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Some pathological sets in the standard theory of Lebesgue measure

BACHELOR THESIS

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Abstract

The goal of this thesis is to describe some pathological sets arising in the standard theory of Lebesgue measure. We first present the Triadic Cantor Set from a topological and a measure point of view. Then we discuss the more general Cantor sets and we introduce the Cantor Dust. We also give a brief overview of Vitali Set and of an example of a measurable non-Borel set. In the last part we conclude talking about the Hausdorff paradox and the Banach-Tarski paradox.

Contents

1 Introduction

1.1 Roadmap

The layout of the thesis is the following. In Section 2, we give a brief list of all the preliminary material necessary for the comprehension of the thesis, starting from the definition of a measure. Section 3 presents The Cantor Triadic Set from several different points of view, it gives an overview of the more general Cantor sets and discusses the Cantor Dust. In Section 4, we state the Vitali set using Zermelo's Axiom. In particular we construct an example of a Lebesgue-measurable non-Borel set. Finally, Section 5 introduces the Hausdorff and Banach-Tarski paradoxes giving a sketch of the proof of these statements.

1.2 Acknowledgements

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2 Measures

2.1 Definitions and elementary properties

In this thesis we use the same terminology as in the lectures notes of Da Lio [1].

Definition 2.1.1. Let X be a non-empty set and let $\mathcal{P}(X)$ represent all the subsets of X. A mapping $\mu : \mathcal{P}(X) \to [0, \infty]$ is called a *measure* if

1. $\mu(\emptyset) = 0$ and
2. μ is σ-subadditive, i.e. if $A \subseteq \bigcup_{k=1}^{\infty} A_k$ then $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$.

Most books call such a mapping μ an *outer measure*. Moreover we say μ to be *finite* if $\mu(X) < \infty$.

Definition 2.1.2 (Carathéodory criterion of measurability)**.** A set $A \subseteq X$ is called *μ-measurable* if for all $B \subseteq X$ we have

$$\mu(B) = \mu(B \cap A) + (B \setminus A).$$

Definition 2.1.3. Let $\mu : \mathcal{P}(X) \to [0, \infty]$ be a measure. Then the set of all μ-measurable sets is a σ-algebra, i.e $\Sigma = \{A \subseteq X : A \text{ is } \mu\text{-measurable } \}$ is a σ-algebra.

We denote the triple (X, Σ, μ) a *measure space.*

Proposition 2.1.4. *Let (X, Σ, μ) be a measure space and let $A_k \in \Sigma, k \in \mathbb{N}$. Then the following conditions hold:*

1. *μ is σ-additive, i.e. for any sequence of mutually disjoint sets we have*
$$\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sum_{k=1}^{\infty} \mu(A_k).$$
2. *For any increasing sequence of sets $A_k \subseteq A_{k+1}$ we have*
$$\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \lim_{k\to\infty} \mu(A_k).$$
3. *For any decreasing sequence of sets $A_{k+1} \subseteq A_k$ such that $\mu(A_1) < \infty$ we have*
$$\mu\Big(\bigcap_{k=1}^{\infty} A_k\Big) = \lim_{k\to\infty} \mu(A_k).$$

Proposition 2.1.5. *The Lebesgue measure is invariant under isometries.*

Definition 2.1.6. A measure μ on $\mathbb{R}^n$ is called *Borel* if every Borel set is μ-measurable. Moreover μ is called *Borel regular* if for every $A \subseteq \mathcal{R}^n$ there exists a Borel set $B \supseteq A$ such that $\mu(A) = \mu(B)$.

Remark 2.1.7. Let (X, d) be a metric space. Then the σ-algebra generated by all open/closed sets of X is called the *Borel σ-algebra of X* and is denoted by $\mathcal{B}(X)$. The elements of $\mathcal{B}(X)$ are called *Borel sets*.

Definition 2.1.8. The *Lebesgue Measure $\mathcal{L}^n$* is the Carathéodory-Hahn extension of the volume defined on the algebra of elementary sets.

