theorem in proving that the Hausdorff measure is Borel.
We now introduce a result that will be used in section (2.3)

Lemma 2.1.8 *Let $\mathcal{L}^n$ be the Lebesgue measure on $\mathbb{R}^n$. Then we have*

$$\mathcal{L}^{n+1} = \mathcal{L}^n \times \mathcal{L}^1$$

Where $\times$ denotes the product measure generated by two measures.

Proof We will use theorem 1.2.3 form the script from M.Struwe [8], which states that the Hahn-Carathéodory extension of a σ-finite premeasure is unique. This statement has to be interpreted in this way: if two measures agree on the algebra in which the premeasure is defined, and one of those two measures is the Hahn-Carathéodory extension of said premeasure, then they must be equal. We have defined $\mathcal{L}^{n+1}$ as the Hahn-Carathéodory extension of the volume in $\mathbb{R}^{n+1}$. If we take a random $(n+1)$-dimensional cube $Q = \prod_{i=1}^{n+1} [a_i, b_i]$, $a_i \leq b_i$ we get:

$$\begin{aligned}
\mathcal{L}^{n+1}(Q) = Vol(Q) = Vol\left(\prod_{i=1}^{n+1} [a_i, b_i]\right) = \prod_{i=1}^{n+1} (b_i - a_i) = \\
= \prod_{i=1}^{n} (b_i - a_i) \cdot (b_{n+1} - a_{n+1}) = \\
= \mathcal{L}^n \left(\prod_{i=1}^{n} [a_i, b_i]\right) \cdot \mathcal{L}^1 ([a_{n+1}, b_{n+1}]) = \\
\overset{(1)}{=} \mathcal{L}^n \times \mathcal{L}^1 \left(\prod_{i=1}^{n} [a_i, b_i] \times [a_{n+1}, b_{n+1}]\right) = \mathcal{L}^n \times \mathcal{L}^1(Q)
\end{aligned}$$

Were in equality (1) we used the first statement of Fubini's theorem (4.1.1 Script M.Struwe[8]). This proves that they agree on the generating Algebra and they must therefore be equal. □

Remark 2.1.9 *By induction over n we can see that* $\mathcal{L}^n = \mathcal{L}^1 \times \mathcal{L}^1 \times \cdots \times \mathcal{L}^1$

Lemma 2.1.10 *Let* $A \subset \mathbb{R}$ *with* $\mathcal{L}^1(A) = 0$. *Then* $([0,1] - A)$ *is dense in* $[0,1]$.

Proof Let U be a non empty open subset of $[0,1]$. Since U is open,U contains at least an interval and therefore we have $\mathcal{L}^1(U) > 0$. This implies that $U \notin A$ (otherwise A would have positive measure). We now have that $U \cap ([0,1] - A) \neq \emptyset$. Since U was an arbitrary open set, we can conclude. □

To conclude this section and link us with the next one we introduce a notion, which will help us immensely in the definition of Hausdorff measure.

Definition 2.1.11 (δ-covering) *Let* $C \subset \mathbb{R}^n$ *and let* $\delta > 0$ *be arbitrary. We say that* $\{U_i\}_{i\in I}$ *is a* ***δ-covering*** *of C if those three conditions hold:*

- $U_i \subset \mathbb{R}^n$ *and* $\forall i \in I : diam(U_i) < \delta$
- $C \subset \bigcup_{i\in I} U_i$
- *I is countable*

Remark 2.1.12 *For countable it is intended either finite or infinitely countable.*

2.2 Hausdorff measure

In this subsection we will introduce the **Hausdorff measure** of a set. There are two reasons why this concept is useful. Firstly it allows us to define the Hausdorff dimension of a set. Secondly it lets us measure sets of "dimension" (interpreted as a more classic intuitive way, see topological dimension defined in the introduction) lower than the framework in which we are working.
To be clearer, imagine that we are working in $\mathbb{R}^3$ and want a significant value for the area of the circle

$$C = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 = 1 \text{ and } z = 0\}.$$

We cannot use the Lebesgue measure $\mathcal{L}^n$ to get a meaningful measure for this object, since $\mathcal{L}^n(C) = 0$. (This can be proven by taking the square $B = [-1,1] \times [-1,1] \times \{0\}$, seeing that $C \subset B$ we have that $\mathcal{L}^n(C) \leq \mathcal{L}^n(B) = 2 \cdot 2 \cdot 0 = 0$). The Hausdorff measure allows us to give a meaningful value for the sets of "dimension" lower than n in $\mathbb{R}^n$. Inverting this line of taught, we can define the Hausdorff measure for non integer values.

Definition 2.2.1 *Let* $C \subset \mathbb{R}^n$ *and let* $0 \leq s < \infty$, $0 < \delta \leq \infty$ *arbitrary. Define*

$$\mathcal{H}^s_\delta(C) = \inf\left\{ \sum_{i=1}^{\infty} \alpha(s) \left(\frac{diam(U_i)}{2} \right)^s ; \{U_i\}_{i\in I} \text{ is a } \delta\text{-covering of } C \right\}$$

Remark 2.2.2 *In lemma (2.1.2) we have proven that* $diam(U) = diam\left(\overline{U}\right)$*. Therefore we can restrict the above definition by only taking the infimum over closed* δ*-coverings:*

$$\mathcal{H}^s_\delta(C) = \inf\left\{\sum_{i=1}^{\infty}\alpha(s)\left(\frac{diam(U_i)}{2}\right)^s ; \{U_i\}_{i\in I} \text{ is a } \delta\text{-covering of } C, U_i \text{ closed}\right\}$$

Remark 2.2.3 *If we take* $\tilde{\delta} < \delta$*, we have that* $\mathcal{H}^s_{\tilde{\delta}}(C) \geq \mathcal{H}^s_\delta(C)$*,* $\forall C \subset \mathbb{R}^n, s \geq 0$*. This is true because every* $\tilde{\delta}$*-covering is also a* δ*-covering. Therefore, when we calculate* $\mathcal{H}^s_{\tilde{\delta}}(C)$*, we take the infimum on a smaller set than when we calculate* $\mathcal{H}^s_\delta(C)$*, resulting in a bigger value.*

From this remark we can finally define the Hausdorff measure:

Definition 2.2.4 (Hausdorff measure) *Let* $C \subset \mathbb{R}^n$ *and let* $0 \leq s < \infty$*. The* ***s-dimensional Hausdorff measure*** *of C on* $\mathbb{R}^n$ *is given by:*

$$\mathcal{H}^s(C) = \lim_{\delta\to 0}\mathcal{H}^s_\delta(C) = \sup_{\delta>0}\mathcal{H}^s_\delta(C)$$

Where the last equality is true thanks to the remark above.

Remark 2.2.5 *The existence of the limit is also a consequence of remark (2.2.3); since* $\mathcal{H}^s_\delta(C)$ *increases (not necessarily monotonously) for* δ *which goes to 0 from the right. It is not always the case that the limit is finite.*

Remark 2.2.6 *The constant* $\alpha(s)$ *is present in the definition of* $\mathcal{H}^s_\delta$ *and its effect cascades into the definition of* $\mathcal{H}^s$*. It is just a normalizing factor that allows us to prove (in the next section) that the Lebesgue measure and the n-dimensional Hausdorff measure agree on* $\mathbb{R}^n(\mathcal{L}^n = \mathcal{H}^n)$*.*

Remark 2.2.7 *If* $s < n$ *then* $\mathcal{H}^s$ *is not a Radon measure in* $\mathbb{R}^n$*. The proof of this statement will be presented later in remark (2.4.3).*

We will now prove the first theorem of this thesis, in which we want to show that the Hausdorff measure is well defined and has some "nice" properties. The proof of this theorem is very much inspired by the proof found in the book of Evans[9], with some clarifications.

Theorem 2.2.8 $\mathcal{H}^s$ *is a Borel regular measure* $\forall\, 0 \leq s < \infty$

Proof Let $0 \leq s$ be fix.
Claim 1 $\mathcal{H}^s_\delta$ is a measure $\forall\, \delta > 0$
Proof Claim 1 Choose $\{U_i\}_{i=1}^\infty \subset \mathbb{R}^n$ and suppose that $\forall i$ there is a δ-covering $\{C^i_j\}_{j=1}^\infty$ of U_i; then $\{C^i_j\}_{i,j=1}^\infty$ covers $\bigcup_{i=1}^{\infty} U_i$. Therefore

$$\mathcal{H}^s_\delta\left(\bigcup_{i=1}^{\infty}U_i\right) \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\alpha(s)\left(\frac{diam\ C^i_j}{2}\right)^s.$$

Taking the inf over all the possible $\{C_j^i\}_{j=1}^{\infty}$ and pulling it into the first sum, we find

$$\mathcal{H}_\delta^s\left(\bigcup_{i=1}^{\infty} U_i\right) \leq \sum_{i=1}^{\infty} \mathcal{H}_\delta^s(U_i).$$

This proves the sub additivity of $\mathcal{H}_\delta^s$. The claim is concluded, since the we can take $\{\emptyset\}$ as a δ-covering of the empty set to prove that $\mathcal{H}_\delta^s(\emptyset) = 0 \,\forall\, s, \delta$.
Claim 2 $\mathcal{H}^s$ is a measure
Proof Claim 2 Choose $\{U_i\}_{i=1}^{\infty} \subset \mathbb{R}^n$. Then

$$\mathcal{H}_\delta^s\left(\bigcup_{i=1}^{\infty} U_i\right) \leq \sum_{i=1}^{\infty} \mathcal{H}_\delta^s(U_i) \leq \sum_{i=1}^{\infty} \mathcal{H}^s(U_i).$$

Where the first inequality was proven in Claim 1 and the second inequality uses the "sup" the definition of $\mathcal{H}^s = \sup_{\delta>0} \mathcal{H}_\delta^s$. We now use the fact that the right hand side of this inequality does not depend on δ by letting δ go to 0.

$$\mathcal{H}^s\left(\bigcup_{i=1}^{\infty} U_i\right) = \lim_{\delta\to 0} \mathcal{H}_\delta^s\left(\bigcup_{i=1}^{\infty} U_i\right) \leq \sum_{i=1}^{\infty} \mathcal{H}^s(U_i).$$

In this way we proved the sub additivity. To conclude the claim note that $\mathcal{H}^s(\emptyset) = \lim_{\delta\to 0} \mathcal{H}_\delta^s(\emptyset) = \lim_{\delta\to 0} 0 = 0$.
Claim 3 $\mathcal{H}^s$ is a metric and Borel measure
Proof Claim 3 Let $A, B \subset \mathbb{R}^n$ with $dist(A, B) > 0$ and let $0 < \delta < \frac{1}{4} dist(A, B)$. Let $\{U_i\}_{i=1}^{\infty}$ be a δ-covering of $A \cup B$. Now put $\mathcal{A} = \{U_j | U_j \cap A \neq \emptyset\}$ and $\mathcal{B} = \{U_j | U_j \cap B \neq \emptyset\}$. We can see:

$$A = A \cap \left(\bigcup_{i=1}^{\infty} U_i\right) = \bigcup_{i=1}^{\infty} (A \cap U_i) \overset{(1)}{=} \bigcup_{U_i \in \mathcal{A}} (A \cap U_i) \subset \bigcup_{U_i \in \mathcal{A}} U_i$$

Where in (1) we use the definition of $\mathcal{A}$. In the same way it can be proved that $B \subset \bigcup_{U_i \in \mathcal{B}} U_i$. Since $dist(A, B) > 0$ we can see that $U_i \cap U_j = \emptyset$ if $U_i \in \mathcal{A}$, $U_j \in \mathcal{B}$. Thanks to this separation we can calculate:

$$\begin{aligned}\sum_{i=1}^{\infty} \alpha(s)\left(\frac{diam(U_i)}{2}\right)^s &\geq \sum_{U_i \in \mathcal{A}} \alpha(s)\left(\frac{diam(U_i)}{2}\right)^s + \sum_{U_i \in \mathcal{B}} \alpha(s)\left(\frac{diam(U_i)}{2}\right)^s \\ &\geq \mathcal{H}_\delta^s(A) + \mathcal{H}_\delta^s(B)\end{aligned}$$

Where the first inequality comes from the separation in $\mathcal{A}$ and $\mathcal{B}$ and the second inequality is the definition of $\mathcal{H}_\delta^s$. Since the right hand side of the equality does not depend on the covering $\{U_i\}_{i=1}^{\infty}$, we can take the infimum (still over $\{U_i\}_{i=1}^{\infty}$, δ-coverings of $A \cup B$) on the left hand side to get:

$$\begin{aligned}\mathcal{H}_\delta^s(A \cup B) &= \inf\left\{\sum_{i=0}^{\infty} \alpha(s)\left(\frac{diam(U_i)}{2}\right)^s ; \{U_i\}_{i\in I} \text{ is a } \delta\text{-covering of } A \cup B\right\} \\ &\geq \mathcal{H}_\delta^s(A) + \mathcal{H}_\delta^s(B).\end{aligned}$$

This result is only valid for $\delta < \frac{1}{4}dist(A, B)$, but this is not a problem, since we want to let δ go to 0. By doing so, we get

$$\mathcal{H}^s(A \cup B) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A \cup B) \geq \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) + \lim_{\delta \to 0} \mathcal{H}^s_\delta(B) = \mathcal{H}^s(A) + \mathcal{H}^s(B).$$

The inequality in the other direction comes from the sub additivity of $\mathcal{H}^s$, proven in claim 2. Therefore we can conclude that

$$\mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B) \; \forall \; A, B \text{ with } dist(A, B) > 0.$$

This implies that $\mathcal{H}^s$ is a metric measure and (by the Carathéodory criterion (thm 2.1.7)) it is also Borel.
Claim 4 $\mathcal{H}^s$ is a Borel regular measure.
Proof Claim 4 For this proof we will use the equivalent definition of $\mathcal{H}^s_\delta$ given in remark (2.2.2).
Let $A \subset \mathbb{R}^n$ such that $\mathcal{H}^s(A) < \infty$. It follows that $\mathcal{H}^s_\delta(A) < \infty \; \forall \; \delta > 0$ (this is because of the sup definition of $\mathcal{H}^s$).
$\forall \; j > 0$, let $\{U_i^j\}_{i=1}^\infty$ be a $\frac{1}{j}$-**closed**-covering of A, with

$$\sum_{i=1}^{\infty} \alpha(s) \left(\frac{diam(U_i^j)}{2} \right)^s \leq \mathcal{H}^s_{\frac{1}{j}}(A) + \frac{1}{j}.$$

This last condition can be fulfilled because of the "inf" in the definition of $\mathcal{H}^s_{\frac{1}{j}}(A)$. Now let $A_j = \bigcup_{i=1}^{\infty} U_i^j$ and $B = \bigcap_{j=1}^{\infty} A_j$. B is a Borel set (since the U_i^j are all closed, they are Borel. By taking infinite union and infinite intersection we remain in the Borel σ-algebra). $A \subset A_j$, $\forall j$ since $\{U_i^j\}_{i=1}^\infty$ is a covering of A for all j. This implies that $A \subset B \Rightarrow \mathcal{H}^s(A) \leq \mathcal{H}^s(B)$. The inequality in the other direction comes from the definition of B

$$\mathcal{H}^s_{\frac{1}{j}}(B) \leq \mathcal{H}^s_{\frac{1}{j}}(A_j) \leq \sum_{i=1}^{\infty} \alpha(s) \left(\frac{diam(U_i^j)}{2} \right)^s \leq \mathcal{H}^s_{\frac{1}{j}}(A) + \frac{1}{j}.$$

Letting j tend to 0 on both sides of the inequality we get the wanted $\mathcal{H}^s(B) \leq \mathcal{H}^s(A)$. This implies $\mathcal{H}^s(A) = \mathcal{H}^s(B)$ and ends the proof of claim 4, since (for an arbitrary A) we have found a B Borel, that has the same Hausdorff measure as A. □

After having proved that the Hausdorff measure is nice to deal with, we start presenting and proving some properties that will help us in dealing with the study of fractal structures.

Theorem 2.2.9 (Scaling property) *Given $\lambda > 0$ and $C \subset \mathbb{R}^n$ we have*

$$\mathcal{H}^s(\lambda C) = \lambda^s \mathcal{H}^s(C)$$

where $\lambda C = \{\lambda x | x \in C\}$ is just the scaling of C by λ.

Proof let $\{U_i\}_{i=1}^{\infty}$ be a δ-covering of C. Then $\{\lambda U_i\}_{i=1}^{\infty}$ is a $\lambda\delta$-covering of λC. Hence

$$\mathcal{H}^s_{\lambda\delta}(\lambda C) \leq \sum_{i=1}^{\infty} \alpha(s)\left(\frac{\lambda\, diam(U_i)}{2}\right)^s = \lambda^s \sum_{i=1}^{\infty} \alpha(s)\left(\frac{diam(U_i)}{2}\right)^s.$$

Taking the infimum over all δ-coverings of C, we get

$$\mathcal{H}^s_{\lambda\delta}(\lambda C) \leq \lambda^s \mathcal{H}^s_{\delta}(C)$$

Now by letting $\delta \to 0$ we get $\mathcal{H}^s(\lambda C) \leq \lambda^s \mathcal{H}^s(C)$. By replacing λ with $\frac{1}{\lambda}$ and C by λC, we get the converse inequality and the end of the proof. □

Definition 2.2.10 *Let $C \subset \mathbb{R}^n$ and $f : C \to \mathbb{R}^m$ be a mapping such that*

$$|f(x) - f(y)| \leq L|x - y|^{\beta}$$

*for some $L > 0$ and $\beta > 0$. Then f is said to satisfy the **Hölder condition** of exponent β and constant L.*

Remark 2.2.11 *If $\beta = 1$ we say that f is **Lipschitz** continuous*

Theorem 2.2.12 *Assume that f satisfies the Hölder condition of exponent β and constant L. Than $\forall\, s > 0$*

$$\mathcal{H}^{\frac{s}{\beta}}(f(C)) \leq \frac{\alpha(\frac{s}{\beta})2^{\frac{s}{\beta}}}{\alpha(s)2^s} L^{\frac{s}{\beta}} \mathcal{H}^s(C)$$

Proof Let $\{U_i\}_{i=1}^{\infty}$ be a δ-covering of C.
Claim 1 $diam(f(C \cap U_i)) \leq L\, diam(U_i)^{\beta}$
Proof Claim 1 Let $x, y \in f(C \cap U_i)$. Than $x = f(\tilde{x}), y = f(\tilde{y})$ for $\tilde{x}$ and $\tilde{y} \in C \cap U_i$. It follows $|x - y| = |f(\tilde{x}) - f(\tilde{y})| \leq L|\tilde{x} - \tilde{y}|^{\beta}$. It follows

$$diam(f(C \cap U_i)) = \sup_{\tilde{x},\tilde{y} \in C \cap U_i} |f(\tilde{x}) - f(\tilde{y})| \leq L \sup_{\tilde{x},\tilde{y} \in C \cap U_i} |\tilde{x} - \tilde{y}|^{\beta} \leq$$

$$\leq L \sup_{\tilde{x},\tilde{y} \in U_i} |\tilde{x} - \tilde{y}|^{\beta} = L\, diam(U_i)^{\beta} \square_{claim1}$$

Thanks to claim 1 we can assert that $\{f(C \cap U_i)\}_{i=1}^{\infty}$ is an ε cover of $f(C)$, where $\varepsilon = L\delta^{\beta}$. By claim 1 we have:

$$\sum_{i=1}^{\infty} \alpha\left(\frac{s}{\beta}\right)\left(\frac{diam(f(C \cap U_i))}{2}\right)^{\frac{s}{\beta}} \leq \frac{\alpha(\frac{s}{\beta})2^{\frac{s}{\beta}}}{\alpha(s)2^s} L^{\frac{s}{\beta}} \sum_{i=1}^{\infty} \alpha(s)\left(\frac{diam(U_i)}{2}\right)^s.$$

By taking the inf over the $\{U_i\}_{i=1}^{\infty}$ we get

$$\mathcal{H}^{\frac{s}{\beta}}_{\varepsilon}(f(C)) \leq \frac{\alpha(\frac{s}{\beta})2^{\frac{s}{\beta}}}{\alpha(s)2^s} L^{\frac{s}{\beta}} \mathcal{H}^s_{\delta}(C).$$

Since $\varepsilon \to 0$ whenever $\delta \to 0$, we just let $\delta \to 0$ and get the wanted result. □

Remark 2.2.13 *If f is a Lipschitz function ($\beta = 1$) , the inequality simplifies to*

$$\mathcal{H}^s(f(C)) \leq L^s \mathcal{H}^s(C)$$

Remark 2.2.14 *If f is an **isometry** (a function, such that $|x-y| = |f(x) - f(y)|$), then both f and f^{-1} are Lipschitz, with both Lipschitz constants $L = 1$(this can be seen just by staring at the definition of Lipschitz function).*

Corollary 2.2.15 *Let f be an isometry, then $\mathcal{H}^s(C) = \mathcal{H}^s(f(C))\ \forall\ C \subset \mathbb{R}^n$*

Proof

$$\mathcal{H}^s(f(C)) \leq \mathcal{H}^s(C) = \mathcal{H}^s(f^{-1}(f(C))) \leq \mathcal{H}^s(f(C))$$
$$\Rightarrow \mathcal{H}^s(C) = \mathcal{H}^s(f(C))$$

Where in the first inequality we used remark (2.2.13) on f and in the second one we used it on f^{-1}. □

Corollary 2.2.16 *$\mathcal{H}^s$ is translation and rotation invariant, because rotations and translations are isometries.*

2.3 $\mathcal{H}^n = \mathcal{L}^n$ on $\mathbb{R}^n$

The reason why we defined the Hausdorff measure using $\alpha(s)$ is to have this nice result, even though we had to struggle a bit in the past section. This equality is not trivial: $\mathcal{L}^n$ is defined as the Hahn-Carathéodory extension of the volume, which was defined on n-dimensional intervals (also called cubes), where $\mathcal{H}^n$ is defined on arbitrary δ-coverings. Once again, this subsection follows the path suggested by Evans book[9].
We start by proving a simpler statement.

Lemma 2.3.1 *$\mathcal{H}^1 = \mathcal{L}^1$ on $\mathbb{R}$*

Proof Let $A \subset \mathbb{R}$ and let $\delta > 0$.

$$\begin{aligned} \mathcal{L}^1(A) &= \inf \left\{ \sum_{i=1}^{\infty} diam(U_j) | A \subset \bigcup_{i=1}^{\infty} U_j \right\} \\ &\leq \inf \left\{ \sum_{i=1}^{\infty} diam(U_j) | \{U_i\}_{i=1}^{\infty} \text{ is a } \delta\text{-covering of } A \right\} \\ &= \mathcal{H}^1_\delta(A). \end{aligned}$$

Now let $I_j = [j\delta, (j+1)\delta]$, $k \in \mathbb{Z}$. We have that $diam(U_i \cap I_j) \leq diam(U_i) \leq \delta$. Therefore $\{U_i \cap I_j\}_{i,j}$ is a δ-covering of A. By using the fact that $diam(D \cup E) \geq diam(D) + diam(E)$ we get

$$diam(U_i) = diam\left(\bigcup_{j\in\mathbb{Z}} (U_i \cap I_j)\right) \geq \sum_{j\in\mathbb{Z}} diam(U_i \cap I_j)$$

From those two facts we get:

$$\begin{aligned}\mathcal{L}^1(A) &= \inf\left\{\sum_{i=1}^{\infty} diam(U_j)|A \subset \bigcup_{i=1}^{\infty} U_j\right\}\\ &\geq \inf\left\{\sum_{i=1}^{\infty}\sum_{j\in\mathbb{Z}} diam(U_i \cap I_j)|A \subset \bigcup_{i=1}^{\infty} U_j\right\}\\ &\geq \mathcal{H}^1_\delta(A)\end{aligned}$$

Where the last inequality comes from the fact that $\{U_i \cap I_j\}_{i,j}$ is a δ-covering of A. Thus we get that $\mathcal{H}^1_\delta = \mathcal{L}^1$ for an arbitrary δ, by letting $\delta \to 0$ we conclude the proof. □

The following lemma and definition will be pivotal in proving the so called Isodiametric inequality (Theorem (2.3.5)).