Claim 2.1.9. *$\mathcal{L}^n$ is Borel regular.*

Definition 2.1.10. A bounded set $A \subseteq \mathbb{R}^n$ is *Jordan measurable* if $\underline{\mu}(A) = \overline{\mu}(A)$, where we define the Jordan inner measure and the Jordan outer measure as follows

$$\underline{\mu}(A) = \sup \{\text{vol}(E) \mid E \subseteq A, E \text{ elementary set}\}$$

and

$$\overline{\mu}(A) = \inf \{\text{vol}(E) \mid E \supseteq A, E \text{ elementary set}\}.$$

3 The Cantor Triadic Set

In this Section we closely follow Hatcher [2] and the notes of Franz [3] for the topological part and the article of Schiavone [4] for the description of the Cantor set from a measure point of view. For the general Cantor set we refer to Struwe [5] and Bramanti [6]. In addition we use the article of Grady [7] to give more details about the fact that the Cantor function being uniformly continuous but not absolutely continuous.

There are two constructions of the Cantor Triadic Set. The first involves the compact set and the second involves the ternary extension.

3.1 Cantor set from a topological point of view

We first introduce briefly the most important definitions of this Section.

Definition 3.1.1. A space X is *compact* if every open cover of X admits a finite subcover.

Proposition 3.1.2. *A closed subset of a compact space is compact in the subspace topology.*

Proof. Let X be a compact space and Y be a closed subset of X. Let $\{O_i\}$ be an open cover of Y in X. We can obtain an open cover of X adding at the given cover the open set $X \setminus Y$. Since X is compact X has a finite subcover. The sets O_i in this finite subcover give then a finite cover for Y. Therefore we conclude that Y is compact. □

Definition 3.1.3. A space X is *totally disconnected* if the connected components of X consist of single elements. This means that for every connected subspace $A \subseteq X$ we have $A = \{x\}$ or $A = \emptyset$.

Proposition 3.1.4. *A set $X \subseteq \mathbb{R}$ is totally disconnected if and only if it does not contain any non-empty open interval.*

Proof. Let X be a set. Suppose X contains a non-empty open interval. This contradicts the definition of totally disconnected set, therefore X cannot be totally disconnected.
For the other direction we suppose that X is not totally disconnected. This means there exists a connected subset Y of X that contains more than one point. We know that in $\mathbb{R}$ a set is connected if and only if it is an interval. Therefore Y is an interval with at least two points. Then $int(Y)$, that is an open interval with at least two points, proves the statement. □

Definition 3.1.5. A point $x \in X$ is called an *isolated point* of a set X if there exists an open neighbourhood U of x such that $U \cap (X \setminus \{x\}) = \emptyset$.

Definition 3.1.6. A closed set in $\mathbb{R}$ with no isolated points is called *perfect.*

We are now ready to proceed with the construction of the Cantor set. Let $C_0 = [0, 1]$. The Cantor Triadic Set is a subspace of $[0, 1]$ and is obtained by repeatedly removing open intervals in X. The first step is to divide C_0 in three equal intervals and eliminating the middle open interval. Therefore we obtain

$$C_1 := [0,1] \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Then we proceed in the same way applying this process to all the new intervals (we started with one interval and now we have already obtained two), that means we take the left interval, we divide it in three new intervals and we remove the central open interval. The same process is applied to the right interval. So we obtain the following:

$$\begin{aligned} C_2 &:= [0,1] \setminus \left(\left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)\right) \\ &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]. \end{aligned}$$

Then we repeat inductively this procedure. In general we obtain that C_n is composed by the disjoint union of 2^n closed intervals of length 3^{-n}. We now construct C_{n+1} from C_n. Given an interval I_n in C_n this has the following form:

$$I_n = \left[x_n, x_n + \frac{1}{3^n}\right] \subseteq C_n.$$

Then C_{n+1} is composed by the intervals

$$\left[x_n, x_n + \frac{1}{3^{n+1}}\right] \quad \text{and} \quad \left[x_n + \frac{2}{3^{n+1}}, x_n + \frac{1}{3^n}\right].$$

We obtain a sequence of compact decreasing sets C_n, i.e. $C_{n+1} \subset C_n$.
The Cantor Triadic Set is defined as the intersection of all these sets

$$C := \bigcap_{n=0}^{\infty} C_n.$$

Claim 3.1.7. *C is non-empty.*

Proof. It is easy to see that the points 0 and 1 are kept in each step of the procedure, since we remove the open middle third, so the end points of the interval are never touched, and therefore they are contained in each C_n for all n. Since points $0, 1$ are in C_n for all n we obtain $0, 1 \in C$. Therefore is C non-empty. □

Claim 3.1.8. *C is compact.*

Proof. C is closed in $[0, 1]$ being the intersection of closed sets C_n (C_n is closed being the finite union of closed intervals). Moreover $[0, 1]$ is compact. These two conditions imply that C is compact, because it is a closed subset of a compact space. □

Claim 3.1.9. *C is totally disconnected.*

Proof. We assume by contradiction that there exists a non-empty open interval I with positive length, defined as follows $I := (x, y) \subseteq C$ with $x < y$. Then by definition this interval is contained in every C_n, $\forall n \in \mathbb{N}$. But the connected components of C_n are 2^n intervals of length 3^{-n}. So I must be contained in one of these intervals. This implies that $|I| = y - x \leq 3^{-n}$ and this must hold for all n. This leads to a contradiction because we can choose n large enough so that $1/3^n$ is less than $|I|$, since the length of I is fixed. □

Claim 3.1.10. *C has no isolated points.*

Proof. Let a be an element in C and let $\epsilon > 0$. Then the neighbourhoods of a have the form $(a - \epsilon, a + \epsilon)$. We take m s.t $\epsilon > \frac{2}{3^m}$, then $a \in C_m$. We recall that C_m is the union of closed intervals of the form $\left[x_m, x_m + \frac{1}{3^m}\right]$, which have length 3^{-m}. Therefore a is contained in one of these intervals. Since also the endpoints of this interval are in C we have that either $x_m \neq a$ or $x_m + 3^{-m} \neq a$. So we can take y that satisfies the following condition:

$$y = \begin{cases} x_m, & \text{if } \; x_m \neq a, \\ x_m + 3^{-m}, & \text{if } \; x_m + 3^{-m} \neq a. \end{cases}$$

Then $y \in (C \setminus \{x\}) \cap (a - \epsilon, a + \epsilon)$ and therefore C has no isolated points. □

Claim 3.1.11. *C is a perfect set.*

Proof. C is a closed set in $\mathbb{R}$ with no isolated points, hence C is perfect. □

3.2 Ternary expansion

Another construction of the Cantor set is based on the ternary expansion in the place of the daily used decimal expansion. We recall briefly how this ternary expansion is described. A number x is defined as follows:

$$x = \sum_{n=-\infty}^{N} d_n \cdot 3^n = d_N \cdot 3^N + \cdots + d_1 \cdot 3 + d_0 + \frac{d_{-1}}{3} + \cdots$$

with $d_n \in \{0, 1, 2\}$.

Example 3.2.1. The number 127.4 expressed as usual in decimal expansion becomes in ternary expansion the number $11201.\overline{1012}_3$. In fact we have:

$$\begin{aligned} 127.4 &= 1 \cdot 10^2 + 2 \cdot 10 + 7 + \frac{4}{10} \\ &= 1 \cdot 3^4 + 1 \cdot 3^3 + 2 \cdot 3^2 + 0 \cdot 3^1 + 1 \cdot 3^0 + \frac{1}{3} + \frac{0}{3^2} + \frac{1}{3^3} + \frac{2}{3^4} + \cdots \\ &= 11201.\overline{1012}_3. \end{aligned}$$

We consider now only the interval $[0, 1]$. The points contained in the Cantor Set have a special characterisation in terms of ternary expansion. To discover it we make the following considerations above the construction of intervals in $[0, 1]$. The middle-third open interval $(\frac{1}{3}, \frac{2}{3})$ consists of base 3 decimals $0.d_1d_2d_3d_4\cdots$ which have $d_1 = 1$. Thus C_1, as defined in the previous Section, consists of base 3 decimals $0.d_1d_2d_3d_4\cdots$ which have $d_1 \neq 1$. Similarly we construct C_2 with elements of the form $0.d_1d_2d_3d_4\cdots$ which have $d_1 \neq 1$ and $d_2 \neq 1$. By proceeding in this way we can construct all C_n's. More generally we represent C as the set of elements that have no digit 1 in their ternary expansion. We have

$$C = \{x \in [0,1] \mid d_i(x) \in \{0,2\} \quad \forall i\}.$$

We may ask why this construction is valid since we know that $\frac{1}{3}$ can be written with $d_1 = 1$, i.e. $\frac{1}{3} = 0.1$ and we also know that $\frac{1}{3} \in C$. How is this possible? We have to pay attention to the uniqueness of the ternary expansion. There is only one way for two different ternary expansions to represent the same number. Let's see which are these two distinct representations. Let $x = 0.d_1d_2d_3d_4\cdots_3$ then we claim that the following two ternary expansions are equal

$$\frac{d_1}{3} + \frac{d_2}{3^2} + \cdots \frac{d_n}{3^n} + 0 + 0 + \cdots$$

and

$$\frac{d_1}{3} + \frac{d_2}{3^2} + \cdots \frac{d_n - 1}{3^n} + \frac{2}{3^{n+1}} + \frac{2}{3^{n+2}} + \cdots .$$