Lemma 2.3.2 *Let $f : \mathbb{R}^n \to [0, +\infty]$ be an $\mathcal{L}^n$ measurable function. Then the "region under the graph" of f:*

$$A = \{(x, y)|x \in \mathbb{R}^n,\ y \in \mathbb{R},\ 0 \leq y \leq f(x)\} \text{ is } \mathcal{L}^{n+1} \text{measurable.}$$

Proof Let us define the sets

$B = f^{-1}(\{+\infty\}) = \{x \in \mathbb{R}^n | f(x) = +\infty\}$

$C = f^{-1}([0, +\infty[) = \{x \in \mathbb{R}^n | 0 \leq f(x) < +\infty\}$

$C_{jk} = f|_C^{-1}\left([\frac{j}{k}, \frac{j+1}{k}[\right) = \{x \in C | \frac{j}{k} \leq f(x) < \frac{j+1}{k}\}$

From the first equality in all three of those lines we can see that the sets are $\mathcal{L}^n$ measurable (since f is an $\mathcal{L}^n$ measurable function). Let it be noted that $C = \bigcup_{j=0}^{\infty} C_{jk}$, $\forall k \in \mathbb{N}$. We finally set

$$D_k = \bigcup_{j=0}^{\infty}\left(C_{jk} \times \left[0, \frac{j}{k}\right]\right) \cup (B \times [0, +\infty])$$

$$E_k = \bigcup_{j=0}^{\infty}\left(C_{jk} \times \left[0, \frac{j+1}{k}\right]\right) \cup (B \times [0, +\infty])$$

Both D_k and E_k are $\mathcal{L}^{n+1}$ measurable: This comes from Fubini's theorem, which states that the product of an $\mathcal{L}^1$ and an $\mathcal{L}^n$ measurable sets is $\mathcal{L}^{n+1}$ measurable. As stated previously C_{jk} and B are $\mathcal{L}^n$ measurable and intervals are $\mathcal{L}^1$ measurable. Therefore a set like $C_{jk} \times \left[0, \frac{j}{k}\right[$ is $\mathcal{L}^{n+1}$ measurable (here we also use Lemma(2.1.8), which states that $\mathcal{L}^{n+1} = \mathcal{L}^n \times \mathcal{L}^1$). Taking then unions leaves the sets D_k and E_k in the σ-algebra of $\mathcal{L}^{n+1}$ measurable sets.

Let it be noted that D_k and E_k are approximations of the set A: D_k from below and E_k from above.
By definition we have that $D_k \subset A \subset E_k$, and we also have that $D_k \subset D_{k+1}$ and that $E_{k+1} \subset E_k$. Therefore we can set $E = \bigcap_{k=1}^{\infty} E_k$ and $D = \bigcup_{k=1}^{\infty} D_k$. We still have that $D \subset A \subset E$ and both D and E are $\mathcal{L}^{n+1}$ measurable (as union and intersection of $\mathcal{L}^{n+1}$ measurable sets). Now we see:

$$\mathcal{L}^{n+1}((E-D)\cap B_R(0)) \leq \mathcal{L}^{n+1}((E_k-D_k)\cap B_R(0)) \leq \frac{1}{k}\mathcal{L}^{n+1}(B_R(0)), \forall R>0.$$

The first inequality comes from the fact that $E \subset E_K$ and $D_k \subset D$, the second by the definition of E_k and D_K. Letting $k \to \infty$ we have that the right hand side goes to 0. Therefore $\mathcal{L}^{n+1}((E-D)\cap B_R(0)) = 0, \forall R > 0$ and so $\mathcal{L}^{n+1}(E-D) = 0$. Since $A \subset E$, it follows that $\mathcal{L}^{n+1}(A-D) = 0$, and therefore A is $\mathcal{L}^{n+1}$ measurable(D was Borel). □

We now introduce the **Steiner symmetrization**, takes a set Ω and a plane P that "cuts" Ω in two parts. We replace the "top" part of Ω with a symmetric copy of the "bottom" part of Ω.

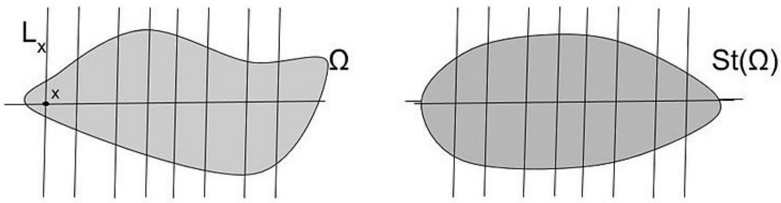

Figure 2.1: A representation of Steiner transformation of a set Ω. [10]

Definition 2.3.3 *Let $a, b \in \mathbb{R}^n, |a| = 1$. Also let $A \subset \mathbb{R}^n$. We define*

$$L_a^b = \{b + ta | t \in \mathbb{R}\}, \textit{ the line thorugh } b \textit{ in direction } a$$

$$P_a = \{x \in \mathbb{R}^n | x \cdot a = 0\}, \textit{ the plane thourgh the origin perpendicular to } a$$

$$S_a(A) = \bigcup_{b \in P_a;\, A \cap L_b^a \neq \emptyset} \left\{ b + ta|;\, |t| \leq \frac{1}{2}\mathcal{H}^1(A \cap L_b^a) \right\}$$

The set $S_a(A)$ is called the ***Steiner symmetrization*** *of A with respect to the plane P_a.*

Lemma 2.3.4 (Properties of Steiner Symmetrization)

(i) $diam(S_a(A)) \leq diam(A)$

(ii) *If A is $\mathcal{L}^n$ measurable, then so is $S_a(A)$; and $\mathcal{L}^n(S_a(A)) = \mathcal{L}^n(A)$*

The proof of this lemma makes use of the fact that $\mathcal{L}^n$ is rotation invariant, lemma (2.3.1) [$\mathcal{H}^1 = \mathcal{L}^1$ on $\mathbb{R}$], lemma(2.3.2), Fubini and Tonelli's theorems. The proof can be read on Evans'book[9].
To complete the proof of the main theorem of this section we will need the following inequality.

Theorem 2.3.5 (Isodiametric inequality) *Let $A \subset \mathbb{R}^n$. Then*

$$\mathcal{L}^n(A) \leq \alpha(n)\left(\frac{diam(A)}{2}\right)^n$$

Remark 2.3.6 *At a first look, this inequality seems trivial, since it can look like we can just take a ball $B = B_{\frac{diam(A)}{2}}$ of diameter $diam(A)$ which contains A. In that case we would have that*

$$\mathcal{L}^n(A) \leq \mathcal{L}^n(B) = \mathcal{L}^n\left(B_{\frac{diam(A)}{2}}\right) = \alpha(n)\left(\frac{diam(A)}{2}\right)^n$$

Where in the last equality we used lemma (2.1.4). The problem is that such a B doesn't always exist.

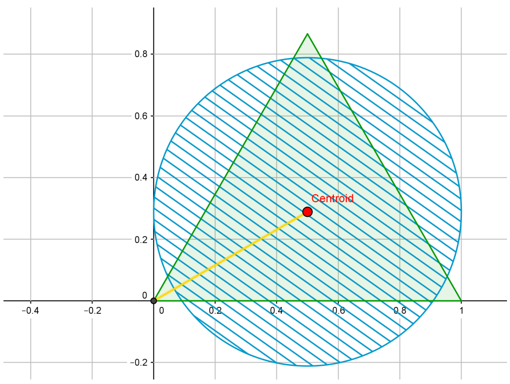

Figure 2.2: Equilateral Triangle and its covering.

Take for example the equilateral triangle of size 1 in $\mathbb{R}^2$. The diameter of the triangle is 1 (length of a side). If we would have to find a circle (blue circle in the image) that covers the triangle, we would centre it in the centroid of the triangle (for symmetry reasons). But the distance between the centroid of the triangle and one of the vertices is $\frac{\sqrt{3}}{3} \approx 0.577 > \frac{1}{2}$.(yellow line in the image).
From this example we can conclude that we cannot find a circle of diameter $diam(A)$ which covers A in every case. What we can always do is find a circle of ***radius*** *$diam(A)$, but in that case the inequality becomes much worse.*

Remark 2.3.7 *The proof of this inequality uses property (ii) from the lemma above, but we decided to omit it in order to focus on the proof of the main theorem of this section.*

Theorem 2.3.8

$$\mathcal{H}^n = \mathcal{L}^n \text{ on } \mathbb{R}^n$$

Proof *Claim 1* $\mathcal{L}^n(A) \leq \mathcal{H}^n(A), \forall A \subset \mathbb{R}^n$
Proof of Claim 1 Let $\delta > 0$ arbitrary. Let $\{U_i\}_{i=1}^{\infty}$ be a δ-covering of A. Then we have

$$\mathcal{L}^n(A) \leq \mathcal{L}^n\left(\bigcup_{i=1}^{\infty} U_i\right) \leq \sum_{i=1}^{\infty} \mathcal{L}^n(U_i) \leq \sum_{i=1}^{\infty} \alpha(n)\left(\frac{diam(U_i)}{2}\right)^n$$

Where in the last inequality we used the Isodiametric inequality. Taking the inf over $\{U_i\}_{i=1}^{\infty}$ coverings we get $\mathcal{L}^n(A) \leq \mathcal{H}^n_\delta(A)$. Since δ was arbitrary, we can let $\delta \to 0$ and conclude the proof of claim1.
Claim 2

$$\mathcal{L}^n(A) = \inf\left\{\sum_{i=1}^{\infty} \mathcal{L}^n(Q_i) | Q_i \text{cube and } \{Q_i\}_{i=1}^{\infty} \text{ is a } \delta\text{-covering of } A\right\}$$

Proof of Claim 2"≤" Since our definition of Lebesgue measure is the Hahn-Carathéodory extension of the volume in $\mathbb{R}^n$, we have that

$$\mathcal{L}^n(A) = \inf\left\{\sum_{i=1}^{\infty} \mathcal{L}^n(Q_i) | Q_i \text{ cube and } \{Q_i\}_{i=1}^{\infty} \text{ is a covering of } A\right\}$$

Therefore the inequality holds, since on the right and side we take the infimum on a smaller set. " ≥ " If we have $\{Q_i\}_{i=1}^{\infty}$ a covering of A in cubes, we just divide this cubes in smaller ones of diameter $\leq \delta$.
Claim 3 $\mathcal{H}^n(A) \leq C_n \mathcal{L}^n(A)$, where $C_n = \alpha(n)\left(\frac{\sqrt{n}}{2}\right)^n$.
Proof of Claim 3 Let Q be a cube of side length $d = \frac{diam(Q)}{\sqrt{n}}$. Then we have

$$\mathcal{L}^n(Q) = d^n = \left(\frac{diam(Q)}{\sqrt{n}}\right)^n.$$

rearranging those term and inserting the definition of C_n we find:

$$\alpha(n)\left(\frac{diam(Q)}{2}\right)^n = C_n \mathcal{L}^n(Q) \quad (1)$$

Therefore we have

$$\begin{aligned}
\mathcal{H}^n_\delta(A) &\leq \inf\left\{\sum_{i=1}^{\infty}\alpha(n)\left(\frac{diam(Q_i)}{2}\right)^n |Q_i \text{cube and } \{Q_i\}_{i=1}^{\infty} \delta\text{-covering of } A\right\} \\
&\overset{(1)}{=} \inf\left\{\sum_{i=1}^{\infty} C_n\mathcal{L}^n(Q_i)|Q_i \text{cube and } \{Q_i\}_{i=1}^{\infty} \text{ is a } \delta\text{-covering of } A\right\} \\
&= C_n \inf\left\{\sum_{i=1}^{\infty} \mathcal{L}^n(Q_i)|Q_i \text{cube and } \{Q_i\}_{i=1}^{\infty} \text{ is a } \delta\text{-covering of } A\right\} \\
&= C_n\mathcal{L}^n(A)
\end{aligned}$$

Where in the last equality we used Claim 2. Since δ was arbitrary, let it tend to 0 and we have $\mathcal{H}^n(A) \leq C_n\mathcal{L}^n(A)$. This implies that $\mathcal{H}^n$ is absolutely continuous with respect to $\mathcal{L}^n$: take A with $\mathcal{L}^n(A) = 0 \Rightarrow$

$$\mathcal{H}^n(A) \leq C_n\mathcal{L}^n(A) = 0 \Rightarrow \mathcal{H}^n(A) = 0.$$

Claim 4 $\mathcal{H}^n(A) \leq \mathcal{L}^n(A)$, $\forall\, A \subset \mathbb{R}^n$

Proof of Claim 4 Let $\delta, \varepsilon > 0$ and let Q_i cubes such that $\{Q_i\}_{i=1}^{\infty}$ is a δ-covering of A and

$$\sum_{i=1}^{\infty}\mathcal{L}^n(Q_i) \leq \mathcal{L}^n + \varepsilon$$

Those cubes can be found because of claim 2 and the definition of infimum. By a corollary of Vitali's covering theorem (See Evans [9],Ch 1.5, Corollary 2),$\forall\, i\, \exists \{B^i_j\}_{j=1}^{\infty}$ collection of disjoint closed balls contained in $\mathring{Q}_i$ such that $diam(B^i_j) \leq \delta$ and

$$\mathcal{L}^n\left(Q_i - \dot{\bigcup_{j=1}^{\infty}} B^i_j\right) = \mathcal{L}^n\left(\mathring{Q}_i - \dot{\bigcup_{j=1}^{\infty}} B^i_j\right) = 0.$$

Where $\mathring{F}$ denotes the interior of the set F and $F - G = \{x \in F | x \notin G\}$. From absolute continuity we get that $\mathcal{H}^n\left(Q_i - \dot{\bigcup_{j=1}^{\infty}} B^i_j\right) = 0$. We can now conclude:

$$\begin{aligned}
\mathcal{H}^n_\delta(A) &\leq \mathcal{H}^n_\delta\left(\bigcup_{i=1}^{\infty} Q_i\right) \leq \sum_{i=1}^{\infty}\mathcal{H}^n_\delta(Q_i) \overset{(2)}{=} \sum_{i=1}^{\infty}\mathcal{H}^n_\delta\left(\dot{\bigcup_{j=1}^{\infty}} B^i_j\right) \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathcal{H}^n_\delta(B^i_j) \\
&\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\alpha(n)\left(\frac{diam(B^i_j)}{2}\right)^n = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathcal{L}^n(B^i_j) = \sum_{i=1}^{\infty}\mathcal{L}^n\left(\dot{\bigcup_{j=1}^{\infty}} B^i_j\right) \\
&= \sum_{i=1}^{\infty}\mathcal{L}^n(Q_i) \leq \mathcal{L}^n(A) + \varepsilon.
\end{aligned}$$

Where in (2) we use the statement proven above. By letting $\delta \to 0$ we get $\mathcal{H}^n(A) \leq \mathcal{L}^n(A) + \varepsilon$ and by letting the arbitrary ε go to 0 we get the wanted inequality. By putting Claim 1 and Claim 4 together we can conclude the proof. □

2.4 Hausdorff dimension

In this section we will finally define the Hausdorff dimension of a set and prove that it is invariant under bi-Lipschitz transformations.