There is not other way to express the same number in another form. This solves our problem: the endpoints of the intervals that compose C_n for any n have always two different expansions. Proceeding with the example above we can also write $\frac{1}{3} = 0.0\bar{2}$, and this procedure can be made for all endpoints of the intervals that compose $C \setminus \{0,1\}$. This means we have removed only the numbers that admit an unique ternary expansion with digit 1.

3.3 An important homeomorphism

At this point we have seen both the two possible characterisations of the Cantor Triadic Set. We can now see and prove the following proposition.

Proposition 3.3.1. *The Cantor Triadic set is homeomorphic to* $\{0,2\}^{\mathbb{N}}$.

Remark 3.3.2. The topology on $\{0,2\}$ is the discrete topology.

Notation. We denote with $\prod_{i=0}^{\infty} X_i$ the product of infinite sequences of spaces $X_1, X_2, X_3 \ldots$.

Definition 3.3.3. The topology defined on $\prod_{i=0}^{\infty} X_i$ is the so called *product topology*. A basis for this topology consists of product of open sets $U_i \subset X_i$ for which $U_i = X_i$ for all except finitely many values of i.

Claim 3.3.4. $\mathcal{B}$ *is a basis for the product topology of* $\{0,2\}^{\mathbb{N}}$, *where* $\mathcal{B}$ *is defined as follows*

$$\mathcal{B} = \{\{a_1\} \times \cdots \times \{a_k\} \times \{0,2\} \times \{0,2\} \times \cdots \mid a_1, \ldots, a_k \in \{0,2\}\}.$$

Proof. We have to check two properties:

1. Every point $x \in \{0,2\}^{\mathbb{N}}$ lies in some set $B \in \mathcal{B}$.
2. For each pair of sets $B_1, B_2 \in \mathcal{B}$ and for each point $x \in B_1 \cap B_2$ there exists a set $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$.

Given a point $x \in \{0,2\}^{\mathbb{N}}$ this has the form $x = x_1 x_2 \cdots x_k x_{k+1} \cdots$. Hence $x \in B = \{x_1\} \times \cdots \times \{x_k\} \times \{0,2\} \times \{0,2\} \times \cdots$ that is clearly an element of $\mathcal{B}$.
Also the second point is straightforward since given an element $x \in B_1 \cap B_2$, then there exists a base element B_3 of $\mathcal{B}$ that has the form $\{x_1\} \times \cdots \times \{x_k\} \times \{0,2\} \times \{0,2\} \times \cdots$. Hence $B_3 \subset B_1 \cap B_2$, because otherwise x cannot belong to the intersection of B_1 with B_2. $\square$

We have already seen that the Cantor set C can be viewed as the product $\prod_{i=0}^{\infty} X_i$, where $X_i = \{0,2\}$ for all $i \in \mathbb{N}$. This is possible since we identify a ternary expansion of the form $0, a_1 a_2 a_3 \dots$ with the sequence $(a_1, a_2, a_3, \cdots)$.

Proof of Proposition 3.3.1. We first define the function that we want to prove is an homeomorphism:

$$f : C \to \{0,2\}^{\mathbb{N}}, \quad x = 0.a_1 a_2 a_3 \cdots \longmapsto (a_1, a_2, a_3, \cdots).$$

The function f is a bijection, since we have already seen that the ternary expansion of an element in the Cantor set is unique. We have to prove that f is an homeomorphism, which means to prove that f is continuous and open. We consider an open set in C, i.e an element in the above defined basis.

$$U = \{a_1\} \times \cdots \times \{a_k\} \times \{0,2\} \times \{0,2\} \times \cdots \in \mathcal{B}.$$