Lemma 2.4.1 *Let $A \subset \mathbb{R}^n$, such that $\mathcal{H}^s_\delta(A) = 0$ for some $\delta > 0$. Then $\mathcal{H}^s(A) = 0$.*

Proof If $s = 0$ we have that$\mathcal{H}^0_\delta = \mathcal{H}^0\ \forall \delta > 0$. The claim is then trivial.
Therefore we can assume that $s > 0$. Let $\varepsilon > 0$. We can find $\{U_i\}_{i=1}^\infty$, a δ-covering of A such that

$$\sum_{i=1}^\infty \alpha(s)\left(\frac{diam(U_i)}{2}\right)^s \leq \mathcal{H}^s_\delta(A) + \varepsilon = 0 + \varepsilon = \varepsilon.$$

Now let $\delta(\varepsilon) = 2\left(\frac{\varepsilon}{\alpha(s)}\right)^{\frac{1}{s}}$. Remark that $\delta(\varepsilon) \to 0$, whenever $\varepsilon \to 0$.
It is also true that $\forall\, i : diam(U_i) < \delta(\varepsilon)$, just by reversing the equation above.
From that je have that $\mathcal{H}_{\delta(\varepsilon)}(A) \leq \varepsilon$. We can now conclude:

$$\mathcal{H}^s(A) = \lim_{\delta\to 0}\mathcal{H}^s_\delta(A) = \lim_{\varepsilon\to 0}\mathcal{H}^s_{\delta(\varepsilon)}(A) \leq \lim_{\varepsilon\to 0}\varepsilon = 0.$$

□

Lemma 2.4.2 *Let $A \subset \mathbb{R}^n$ and $0 \leq \boldsymbol{s} < \boldsymbol{t} < +\infty$*

(i) *If $\mathcal{H}^s(A) < +\infty$, then $\mathcal{H}^t(A) = 0$.*

(ii) *If $\mathcal{H}^t(A) > 0$, then $\mathcal{H}^s(A) = +\infty$.*

Proof **(i)** Let $\mathcal{H}^s(A) < +\infty$. Let $\{U_i\}_{i=1}^\infty$ be a δ-covering of A such that

$$\sum_{i=1}^\infty \left(\frac{diam(U_i)}{2}\right)^s \leq \mathcal{H}^s_\delta(A) + 3.$$

This is possible because of the definition of $\mathcal{H}^s_\delta(A)$ and by taking $\varepsilon = 3$.

$$\begin{aligned}
\mathcal{H}^t_\delta(A) &\leq \sum_{i=1}^\infty \alpha(t)\left(\frac{diam(U_i)}{2}\right)^t = \\
&= \frac{\alpha(t)}{\alpha(s)}2^{s-t}\sum_{i=1}^\infty \alpha(s)\left(\frac{diam(U_i)}{2}\right)^s (diam(U_i))^{t-s} \\
&\leq \frac{\alpha(t)}{\alpha(s)}2^{s-t}\sum_{i=1}^\infty \alpha(s)\left(\frac{diam(U_i)}{2}\right)^s \delta^{t-s} \\
&= \frac{\alpha(t)}{\alpha(s)}2^{s-t}\delta^{t-s}\sum_{i=1}^\infty \alpha(s)\left(\frac{diam(U_i)}{2}\right)^s \\
&\leq \frac{\alpha(t)}{\alpha(s)}2^{s-t}\delta^{t-s}(\mathcal{H}^s_\delta(A) + 3)
\end{aligned}$$

Since $\mathcal{H}_\delta(A) < +\infty$, we have that the right hand side tends to 0, whenever $\delta \to 0$. This concludes the proof of **(i)**. **(ii)** Is the exact logical inversion of **(i)**, therefore it is also proved. □

Remark 2.4.3 *We can now easily prove remark (2.2.7), which stated that $\mathcal{H}^s$ is not a radon measure for $s < n$.*

Proof let $n \geq 1$ (otherwise there would not be an $s < n$ and the proposition would not make sense). Let B be the closed unit ball on $\mathbb{R}^n$. Then $\mathcal{H}^n(B) = \mathcal{L}^n(B) = \alpha(n)$. since $0 < \alpha(n) < +\infty \forall n > 0$, we have that $0 < \mathcal{H}^n(B) < +\infty$. Now we apply Lemma (2.4.2)(ii) and get that $\mathcal{H}^s(B) = +\infty$. This implies that $\mathcal{H}^s$ is not Radon, since B was closed and bounded, therefore compact. □

Remark 2.4.4 *Thanks to this last lemma we can finally define the Hausdorff dimension of a set: it s in fact notable that there is a one and only value in which $\mathcal{H}^s(F)$ could be different than 0 or $+\infty$.*

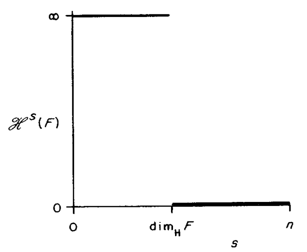

Figure 2.3: Graph of $\mathcal{H}^s(F)$ with respect to s .Note the "jump" in Hausdorff measure by $s = dim_{\mathcal{H}}(F)$. (From Falconer [3], page 28, figure 2.3)

Definition 2.4.5 (Hausdorff dimension) *The Hausdorff dimension of a set $A \subset \mathbb{R}^n$ is*

$$dim_{\mathcal{H}}(A) = \inf\{0 \leq s < +\infty | \mathcal{H}^s(A) = 0\} = \sup\{0 \leq s < +\infty | \mathcal{H}^s(A) = +\infty\}.$$

Remark 2.4.6 *The Hausdorff dimension of a set does not have to be an integer number, we will see examples of such sets in Chapter 3*

Remark 2.4.7 *From the previous lemma, $dim_{\mathcal{H}}(A)$ is well defined. We also get for free the first condition, which helps us calculate the Hausdorff dimension of a set:*

$$0 < \mathcal{H}^s(A) < +\infty \Rightarrow dim_{\mathcal{H}}(A) = s.$$

The other implication doesn't always hold: we have that $dim_{\mathcal{H}}(\mathbb{R}^n) = n$ but $\mathcal{H}^n(\mathbb{R}^n) = \mathcal{L}^n(\mathbb{R}^n) = +\infty$. Other counterexamples can be found also for compact sets. Rewriting the above statement in a more mathematical way, we can say that

$$dim_{\mathcal{H}}(A) = s \Rightarrow \mathcal{H}^s(A) \in [0, +\infty]$$

Example 2.4.8 *For a simple example let us consider a flat unit disc in* $\mathbb{R}^3$ *as in the preface of section (2.2)*

$$C = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 = 1 \text{ and } z = 0\}.$$

We have that $\mathcal{H}^1(C) = +\infty$, $\mathcal{H}^2(C) = \pi, \mathcal{H}^3(C) = 0$. *Therefore the disc C must have Hausdorff dimension* $dim_{\mathcal{H}}(C) = 2$.

Remark 2.4.9 *For a set* A *in* $\mathbb{R}^n$ *we have* $dim_{\mathcal{H}}(A) \leq n$.

Proof We will prove that $\mathcal{H}^{n+t}(A) = 0 \; \forall t > 0$. this claim is sufficient because of the explanations given above.

$$\mathcal{H}^{n+t}(A) = \mathcal{H}^{n+t}\left(\bigcup_{k=1}^{\infty} A \cap B_k(0)\right) \leq \sum_{k=1}^{\infty} \mathcal{H}^{n+t}(A \cap B_k(0))$$

We also have that

$$\mathcal{H}^n(A \cap B_k(0)) \leq \mathcal{H}^n(B_k(0)) = \mathcal{L}^n(B_k(0)) = \alpha(n)k^n < +\infty$$

By applying lemma (2.4.2)(i) we have that $\mathcal{H}^{n+t}(A \cap B_k(0)) = 0 \; \forall k$.
By inserting this fact into the first equation of this proof we get

$$\mathcal{H}^{n+t}(A) \leq \sum_{k=1}^{\infty} \mathcal{H}^{n+t}(A \cap B_k(0)) = \sum_{k=1}^{\infty} 0 = 0.$$

□

Remark 2.4.10 *Any* ***open*** *set* $A \subset \mathbb{R}^n$ *has* $dim_{\mathcal{H}}(A) = n$.

Proof If A is open, we can always find a ball $B \subset A$. We now have:

$$\mathcal{H}^n(A) \geq \mathcal{H}^n(B) = \mathcal{L}^n(B) > 0$$

Therefore $dim_{\mathcal{H}}(A) \geq n$ and by applying the previous remark we get equality. □

Remark 2.4.11 *Any* ***countable*** *set* $A \subset \mathbb{R}^n$ *has* $dim_{\mathcal{H}}(A) = 0$.

Proof Let $A = \{x_n, n \in \mathbb{N}\}$. We will show that $H^s(A) = 0 \forall s > 0$ this is enough because of the definition of Hausdorff dimension.
Let $\delta > 0$. Choose $\left\{B_{\frac{\delta}{2^{\frac{n}{s}}}}(x_n)\right\}_{n=1}^{\infty}$ be a δ-covering of A. It follows that

$$\mathcal{H}^s_\delta(A) \leq \sum_{n=1}^{\infty} \left(\frac{\delta}{2^{\frac{n}{s}}}\right)^s = \sum_{n=1}^{\infty} \delta^s \cdot 2^{-n} = \delta^s$$

Since $s > 0$, the right hand side tends to 0 whenever $\delta \to 0$. This concludes the proof. □

Remark 2.4.12 *If A is a smooth m-dimensional sub-manifold of $\mathbb{R}^n$, then we have $dim_{\mathcal{H}}(A) = m$, as expected. This can be deduced from the relationship between Hausdorff and Lebesgue measures, but the calculation is overly complicated for this thesis.*

Remark 2.4.13 $A \subset B \Rightarrow dim_{\mathcal{H}}(A) \leq dim_{\mathcal{H}}(B)$. *This comes form the monotonicity of the Hausdorff measure.*

We will now prove a couple of properties, that will be useful in calculating the Hausdorff dimension of auto-similar sets.

Lemma 2.4.14 *Let $A \subset \mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$ satisfy the Hölder condition of exponent β. Then*

$$dim_{\mathcal{H}}(f(A)) \leq \frac{1}{\beta} dim_{\mathcal{H}}(A).$$

Proof If $s > dim_{\mathcal{H}}(A)$ we have that $\mathcal{H}^s(A) = 0$. By theorem (2.2.12) we can see

$$\mathcal{H}^{\frac{s}{\beta}}(f(A)) \leq \frac{\alpha(\frac{s}{\beta})2^{\frac{s}{\beta}}}{\alpha(s)2^s} L^{\frac{s}{\beta}} \mathcal{H}^s(A) = 0 \leq \mathcal{H}^{\frac{s}{\beta}}(f(A))$$

This proves equality overall and (by the remarks above) implies that $dim_{\mathcal{H}}(f(A)) \leq \frac{s}{\beta}\ \forall s > dim_{\mathcal{H}}(A)$. Letting $s \to dim_{\mathcal{H}}(A)$ we get the wanted inequality. □

Definition 2.4.15 *Let $A \subset \mathbb{R}^n$.A function $f : A \to \mathbb{R}^m$ is said to be* ***bi-Lipschitz*** *if $\exists L_1, L_2 \in \mathbb{R}$ such that*

$$L_1|x-y| \leq |f(x) - f(y)| \leq L_2|x-y|$$

Remark 2.4.16 *f bi-Lipschitz $\Rightarrow$ f injective.*

Proof Assume that $f(x) = f(y)$. Hence

$$|x-y| \leq \frac{1}{L_1}|f(x) - f(y)| = 0 \Rightarrow |x-y| = 0 \Rightarrow x = y.$$

□

Remark 2.4.17 *f bi-Lipschitz $\Rightarrow f^{-1} : f(A) \to A$ is Lipschitz*

Proof Note that the function f^{-1} is well defined because of the previous remark. Now let $a = f(x)$ and $b = f(y) \in f(A)$. We have

$$|f^{-1}(a) - f^{-1}(b)| = |f^{-1}(f(x)) - f^{-1}(f(x))| = |x-y| \leq \frac{1}{L_1}|f(x) - f(y)| = \frac{1}{L_1}|a-b|.$$

□

Thanks to these small remarks, we can prove a not so trivial result.