Then we have $f^{-1}(U) = \{x \in C : x = 0.a_1 \cdots a_k d_1 d_2 d_3 \cdots | d_i \in \{0,2\} \ \forall i \in \mathbb{N}\}$. We have to prove that this set is open in C. We know that the set $I_k = [0.a_1 \cdots a_k, 0.a_1 \cdots a_k 222 \cdots]$ is an interval in C_k. Then we have that $I_k \cap C$ is open in C. Moreover $I_k \cap C = f^{-1}(U)$. Hence $f^{-1}(U)$ is open and f is continuous. To prove that f is open we take an open set in C. Open sets in C are the intersections of open intervals in $\mathbb{R}$ with C. Then the image of this open set under f is exactly an element of the basis, so an open set in $\{0,2\}^{\mathbb{N}}$. Hence f is open and we have proved that f is an homeomorphism. $\square$

3.4 An uncountable $\mathcal{L}^n$-null set

In addition in this Section we follow the results of Sheet 4 [8].

Proposition 3.4.1. *The Cantor Triadic Set has null-measure, i.e* $\mathcal{L}^1(C) = 0$.

Proof. By the construction of the Cantor set we have that $C := \bigcap_{n=0}^{\infty} C_n$, where we have defined C_n as a Borel subset of $[0, 1]$ which is formed by the disjoint union of 2^n intervals of length 3^{-n}. Thus by additivity we have

$$\mathcal{L}^1(C_n) = 2^n \cdot 3^{-n} = \left(\frac{2}{3}\right)^n.$$

Since we have a sequence of decreasing subsets $C_{n+1} \subset C_n$ that satisfy the property $\mathcal{L}^1(C_1) < +\infty$ we then have

$$\mathcal{L}^1(C) = \mathcal{L}^1\Big(\bigcap_{n=0}^{\infty} C_n\Big) = \lim_{n\to\infty} \mathcal{L}^1(C_n) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0.$$

□

Proposition 3.4.2. *The Cantor Triadic Set is uncountable.*

To prove this Proposition we need first to define the Cantor-Lebesgue function.

Definition 3.4.3. We define the *Cantor-Lebesgue function* as follows

$$F : C \to [0, 1], \quad x = \sum_{i=1}^{\infty} \frac{d_i}{3^i} \longmapsto \sum_{i=1}^{\infty} \frac{d_i}{2} \frac{1}{2^i}.$$

Claim 3.4.4. *The Cantor-Lebesgue function satisfies $F(0) = 0$ and $F(1) = 1$.*

Proof. The ternary expansion of 0 is the following: $0 = \sum_{i=1}^{\infty} 0 \cdot \frac{1}{3^i}$, therefore by definition of F we obtain $F(0) = \sum_{i=1}^{\infty} 0 \cdot \frac{1}{2^{i+1}} = 0$. For 1 we have the expansion $1 = 0.\bar{2}$, that is $1 = \sum_{i=1}^{\infty} \frac{2}{3^i}$, then we have

$$F(1) = \sum_{i=1}^{\infty} 2 \cdot \frac{1}{2^{i+1}} = \sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} \cdot \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} = 1.$$

□

Claim 3.4.5. *The Cantor-Lebesgue function is well defined.*

Proof. We have seen above that in general ternary expansion is not unique. However, in the construction of the Cantor set this expansion is restricted only to coefficients $\{0, 2\}$. Therefore the expansion is unique, which proves that F is well defined on C. □

Claim 3.4.6. *The Cantor-Lebesgue function is continuous.*

Proof. Let $\epsilon > 0$. We take an $x \in C$ and a sequence $\{x_n\}_{n=0}^{\infty}$ in C that converges to x. This is possible since C has no isolated points, that means that every point in C is a limit point of C. The idea is to make δ small enough so that the ternary expansions of x_n and of x agree sufficiently far. We choose $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \epsilon$, and let $\delta = \frac{1}{3^M}$. By definition of converging sequence we have that $\exists M > N$ such that $|x_n - x| < \frac{1}{3^M}$ for all $n > M$. Therefore x and x_n must be in the same interval of C_n for all $n > M$. This means that $d_i(x) = d_i(x_n)$ for any $i \leq M$. Therefore, using the triangle inequality and geometric series we obtain:

$$\begin{aligned}
\big|F(x_n) - F(x)\big| &= \left|\sum_{i=1}^{\infty} \frac{d_i(x_n)}{2}\frac{1}{2^i} - \sum_{i=1}^{\infty} \frac{d_i(x)}{2}\frac{1}{2^i}\right| \\
&= \left|\sum_{i=1}^{M} \frac{d_i(x_n) - d_i(x)}{2^{i+1}} + \sum_{i=M+1}^{\infty} \frac{d_i(x_n) - d_i(x)}{2^{i+1}}\right| \\
&= \left|\sum_{i=M+1}^{\infty} \frac{d_i(x_n) - d_i(x)}{2^{i+1}}\right| \leq \sum_{i=M+1}^{\infty} \frac{2}{2^{i+1}} = \sum_{i=M+1}^{\infty} \frac{1}{2^i} \\
&= \frac{1}{2^{M+1}} \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{2^M} < \frac{1}{2^N} < \epsilon.
\end{aligned}$$

Hence F is continuous. □

Claim 3.4.7. *The Cantor-Lebesgue function is surjective.*

Proof. Take an $y \in [0,1]$. The binary expansion of y is $y = \sum_{k=1}^{\infty} b_k \cdot \frac{1}{2^k}$, with $b_k \in \{0,1\}$. Then we define $c_k := 2b_k$ for all $k \geq 1$. In this case $x = \sum_{k=1}^{\infty} c_k \cdot \frac{1}{3^k}$ is by definition an element of C, because $c_k \in \{0,2\}$. Therefore it holds

$$F(x) = F\left(\sum_{k=1}^{\infty} \frac{c_k}{3^k}\right) = \sum_{k=1}^{\infty} \frac{c_k}{2^{k+1}} = \sum_{k=1}^{\infty} \frac{b_k}{2^k} = y.$$

Hence F is surjective. □

Claim 3.4.8. *The Cantor-Lebesgue function is monotone.*

Proof. Take $x, y \in C$, such that $x < y$. The ternary expansion of x and y must be different at some point t, otherwise $x = y$. In addition at this point the only possibility is that $d_t(x) < d_t(y)$, hence $F(x) < F(y)$. □

Proof of Proposition 3.4.2. We have demonstrated that F is a continuous map that goes from $\mathcal{C}$ onto $[0,1]$ and that F is surjective, but we also know that $[0,1]$ is uncountable, hence $\mathcal{C}$ is uncountable. □

We can also expand the definition of the Cantor-Lebesgue function to the whole interval $[0, 1]$ and not only for the points in C. We construct this function as follows defining $N \in \mathbb{N}$ to be the smallest positive integer such that $d_i(x) = 1$ for $x \notin \mathcal{C}$.

$$F_1 : [0,1] \to [0,1], \quad x = \sum_{i=1}^{\infty} \frac{d_i}{3^i} \longmapsto \begin{cases} \displaystyle\sum_{i=1}^{\infty} \frac{d_i}{2} \frac{1}{2^i} & \text{if } x \in C, \\ \displaystyle\sum_{i=1}^{N-1} \frac{d_i}{2} \frac{1}{2^i} + \frac{1}{2^N} & \text{otherwise.} \end{cases}$$

Claim 3.4.9. *The extended Cantor Ternary function F_1 is well defined.*

Proof. Considering the extended Cantor Ternary function F_1 in place of the function F the proof requires also to check that given different ternary expansions of the same number x then $F_1(x)$ gives the same result for these expansions. Then we need also to examine that all the values that $F_1(x)$ can take are in the interval $[0, 1]$. We have seen above that there is a unique way to represent a number x in different ternary expansions. We define x_1, x_2 to be the two possible representations. We have

$$x_1 = \frac{d_1}{3} + \frac{d_2}{3^2} + \cdots \frac{d_n}{3^n} + 0 + 0 + \cdots$$

and

$$x_2 = \frac{d_1}{3} + \frac{d_2}{3^2} + \cdots \frac{d_n - 1}{3^n} + \frac{2}{3^{n+1}} + \frac{2}{3^{n+2}} + \cdots .$$