Theorem 2.4.18 *Let $A \subset \mathbb{R}^n$.*

(i) *If $f : A \to \mathbb{R}^m$ is Lipschitz, then $dim_{\mathcal{H}}(f(A)) \leq dim_{\mathcal{H}}(A)$*

(ii) *If $f : A \to \mathbb{R}^m$ is bi-Lipschitz, then $dim_{\mathcal{H}}(f(A)) = dim_{\mathcal{H}}(A)$*

Proof **(i)** f Lipschitz $\Rightarrow$ f satifies the Holder condition with $\beta = 1$. The claim follows form the application of lemma (2.4.13).
(ii) from (i), we have that $dim_{\mathcal{H}}(f(A)) \leq dim_{\mathcal{H}}(A)$. Using remark (2.4.17), we know that f^{-1} is Lipschitz. Therefore we can apply point (i) also on f^{-1} and get

$$dim_{\mathcal{H}}(A) = dim_{\mathcal{H}}\left(f^{-1}(f(A))\right) \leq dim_{\mathcal{H}}(f(A)).$$

We have therefore proved both inequality and by putting them together we get (ii). □

Remark 2.4.19 *The previous theorem shows that the Hausdorff dimension of a set is invariant under bi-Lipschitz transformation. This means that there cannot be a bi-Lipschitz transformation between two sets of different Hausdorff dimension. This fact can be helpful in proving that a set does not have a certain Hausdorff dimension (by finding another set with that dimension and constructing a bi-Lipschitz function between the two sets).*

This condition resembles the topological notion of invariant under continuous mapping, with bi-Lipschitz functions in place of homeomorphisms. Let it be noted that every bi-Lipschitz function is also continuous, therefore Hausdorff dimension does not usually provide us with hints on the topological structure of the set. An exception to this statement is the following lemma.

Lemma 2.4.20 *Let $A \subset \mathbb{R}^n$ with $dim_{\mathcal{H}}(A) < 1$. Then A is totally disconnected.*

Proof Let $x, y \in A$ be different. Define the mapping

$$\begin{aligned} f : \mathbb{R}^n &\to [0, +\infty[\\ z &\mapsto f(z) = |z - x| \end{aligned}$$

By the triangular inequality we get that $|f(z) - f(w)| \leq |z - w|$. Therefore f is Lipschitz with Lipschitz constant 1. By applying theorem (2.4.18)(i) we get that $dim_{\mathcal{H}}(f(A)) \leq dim_{\mathcal{H}}(A) < 1$. This implies that $f(A) \subset \mathbb{R}$ with $0 = \mathcal{H}^1(f(A)) = \mathcal{L}^1(f(A))$. Thanks to lemma (2.1.10) we know that $f(A)$ has a dense complement in $[0, 1]$.
Now let $r \notin f(A)$ and $0 < r < f(y)$ (this r exists because x and y are distinct). We now have

$$A = \{z \in A : |z - x| < r\} \cup \{z \in A : |z - x| > r\}.$$

Therefore A is contained in two disjoint open sets, with x belonging to one and y belonging to the other. This implies that x and y lie in different connected components of A. This concludes the proof. □

Example 2.4.21 *We will see that the Cantor triadic set has Hausdorff dimension* $\frac{\ln(2)}{\ln(3)} < 1$*. This can be a proof that the Cantor triadic set is totally disconnected.*

Chapter 3

Calculating Hausdorff dimension

In this chapter we will focus on some methods for calculating the Hausdorff dimension of some compact sets. This will bring us to the definition of box-counting dimension and the concept of **Iterated function system (IFS)**.

3.1 Direct Method

As we have seen in the previous chapter, we have the following implication:

$$0 < \mathcal{H}^s(A) < +\infty \Rightarrow dim_{\mathcal{H}}(A) = s.$$

The idea is now to find an upper and a lower bound for $\mathcal{H}^s(A)$ for a given s and to use the proposition above to conclude that $dim_{\mathcal{H}}(A) = s$. Let it be noted that this method is not particularly effecting, since finding the lower bound usually requires some effort. Nevertheless it will help us calculate the Hausdorff dimension of two sets: the **Cantor dust** and the **Cantor triadic set**.

3.1.1 Cantor dust

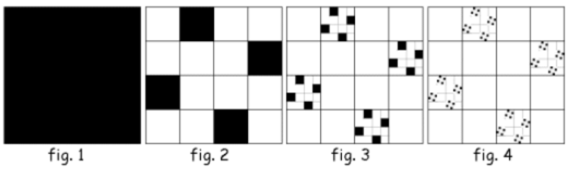

Figure 3.1: Cantor Dust. [11]

Let $A_0 = [0,1]^2 \subset \mathbb{R}^2$. Next divide A_0 into 16 equal squares and delete 12 of them as according to the picture. Repeat this step on the remaining 4 squares. We have that

A_0 is the union of 1 square of side length 1.

A_1 is the union of 4 square of side length $\frac{1}{4}$.

A_2 is the union of 16 square of side length $\frac{1}{16}$.
$\vdots$
A_k is the union of 4^k square of side length $4^{-k.}$

By construction we also have that $A_{k+1} \subset A_k$. Finally we set $A = \bigcap_{k=0}^{\infty} A_k$.
The set A is called **Cantor dust**.

Theorem 3.1.1 $dim_{\mathcal{H}}(A) = 1$

Proof We will show $1 \leq \mathcal{H}^1(A) \leq \sqrt{2}$
"$\mathcal{H}^1(A) \leq \sqrt{2}$" Since A_k is the union of 4^k square of side length $4^{-k.}$, we can cover each one of those squares with a ball of diameter $diam(B_i) = \sqrt{2} \cdot 4^{-k}$. Since $A \subset A_k$ and $\alpha(1) = 2$ we can conclude:

$$\begin{aligned} \mathcal{H}^1_\delta(A) \leq \mathcal{H}^1_\delta(A_k) &= \inf\left\{\sum_{i=1}^{\infty} \alpha(1)\left(\frac{diam(U_i)}{2}\right); \{U_i\}_{i\in I} \text{ is a } \delta\text{-covering of } C\right\} \\ &\leq \sum_{i=1}^{4^k} 2\left(\frac{diam(B_i)}{2}\right) = \sum_{i=1}^{4^k} 2\left(\frac{\sqrt{2}\cdot 4^{-k}}{2}\right) = 4^k \cdot 2\left(\frac{\sqrt{2}\cdot 4^{-k}}{2}\right) = \sqrt{2}. \end{aligned}$$

This was done assuming that $\delta > diam(B_i)$. Since the right hand side is independent of δ, we can let $\delta \to 0$ and conclude the claim.
"$1 \leq \mathcal{H}^1(A)$" Let $\pi : \begin{cases} \mathbb{R}^2 & \to \mathbb{R} \\ (x, y) & \mapsto x \end{cases}$ be the projection into the x-axis. By construction we have that $\pi(A) = [0, 1]$ (it is enough to look at the picture). Now let $\{U_i\}_{i=1}^{\infty}$ be a δ-covering of A. Without loss of generality, we can assume that $U_i = B_{r_i}(z_i)$, for some $z_i = (x_i, y_i))$ (This can be done thanks to a reasoning similar to the one explained in the proof of theorem (2.3.8) claim 2). Then we have

$$[0,1] = \pi(A) \leq \pi\left(\bigcup_{i=1}^{\infty} B_{r_i}(z_i)\right) \leq \bigcup_{i=1}^{\infty} \pi(B_{r_i}(z_i)) = \bigcup_{i=1}^{\infty}]x_i - r_i, x_i + r_i[$$

and hence

$$1 = \mathcal{L}^1([0,1]) \leq \sum_{i=1}^{\infty} \mathcal{L}^1(]x_i - r_i, x_i + r_i[) = 2\sum_{i=1}^{\infty} r_i$$

It follows that

$$\sum_{i=1}^{\infty} \alpha(1)\left(\frac{diam(U_i)}{2}\right) = \sum_{i=1}^{\infty} 2 \cdot r_i = 2\sum_{i=1}^{\infty} r_i \geq 1$$

Taking the inf over the δ-coverings of A yields us the wanted result. □

3.1.2 Cantor triadic set

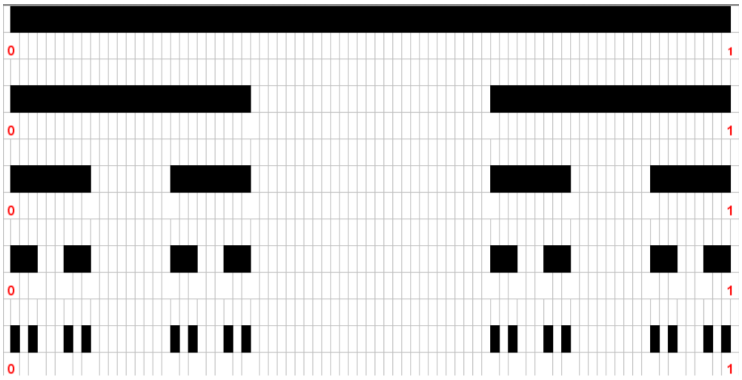

Figure 3.2: Cantor Triadic Set. [12]

The Cantor triadic set is defined in a similar way as the cantor dust. We start with $S_0 = [0,1]$ and we remove the middle third of the set, this way $S_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Repeating this process on the two remaining intervals we get S_2 and so on. We have that S_k is a union of 2^k intervals of length 3^{-k} and let us denote $\{I_j\}_{j=1}^{2^k}$ those intervals. The definition of the cantor triadic set is therefore

$$S = \bigcap_{k=0}^{\infty} S_k.$$

It can be proven that S is an example of uncountable set with Lebesgue measure 0. This implies(since $\mathcal{H}^1 = \mathcal{L}^1$ on $[0,1]$) that $\mathcal{H}^1(S) = 0$ and, as a consequence, that $dim_{\mathcal{H}}(S) \leq 1$. The proof of the following theorem is inspired by the proof given by Jay Shah[19].

Theorem 3.1.2

$$dim_{\mathcal{H}}(S) = \frac{\ln(2)}{\ln(3)} := s.$$

Proof *Claim 1* $\mathcal{H}^t(S) = 0 \; \forall t > s$.
Proof of Claim 1 Let $k > 0$ and let $\{I_j\}_{j=1}^{2^k}$ be the intervals of length 3^{-k} composing S_k. Since $S \subset S_k$, we have that $\{I_j\}_{j=1}^{2^k}$ is also a covering of S and

$$\sum_{j=1}^{2^k} diam(I_j)^t = \sum_{j=1}^{2^k} (3^{-k})^t = 2^k \cdot 3^{-kt} = e^{k(\ln(2) - t\ln(3))}$$

Since $(\ln(2) - t\ln(3)) < 0$ by the choice of t, it follows that the right hand side goes to 0, whenever $k \to \infty$. By this fact we can conclude claim 1.
Claim 1 $\Rightarrow dim_{\mathcal{H}}(S) \leq s$ by the definition of Hausdorff dimension.
The next claim will be used for proving Claim 3.

<u>*Claim 2*</u> $\forall \varepsilon > 0\, \forall \alpha \in]0,1[\, \forall \{C_n\}_{n=1}^{\infty} \subset [0,1] \exists$ finitely many $D_1, \ldots, D_m$ such that

$$\bigcup_{n=1}^{\infty} C_n \subset \bigcup_{j=1}^{m} D_j \text{ and } \sum_{j=1}^{m} diam(D_j)^{\alpha} \leq \sum_{n=1}^{\infty} diam(C_n)^{\alpha} + \varepsilon.$$

<u>*Proof of Claim 2*</u> Let C_n be any subset of $[0,1]$. Since $[0,1]$ is bounded, we have that $a_n = \inf C_n$ and $b_n = \sup C_n$ exist and are finite. Let $I_n = [a_n, b_n]$. We have that $diam(C_n) = diam(I_n)$ and $C_n \subset I_n$. Therefore we can assume without loss of generality that C_n are all closed intervals.
Now for every C_n we can take an open interval U_n with $C_n \subset U_n$ and $diam(U_n) = diam(C_n) + \left(\frac{\varepsilon}{2^n}\right)^{\frac{1}{\alpha}}$. We now have that

$$diam(U_n)^{\alpha} \leq diam(C_n)^{\alpha} + \frac{\varepsilon}{2^n}.$$

Since $[0,1]$ is compact and the U_n are open, we can find a finite sub cover $D_1, \ldots, D_m$ with $D_i = U_n$ for some n and $\bigcup_{n=1}^{\infty} C_n \subset \bigcup_{n=1}^{\infty} U_n = \bigcup_{j=1}^{m} D_j$. Taking the sum of the diameters concludes the proof.
<u>*Claim 3*</u> $\mathcal{H}^s(S) > 0$
<u>*Proof of Claim 3*</u> Let $\{C_n\}_{n=1}^{\infty}$ be a covering of S. We can find countable $D_1, \ldots, D_m$ as in Claim 2. Now choose k such that

$$\left(\frac{1}{3}\right)^k \leq \min_{j=1}^{m}\{diam(D_j)\}$$

Define N_l as the number of sets D_j such that $3^{-l} \leq diam(D_j) < 3^{-l+1}$, for every $l = 1, \ldots, k$. By using this definition we have

$$\sum_{j=1}^{m} diam(D_j)^s \geq \sum_{l=1}^{k} N_l 3^{-ls} = \sum_{l=1}^{k} N_l 2^{-l} \tag{3.1}$$

Where in the last equality we used the definition of s. Now if we find a lower bound for the right hand side, we can conclude.
Suppose that D_j satisfies $3^{-l} \leq diam(D_j) < 3^{-l+1}$. Then D_j can intersect at most 2 of the intervals composing S_l (because the space between the intervals composing S_l is 3^{-l}). Each one of those intervals produces 2^{k-l} intervals when going from S_l to S_k. This implies that D_j contains at most $2 \cdot 2^{k-l} = 2^{k-l+1}$ intervals in S_k. Since S_k is composed by 2^k intervals, we get the inequality:

$$2^k \leq \sum_{l=1}^{k} N_l 2^{k-l+1} \Rightarrow \frac{1}{2} \leq \sum_{l=1}^{k} N_l 2^{-l}.$$

We have therefore found a lower bound for $\sum_{l=1}^{k} N_l 2^{-l}$ and (by using equation (3.1)) also a lower bound for $\sum_{j=1}^{m} diam(D_j)^s$. Since the covering was arbitrary, we have that the lower bound holds also for $\mathcal{H}^s(S)$ and we can therefore conclude the proof. □

3.2 An upper bound for the Hausdorff dimension

Throughout the years, there have been many definitions of dimension, as we have briefly stated in the introduction. One of these definitions will help us in finding an upper bound for the Hausdorff dimension of a set and it is called the **Box-counting** dimension. The advantage of working with this dimension is that it is much easier to calculate then the Hausdorff dimension.
The idea behind this dimension is the following: imagine we have a set A and we want to cover A in cubes of length $\frac{1}{n}$. How many cubes would be need? We will see that this is independent of the frame of reference $\mathbb{R}^n$. Let us start with a definition:

Definition 3.2.1 *Let $A \subset \mathbb{R}^n$ bounded and let $\delta > 0$. We define*

$$\mathcal{N}(A,\delta) = \min\{n \in \mathbb{N} | A \textit{ can be covered by } n \textit{ cubes of side length } \delta\}.$$

Remark 3.2.2 *This is well defined since A is bounded, therefore it can be contained in a big cube, which can be divided into a finite number of smaller cubes of side length δ.*

Example 3.2.3 *Usually it is easier to set $\delta = \frac{1}{n}$.*
If we take $A = [0,1] \subset \mathbb{R}$, it can be seen that the best way of dividing it is to write $[0,1] = \bigcup_{i=1}^{n} \left[\frac{i-1}{n}, \frac{i}{n}\right]$, therefore we have that $\mathcal{N}\left(A, \frac{1}{n}\right) = n$.
In the same way, if we would have $A = [0,1] \times \{0\} \subset \mathbb{R}^2$ we can write $A \subset \bigcup_{i=1}^{n} \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[-\frac{1}{2n}, \frac{1}{2n}\right]$ is an efficient way of covering A, hence $\mathcal{N}\left(A, \frac{1}{n}\right) = n$ still.
This invariance is what will help us defining the box-counting dimension.

Definition 3.2.4 *Let $A \subset \mathbb{R}^n$ be bounded,*

(lower box-counting dimension) $\underline{dim_B}(A) = \liminf_{\delta \to 0} \frac{\ln(\mathcal{N}_\delta(A))}{-\ln(\delta)}$

(upper box-counting dimension) $\overline{dim_B}(A) = \limsup_{\delta \to 0} \frac{\ln(\mathcal{N}_\delta(A))}{-\ln(\delta)}$

(box-counting dimension) $dim_B(A) = \lim_{\delta \to 0} \frac{\ln(\mathcal{N}_\delta(A))}{-\ln(\delta)}$

The last definition only makes sense when $\underline{dim_B}(A) = \overline{dim_B}(A)$.

Remark 3.2.5 *It can be shown that $dim_B(A)$ does not change if we take arbitrary sets of diameter less than δ instead of cubes when calculating the minimum in*

$\mathcal{N}(A,\delta)$. The same thing can be said if we take the collection of cubes in the δ- ***coordinate mesh*** *of $\mathbb{R}^n$; i.e. cubes of the form:*

$$Q = [m_1\delta, (m_1+1)\delta] \times [m_2\delta, (m_2+1)\delta] \times \dots [m_n\delta, (m_n+1)\delta],\ m_i \in \mathbb{Z}\ \forall i.$$

This list can be extended further, but in praxis we take the most convenient definition, and it is usually the last one.

Example 3.2.6 *Let us once again consider the Cantor triadic set S. We will prove that*

$$\underline{dim}_B(S) = \overline{dim}_B(S) = \frac{\ln(2)}{\ln(3)} = dim_{\mathcal{H}}(S)$$

By the usual covering of the intervals $\{I_j\}_{j=1}^{2^k}$ of S_k, we have that $\mathcal{N}_\delta(S) \le 2^k$ for $\delta \in]3^{-k}, 3^{-k+1}]$. Therefore it follows :

$$\overline{dim}_B(S) = \limsup_{\delta\to 0} \frac{\ln(\mathcal{N}_\delta(S))}{-\ln(\delta)} \le \limsup_{k\to+\infty} \frac{\ln(2^k)}{-\ln(3^{-k+1})} = \limsup_{k\to+\infty} \frac{\ln(2^k)}{\ln(3^{k-1})} =$$
$$= \limsup_{k\to+\infty} \frac{k\ln(2)}{(k-1))\ln(3)} = \frac{\ln(2)}{\ln(3)}$$

By a similar argument as in the proof of the Hausdorff dimension of the triadic set, we can say that any interval of length δ, with $\delta \in [3^{-k-1}, 3^{-k}[$, intersects at most one of the intervals of length 3^{-k} constructing S_k. Since there are 2^k intervals composing S_k, we require at least 2^k intervals of length δ to cover S. This yields us to the bound $\mathcal{N}_\delta(S) \ge 2^k$ and by a calculation similar to the one above we get $\underline{dim}_B(S) \ge \frac{\ln(2)}{\ln(3)}$. This concludes the proof:

$$\frac{\ln(2)}{\ln(3)} \le \underline{dim}_B(S) \le dim_B(S) \le \overline{dim}_B(S) \le \frac{\ln(2)}{\ln(3)}.$$

Remark 3.2.7 *It is not always the case that $dim_{\mathcal{H}} = dim_B$, the next example will be a proof of this statement.*

Example 3.2.8 *Let $A = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$. We have that $dim_{\mathcal{H}}(A) = 0$, since A is a countable set. We claim that $dim_B(A) = \frac{1}{2}$.*
Let us consider a set U with $diam(U) = \delta < \frac{1}{2}$ and lets define the set
$A_k = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{k-1}, \frac{1}{k}\right\} \subset A$. Now let k such that

$$\frac{1}{k-1} - \frac{1}{k} = \frac{1}{k(k-1)} > \delta \ge \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Thanks to the first inequality, we discover that U contains at most one point of A_k and therefore we need at least k sets of diameter δ to cover A. This yields (by using the second part of the inequality)

$$\frac{\ln(\mathcal{N}_\delta(A))}{-\ln(\delta)} \ge \frac{\ln(k)}{\ln(k(k+1))}$$

By using the theorem of Bernoulli de l'Hôpital, we can conclude that

$$\underline{dim_B}(A) = \liminf_{\delta\to 0} \frac{\ln(\mathcal{N}_\delta(A))}{-\ln(\delta)} \geq \liminf_{k\to\infty} \frac{\ln(k)}{\ln(k(k+1))} = \liminf_{k\to\infty} \frac{k(k+1)}{2\cdot k(k+1)} = \frac{1}{2}$$

For the other inequality, take again k such that $\frac{1}{k-1} - \frac{1}{k} = \frac{1}{k(k-1)} > \delta \geq \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. *Then you only need* $(k+1)$ *intervals of length* δ *to cover the interval* $[0, \frac{1}{k}]$, *which contains all but* $(k-1)$ *points of A. For said remaining* $(k-1)$ *points just take any* $(n-1)$ *intervals of length* δ *which contain them. Thanks to this construction, we can assert that*

$$\frac{\ln(\mathcal{N}_\delta(A))}{-\ln(\delta)} \leq \frac{\ln((k+1)+(k-1))}{\ln(k(k-1))} = \frac{\ln(2k)}{\ln(k(k-1))}$$

a calculation similar to the previous one shows

$$\overline{dim_B}(A) = \limsup_{\delta\to 0} \frac{\ln(\mathcal{N}_\delta(A))}{-\ln(\delta)} \leq \limsup_{k\to\infty} \frac{\ln(2k)}{\ln(k(k-1))} = \limsup_{k\to\infty} \frac{k(k-1)}{2\cdot k(k-1)} = \frac{1}{2}$$

We can therefore conclude

$$\frac{1}{2} \leq \underline{dim_B}(S) \leq dim_B(S) \leq \overline{dim_B}(S) \leq \frac{1}{2}.$$

Remark 3.2.9 *From this examples, it follows that* $dim_B\left(\bigcup_{i=1}^{\infty} A_i\right) \neq \sup_{i=1}^{\infty} dim_B(A_i)$, *which is a property that we would like to have in a "dimension".*
The Box-counting dimension also has the problem that it is not always defined, specifically when $\underline{dim}_B \neq \overline{dim}_B$ *and for unbounded sets.*

Nevertheless, the Box-counting dimension gives us a useful boundary for the Hausdorff dimension:

Theorem 3.2.10 *Let* $A \subset \mathbb{R}^n$. *We have that* $dim_{\mathcal{H}}(A) \leq dim_B(A)$.

Proof This proof follows closely the path laid in the lecture notes from Prof. Orsina.[13]
Let $\tilde{s} = dim_B(A)$. Let $\varepsilon > 0$ and $\delta(\varepsilon)$ such that

$$\tilde{s} - \varepsilon \leq \frac{\ln(\mathcal{N}_\delta(A))}{-\ln(\delta)} \leq \tilde{s} + \varepsilon, \qquad \forall \delta < \delta(\varepsilon)$$

Remark that such a $\delta(\varepsilon)$ exists because of the lim in the definition of $dim_B(A)$. By multiplying by $-\ln(\delta)$(which is positive for $\delta < 1$) and taking the exponential, we get

$$\frac{1}{\delta^{\tilde{s}-\varepsilon}} \leq \mathcal{N}(A,\delta) \leq \frac{1}{\delta^{\tilde{s}+\varepsilon}}, \qquad \forall \delta < \delta(\varepsilon) \tag{3.2}$$

Now we fix $\delta < \delta(\varepsilon)$ and we define $M_\delta = \mathcal{N}(A, \delta)$.
By the definition of $\mathcal{N}(A, \delta)$, there is a finite amount of cubes $\{Q_i\}_{i=1}^{M_\delta}$ of side length δ with $A \subset \bigcup_{i=1}^{M_\delta} Q_i$. Since $diam(Q_i) = \sqrt{n}\delta$, we know that $\{Q_i\}_{i=1}^{M_\delta}$ is a $\sqrt{n}\delta$-covering of A.
Now $\forall s > \tilde{s}$ choose ε such that $\tilde{s} + \varepsilon < s$. For those values we can now write:

$$\begin{aligned}\mathcal{H}^s_{\sqrt{n}\delta}(A) &\leq \sum_{i=1}^{M_\delta} \alpha(s) \left(\frac{diam(Q_i)}{2}\right)^s = \sum_{i=1}^{M_\delta} \alpha(s) \left(\frac{\sqrt{n}\delta}{2}\right)^s \\ &= M_\delta \alpha(s) \delta^s \left(\frac{\sqrt{n}}{2}\right)^s \leq \alpha(s) \left(\frac{\sqrt{n}}{2}\right)^s \delta^{s-(\tilde{s}+\varepsilon)}\end{aligned}$$

Where the last inequality comes from equation (3.2). Since both $\sqrt{n}\delta$ and $\delta^{s-(\tilde{s}+\varepsilon)}$ tend to 0 ad $\delta \to 0$, we get that

$$\mathcal{H}^s(A) = 0, \ \forall s > \tilde{s} = dim_B(A).$$

We can therefore conclude that $dim_{\mathcal{H}}(A) \leq dim_B(A)$. □

We will use this inequality in the next chapter to show that
$dim_{\mathcal{H}}(A) = dim_B(A)$ for A fix point of an **I**terated **F**unctions **S**ystem.

3.3 Self-similarity and Iterated Function Systems

In this last section of the thesis we will follow the construction given in the course by Prof.Orsina [13] in order to describe self-similar compact subsets of $\mathbb{R}^n$ as attractors of Iterated Function Systems (from now on IFS).

3.3.1 Banach Fixed Point Theorem

Definition 3.3.1 *A metric space (M, d) is said to be **complete**, if every Cauchy sequence in M converges in M.*

Definition 3.3.2 *Given a metric space (M, d), a function $f : M \to M$ is a **contraction** if $\exists 0 < C < 1$ such that*

$$d(f(x), f(y)) \leq Cd(x, y).$$

Definition 3.3.3 *Let $f : M \to M$. $\tilde{x}$ is a **fixed point of** f if $f(\tilde{x}) = \tilde{x}$.*

Remark 3.3.4 *Fixed Points are defined not only for contractions. For example a rotation around the origin in $\mathbb{R}^n$ has the origin itself as a unique fixed point but is not a contraction.*

Remark 3.3.5 *In the case that f is not a contraction the number of fixed points of f can vary from 0 to uncountably infinite. Take for example $f = id_{\mathbb{R}^n}$. Even though f is almost a contraction ($C = 1$) it has all of $\mathbb{R}^n$ as a set of fixed points. On the other hand, if we take f as a translation, we have that f does not have any fixed points.*

Theorem 3.3.6 (Banach Fixed Point Theorem) *let f be a contraction on (M, d) complete metric space. The f has a unique fixed point $\tilde{x} \in M$*

Remark 3.3.7 *The idea behind the proof is to take* ***any*** *initial value x_0 and define the sequence $x_n = f(x_{n-1})$ inductively. Still by induction and by making use the fact that f is a contraction, it can be proved that $d(x_n, x_{n-1}) \leq C^{n-1} d(x_1, x_0)$, where C is the constant of the contraction f. Now by using the triangular inequality, it can be shown that $d(x_n, x_m) \leq \frac{C^m}{1-C} d(x_0, x_1)$, which tends to 0 as $m \to \infty$. This proves that $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence and, by completeness of M, that it has a limit $\tilde{x}$. Uniqueness is proved in the usual way.*

Remark 3.3.8 *The fact that in the proof we can take* ***any*** *initial point x_0 is truly remarkable and will be an important characteristic of self-similar set.*

3.3.2 Hausdorff distance

The purpose of this subsection is to show that the set of compact subsets of $\mathbb{R}^n$ can be seen as a complete metric space by choosing the right distance, the Hausdorff distance. This fact will allow us to apply Banach Fixed point theorem on some specific functions and describe some self similar sets as the unique fixed point of such functions.
We will take for granted some basic properties of compact sets, which can be found in the script of Topology by Prof. W.Werner[14].

Definition 3.3.9 *Let (M, d) be a metric space. Then set*

$$\mathcal{K}(M) = \{K \subset M | K \neq \emptyset \text{ and } K \text{ compact}\}.$$

<u>Notation</u> *if there is no ambiguity, we just write $\mathcal{K}$ instead of $\mathcal{K}(M)$.*

We will now build up the definition of Hausdorff distance on $\mathcal{K}$.

Definition 3.3.10 *Let $x \in M$ and $K \in \mathcal{K}(M)$. define*

$$\tilde{d}(x, K) = \min\{d(x, y) | y \in K\}.$$

This is well defined, since $d(x, .)$ is continuous and K is compact.

Remark 3.3.11 *Let $K \in \mathcal{K}$ fix. Then the function $\tilde{d}(., K) : M \to [0, +\infty[$ is also continuous. This can be seen by taking a sequence $(x_n)_{n=1}^{\infty}$ which converges to a fix x_0 and showing that $\tilde{d}(x_0, K) = \lim_{n\to\infty} \tilde{d}(x_n, K)$*

Definition 3.3.12 *Let (M,d) be metric and let $K,H \in \mathcal{K}$. The **oriented distance** (or signed distance) between K and H is*

$$d_{or}(K,H) = \max\{\tilde{d}(x,H) | x \in K\}.$$

The maximum is obtained thanks to the compactness of K and the previous remark.

Remark 3.3.13 *The importance of* min *and* max *instead of* inf *and* sup *in the previous two definitions is the fact that we can now find x and y in K and H respectively, such that $d_{or}(K,H) = d(x,y)$.*

Lemma 3.3.14 $d_{or}(K,H) = 0 \Rightarrow K \subset H$.

Proof By the definition of d_{or}, we have that $\tilde{d}(x,H) = 0 \forall x \in K$. Now by the definition of $\tilde{d}$, we know that there is a $y \in H$ such that $d(x,y) = \tilde{d}(x,H) = 0$. Since d is a metric, this implies that $x = y$. Therefore, for every $x \in K$ we have found an $y \in H$ such that $x = y$. This concludes the proof. □

Remark 3.3.15 *Even though d_{or} is called a "distance", it is not one. The problem is that d_{or} is not symmetric. The definition of Hausdorff distance yields a solution to this problem.*

Definition 3.3.16 *Let (M,d) be metric and let $K,H \in \mathcal{K}$. The the **Hausdorff distance** between K and H is*

$$d_h(K,H) = \max\{d_{or}(K,H), d_{or}(H,K)\}.$$

Theorem 3.3.17 *$(\mathcal{K}, d_h)$ is a metric space.*

Proof The well definedness of d_h was discussed above. It remains to be proved that d_h is a metric. Non negativity comes from the fact that d is a metric on M and symmetry comes straight from the definition of d_h.
It remains to be proved that $d_h(H,K) = 0 \Rightarrow H = K$ and that the triangular inequality is satisfied. The triangular inequality will not be covered by this thesis but we refer to the lecture notes of Prof.Orsina ([13], p.24). We will prove the first statement.
Let $H,K \in \mathcal{K}$ such that $d_h(H,K) = 0$. By the definition of d_h we have that both $d_{or}(H,K) = 0$ and $d_{or}(K,H) = 0$. Applying twice lemma (3.3.14), we have that $H \subset K$ and $K \subset H$. □

We will now work our way up to an equivalent definition of the Hausdorff distance on $M = \mathbb{R}^n$ with d as the usual euclidean distance.

Definition 3.3.18 *Let $K \in \mathcal{K}$.The ε-dilation of K is given by*

$$K + \varepsilon = \left\{x \in \mathbb{R}^n | \tilde{d}(x,K) \leq \varepsilon\right\}$$

Lemma 3.3.19 *Let $H,K \in \mathcal{K}$, $\varepsilon > 0$. Then*

$$(K+\varepsilon) \cup (H+\varepsilon) \subset (K \cup H) + \varepsilon$$

Proof Let $x \in K + \varepsilon$. Then by definition of $K + \varepsilon$ we have that $\tilde{d}(x, K) \leq \varepsilon$. By plugging in the definition of $\tilde{d}$, we arrive to $\min\{d(x,y)|y \in K\} \leq \varepsilon$. Hence

$$\tilde{d}(x, K \cup H) = \min\{d(x,y)|y \in K \cup H\} \leq \min\{d(x,y)|y \in K\} \leq \varepsilon.$$

Where the middle inequality derives from taking the min on a smaller set. Therefore $x \in (K \cup H) + \varepsilon$. The same reasoning can be repeated for $x \in H + \varepsilon$. □

Lemma 3.3.20 *Let $K \in \mathcal{K}$. Then*

$$d_h(H,K) = \min\{\varepsilon > 0 | H \subset K + \varepsilon \text{ and } K \subset H + \varepsilon\}$$

Proof [sketch] This proof goes by demonstrating two small claims. First let $H, K \in \mathcal{K}$. It can be proved that $K + \varepsilon$ and $H + \varepsilon$ are both compact.(Boundedness is clear, since $K + \varepsilon$ is just a "collar" of radius ε around the bounded K; sequence-Closeness can be shown by triangular inequality). Finally, thanks to lemma (3.3.19), it can be shown that

$$d_h(H,K) \leq \varepsilon \Leftrightarrow H \subset K + \varepsilon \text{ and } K \subset H + \varepsilon \tag{3.3}$$

□

This lemma is pivotal in proving the following theorem

Theorem 3.3.21 $(\mathcal{K}, d_h)$ *is a* ***complete*** *metric space.*

Remark 3.3.22 *For the proof of this theorem we refer to the lecture notes of Prof.Orsina ([13],p.28), even though the bulk of the work is to arrive to the right definition as we just did.*

3.3.3 IFS

In all of this subsection we will deal with $M = \mathbb{R}^n$, therefore we set $\mathcal{K} = \mathcal{K}(\mathbb{R}^n)$. We also refer to $\mathcal{K}$ as the metric space $(\mathcal{K}, d_h)$.

Definition 3.3.23 *Given a continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$, we can define:*

$$\begin{aligned} \tilde{f} : \mathcal{K} &\to \mathcal{K} \\ K &\mapsto f(K) \end{aligned}$$

This function is well defined because $f(K)$ is still compact for f continuous.

Lemma 3.3.24 *f Lipchitz with Lipschitz constant $L \Rightarrow \tilde{f}$ Lipschitz with lipschitz constant L (on $\mathcal{K}$)*

Proof Let $K, H \in \mathcal{K}$. Then

$$d_{or}(\tilde{f}(K), \tilde{f}(H)) = \max_{z \in \tilde{f}(K)} \min_{w \in \tilde{f}(H)} d(z,w) \overset{(1)}{=}$$

$$\overset{(1)}{=} \max_{x \in K} \min_{y \in H} d(f(x), f(y)) \leq L \max_{x \in K} \min_{y \in H} d(x,y) = L d_{or}(K,H).$$

Where in (1) we used the definition of $\tilde{f}$. We can repeat the same process for $d_{or}(\tilde{f}(H), \tilde{f}(K))$ and then, by taking the max we get that

$$d_h(\tilde{f}(H), \tilde{f}(K)) \leq L d_h(H,K).$$

□

Remark 3.3.25 *From the previous lemma, we can say that if f is a contraction, then also $\tilde{f}$ is a contraction(since they have the same Lipschitz constant). By the fact that $\mathcal{K}$ is complete we can apply Banach Fixed point theorem and get that $\tilde{f}$ has a unique fix point $\overline{K}$ in $\mathcal{K}$. The problem is that this fix point is uninteresting:*
Since K is a compact subset of a complete space, it is also complete. Then since f is a contraction on K complete, we have that f has a fixed point $\overline{x} \in K$. The set $\{\overline{x}\}$ is compact in K, therefore it is in $\mathcal{K}$ and we have that $\tilde{f}(\{\overline{x}\}) = f(\{\overline{x}\}) = \{f(\overline{x})\} = \overline{x}$. This implies that $\{\overline{x}\}$ is a fixed point of $\tilde{f}$ and, by uniqueness, that it is $\overline{K}$.

The interesting part is not taking only one contraction f, but a series of contractions $f_1, \ldots, f_m$ and "composing" them in a smart way.

Definition 3.3.26 (IFS) *Given contractions $f_1, \ldots, f_m$ we say that $\mathcal{F} = \{f_1, \ldots, f_m\}$ is an iterated function system.*

Definition 3.3.27 *Given an IFS $\mathcal{F}$, we can define*

$$\Phi : \mathcal{K} \to \mathcal{K}$$
$$K \mapsto \Phi(K) = \bigcup_{i=1}^{m} f_i(K)$$

This is well defined since a finite union of compact sets is still compact.

Theorem 3.3.28 *Let an IFS $\mathcal{F} = \{f_1, \ldots, f_m\}$ with Lipschitz constants $\{C_1, \ldots C_m\}$ and let Φ defined as above.*
Then Φ is a contraction with contraction constant $C = C_\Phi \leq \max_{i=1}^{m}\{C_i\} < 1$.

Proof We use that that

$$d_h(A \cup B, C \cup D) \leq \max\{d_h(A,C), d_h(B,D)\} =: \varepsilon.$$

for any given $A, B, C, D \in \mathcal{K}$, which we will now prove.
Since $d_h(A,C) \leq \varepsilon$ and $d_h(B,D) \leq \varepsilon$, we have that $A \subset C + \varepsilon$ and $B \subset D + \varepsilon$. Therefore

$$A \cup B \subset (C + \varepsilon) \cup (D + \varepsilon) \subset (C + D) + \varepsilon.$$

Where in the last inclusion we used lemma (3.3.19). Exchanging A, B with C, D we get an inverse inequality and the claim follows by equation (3.3). Now let $H, K \in \mathcal{K}$. Applying previous fact $2m$ times we have that

$$d_h(\Phi(H), \Phi(K)) = d_h\left(\bigcup_{i=1}^{m} f_i(H), \bigcup_{i=1}^{m} f_i(K)\right) \leq \max_{i=1}^{m}\{d_h(f_i(H), f_i(K))\}.$$

Now we use lemma 3.3.24 and conclude the proof. □

Definition 3.3.29 *Given an IFS $\mathcal{F}$ and Φ defined as above. Since the previous theorem has proven that Φ is a contraction, we can apply Banach fixed point theorem on Φ. Let $\overline{K}$ be the unique fixed point of Φ. We say that $\overline{K}$ is the* ***attractor*** *of Φ.*

Remark 3.3.30 *By the application of Banach fixed point theorem and its proof, we have that $\overline{K} = \lim_{n\to\infty} \Phi^n(A)$, where $A \in \mathcal{K}$ is any starting set and $\Phi^n(A) = \Phi \circ \cdots \circ \Phi(A)$.*

Lemma 3.3.31 *let x_i be a fixed point of f_i. Then $x_i \in \overline{K}$.*

Proof Simply ba calculating $\Phi(\{x_i\}) = \bigcup_{j=1}^{m} f_j(x_i) = \{x_i\} \cup \bigcup_{j=1, j\neq i}^{m} f_j(x_i)$ we see that $x_i \in \Phi(\{x_i\})$. By repetition of this calculation ad the independence of the convergence starting point the the proof of Banach fixed point theorem, the statement follows. □

The beauty of working with IFSs is the fact that we can easily compute the box-counting dimension of their attractors, if the IFS satisfies a couple of additional conditions.

Theorem 3.3.32 *Given an IFS $\mathcal{F}$,$\{C_1, \ldots, C_m\}$ be its Lipschitz constants and Φ as usual. Let $\overline{K}$ be the attractor of Φ and let $\tilde{\mathbf{s}} = dim_b(\overline{K})$. If $\mathcal{F}$ satisfies:*

(i) *$f_i(x) = C_i x + t_i$ for some $t_i \in \mathbb{R}^n$.*

(ii) *$\exists \tilde{Q}$ cube such that $\Phi(\tilde{Q}) \subset \tilde{Q}$.*

(iii) *$f_i(\tilde{Q}) \cap f_i(\tilde{Q}) = \varnothing$ for $j \neq i$*

Then we have that $\tilde{\mathbf{s}}$ ***is the unique solution of the equation***

$$\sum_{i=1}^{m} C_i^{\tilde{s}} = 1. \tag{3.4}$$

Remark 3.3.33 *Conditions **(i),(ii),(iii)** are satisfied by the "classical" fractal structures, since they are defined inductively by a contraction (in size) and translation, which is the requirement **(i)**. Requirement **(ii)** and **(iii)** ensure that we don't overlap too much when applying* Φ *(which could bring to a lowering of the box counting dimension when repeated infinitely many times).*

The next key theorem is what brings this construction together and allows us to calculate the Hausdorff dimension of an IFS

Theorem 3.3.34 *Let* $\mathcal{F}$ *and IFS as in the previous theorem. Let* $\overline{K}$ *be its attractor. Then*

$$dim_{\mathcal{H}}\left(\overline{K}\right) \geq dim_b(\overline{K})$$

Remark 3.3.35 *Considering that theorem (3.2.10) was valid for every subset of* $\mathbb{R}^n$, *we get that*

$$dim_{\mathcal{H}}\left(\overline{K}\right) = dim_b(\overline{K})$$

and they both are the unique solution fo equation (3.4).

In the case that a set A is composed by n copies of side length λ of itself, some authors estimate the Hausdorff dimension of A by the **heuristic** $s_{eu} = -\frac{\ln(n)}{\ln(\lambda)}$. The reasoning behind this definition is the scaling property of the Hausdorff measure (theorem (2.2.9)):

$$\mathcal{H}^s(A) = \mathcal{H}^s\left(\dot{\bigcup_{i=1}^{n}}\lambda A\right) = \sum_{i=1}^{n}\mathcal{H}^s(\lambda A) = n\lambda^s\mathcal{H}^s(A).$$

By simplifying $\mathcal{H}^s(A)$ (which we can do since s is supposed to be the Hausdorff dimension of A) we get

$$\frac{1}{n} = \lambda^s \Rightarrow s = -\frac{\ln(n)}{\ln(\lambda)}.$$

There are two problems with this calculation. The first is that $s = dim_{\mathcal{H}}(A) \not\Rightarrow \mathcal{H}^s(A) \neq 0$ or $\mathcal{H}^s(A) \neq +\infty$. Therefore the simplification cannot always occur. The second is the representation $A = \dot{\bigcup}_{i=1}^{n}\lambda A$, which is not rigorous (it is the union of the "same" set n times).
Nevertheless, we have that s_{eu} and dim_b(and therefore also $\dim_{\mathcal{H}}$) agree once we consider a better representation a A given by the IFS:
"A is composed by n copies of size λ of itself" means that A is the fixed point of an IFS of the form $\mathcal{F} = \{f_1, \ldots, f_n\}$ with Lipschitz constants $\{C_1, \ldots C_m\}$ all equal to λ. By using equation (3.4) we get:

$$1 = \sum_{i=1}^{n} C_i^{\tilde{s}} = \sum_{i=1}^{n}\lambda^{\tilde{s}} = n\lambda^{\tilde{s}} \Rightarrow \tilde{s} = -\frac{\ln(n)}{\ln(\lambda)} = s_{eu}.$$

Therefore the heuristic is justified in the case of an IFS.

3.3.4 Calculating Hausdorff dimension of some IFS

3.3.4.1 Koch Curve

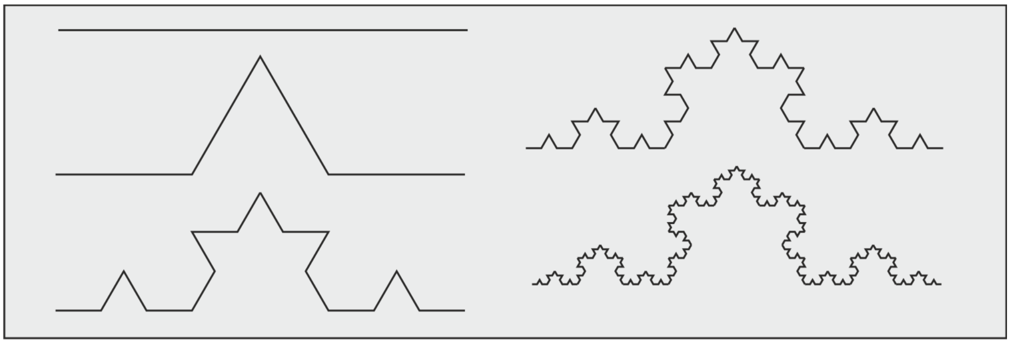

Figure 3.3: Koch Curve [15]

The Koch curve K is a subset of $\mathbb{R}^2$ which is defined by taking the interval $[0,1] \times \{0\}$, dividing it in 3 parts and replacing the middle part by two segments of length $\frac{1}{3}$, the first one with an angle of $\frac{2\pi}{3}$ with respect to the horizontal line and the with an angle of $\frac{2\pi}{3}$ with respect to the previous segment. Repeating this process on the four segments left yields to 16 segments, repeating an infinite amount of times gives us the Koch Curve.
We can therefore define K as the fixed point of an IFS. The right choice of functions can be seen by describing the process in the first step, with f_i describing the transformation from the beginning segment (L) to the i-th segment, which we will call L_i.

$L_1 = [0, \frac{1}{3}] \times \{0\}$ is just the rescaling of L, therefore $f_1(x) = \frac{1}{3}x$

$L_4 = [\frac{2}{3}, 1] \times \{0\}$ adds a translation to the process, hence $f_4(x) = \frac{1}{3}x + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$

L_2 adds a rotation, resulting in $f_2(x) = \frac{1}{3}\begin{pmatrix} \cos\left(\frac{2\pi}{3}\right) & \sin\left(\frac{2\pi}{3}\right) \\ -\sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{pmatrix} x + \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}$

similarly for L_3: $f_3(x) = \frac{1}{3}\begin{pmatrix} \cos\left(\frac{-2\pi}{3}\right) & \sin\left(\frac{-2\pi}{3}\right) \\ -\sin\left(\frac{-2\pi}{3}\right) & \cos\left(\frac{-2\pi}{3}\right) \end{pmatrix} x + \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{6} \end{pmatrix}$

All 4 of those functions are contractions with Lipschitz constant $\frac{1}{3}$ (rotations and translations are isometries, the only factor is always the $\frac{1}{3}$ before the s). By using equation (3.4), we have

$$\frac{1}{3}^{\tilde{s}} + \frac{1}{3}^{\tilde{s}} + \frac{1}{3}^{\tilde{s}} + \frac{1}{3}^{\tilde{s}} = 1 \Rightarrow \tilde{s} = dim_{\mathcal{H}}(K) = \frac{\ln(4)}{\ln(3)} \approx 1.2619 < 2.$$

3.3.4.2 Vicsek Fractal

Figure 3.4: Vicsek Fractal [16]

The Vicsek fractal V is once again a subset of $\mathbb{R}^2$. It is defined by taking the square $[0,1]^2$ and dividing it in 9 squares. We then keep the only the 5 of them accordingly to the scheme given in the picture above. by repeating this process an infinite amount of time, we get V; which is composed of $n = 5$ copies of itself with side length $\lambda = \frac{1}{3}$ with respect to the whole V.
In the last paragraph of last section we have shown that the the heuristic $s_{eu} = -\frac{\ln(n)}{\ln(\lambda)} = dim_{\mathcal{H}}(V)$ is valid. It follows that

$$dim_{\mathcal{H}}(V) = -\frac{\ln(5)}{\ln\left(\frac{1}{3}\right)} = \frac{\ln(5)}{\ln(3)} \approx 1.4649 < 2.$$

3.3.4.3 Menger Sponge

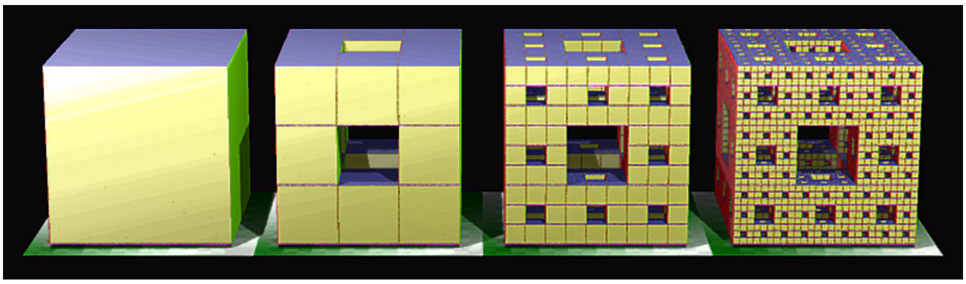

Figure 3.5: Menger Sponge [17]

Finally, we treat a set $S \subset \mathbb{R}^3$ called Menger sponge. We start by taking the set cube$[0,1]^3 \subset \mathbb{R}^3$, dividing it into 27 equal cubes of side length $\frac{1}{3}$ and taking 20 of them in the way illustrated by the picture.(We remove the central cube from each "side" and the cube in the middle). Repeat on the remaining 20 cubes the same procedure infinitely many times and get the set S. Therefore S is the union of $n = 20$ copies of itself of side length $\lambda = \frac{1}{3}$. By the reasoning presented in the previous example, we can conclude that

$$dim_{\mathcal{H}}(S) = \frac{\ln(20)}{\ln(3)} \approx 2.7268 < 3.$$

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