Now we have to consider the following three cases.
Case 1: we first contemplate the case where one of the d_k is equal 1 for $k < n$. In this example for both expansions we have

$$F_1(x_1) = F_1(x_2) = \sum_{i=1}^{N-1} \frac{d_n}{2^{i+1}} + \frac{1}{2^N},$$

where as always N is defined to be the smallest positive integer such that $d_N(x) = 1$. Moreover here $N < n$.
Case 2: we now assume that $d_k(x) \neq 1$ for all $k \in \{1, \cdots, n\}$. Moreover we assume that $d_n(x) = 2$. Therefore we have

$$F_1(x_1) = \sum_{i=1}^{\infty} \frac{d_i}{2} \frac{1}{2^i} = \sum_{i=1}^{n} \frac{d_i}{2} \frac{1}{2^i} = \sum_{i=1}^{n-1} \frac{d_i}{2} \frac{1}{2^i} + \frac{2}{2} \frac{1}{2^n} = \sum_{i=1}^{n-1} \frac{d_i}{2} \frac{1}{2^i} + \frac{1}{2^n}.$$

For x_2 we have that $d_n(x) - 1 = 1$, hence $N = n$ and we obtain by definition

$$F_1(x_1) = \sum_{i=1}^{n-1} \frac{d_i}{2} \frac{1}{2^i} + \frac{1}{2^n}.$$

Also in this case we obtain for both representations the same result, as requested.
Case 3: we assume as above that $d_k(x) \neq 1$ for all $k \in \{1, \cdots, n\}$, but this time we let $d_n(x)$ be equal to 1. Hence we obtain

$$F_1(x_1) = \sum_{i=1}^{n-1} \frac{d_i}{2} \frac{1}{2^i} + \frac{1}{2^n}$$

and

$$F_1(x_1) = \sum_{i=1}^{n-1} \frac{d_i}{2} \frac{1}{2^i} + 0 + \sum_{i=n+1}^{\infty} \frac{2}{2} \frac{1}{2^i} = \sum_{i=1}^{n-1} \frac{d_i}{2} \frac{1}{2^i} + \frac{1}{2^n},$$

where for the last equality we have used geometric series properties. Therefore we have obtained in all situations the same result.
We now check that $0 \leq F_1(x) \leq 1$. We first notice that all addends are either positive or 0 therefore $F_1(x) \geq 0$.
Let $F_1(x) = \sum_{i=1}^{\infty} \frac{d_i}{2} \frac{1}{2^i}$. Since $d_i(x) \in \{0, 1, 2\}$, we have that $\frac{d_i(x)}{2} \leq 1$, therefore

$$F_1(x) = \sum_{i=1}^{\infty} \frac{d_i}{2} \frac{1}{2^i} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} = 1.$$

Similarly if $F_1(x) = \sum_{i=1}^{N-1} \frac{d_i}{2} \frac{1}{2^i} + \frac{1}{2^N}$ then

$$F_1(x) = \sum_{i=1}^{N-1} \frac{d_i}{2} \frac{1}{2^i} + \frac{1}{2^N} \leq \sum_{i=1}^{N-1} \frac{1}{2^i} + \frac{1}{2^N} = 1 - \frac{1}{2^{N-1}} + \frac{1}{2^N} = \frac{2^N - 1}{2^N} \leq 1.$$

Hence the extended Cantor Ternary function is well defined. □

Claim 3.4.10. *The extended Cantor-Lebesgue function F_1 is an increasing function.*

Proof. Take $x, y \in C$, such that $x < y$. The ternary expansion of x and y must be different at some point t, otherwise $x = y$. In addition at this point the only possibility is that $d_t(x) < d_t(y)$ and $d_k(x) = d_k(y)\ \forall k \in \{1, \cdots, t\}$. Then we consider the following three cases.
Case 1: if $d_k(x) = d_k(y) = 1$ for $1 \leq k < t$ then we have $k = N$ and

$$F_1(x) = F_1(y) = \sum_{i=1}^{N-1} \frac{d_i}{2} \frac{1}{2^i} + \frac{1}{2^N}.$$

Hence $F_1(y) \geq F_1(x)$.
Case 2: we now assume that $d_k(x) = d_k(y) \neq 1$ for all $k \in \{1, \cdots, t\}$. Moreover we assume that $d_t(y) = 2$. Therefore we have

$$F_1(y) = \sum_{i=1}^{N-1} \frac{d_i}{2} \frac{1}{2^i} + \frac{2}{2} \frac{1}{2^N} + R$$

and

$$F_1(x) = \sum_{i=1}^{N-1} \frac{d_i}{2} \frac{1}{2^i} + \frac{1}{2^N}$$

or

$$F_1(x) = \sum_{i=1}^{N-1} \frac{d_i}{2} \frac{1}{2^i} + 0 + S$$

where S and R are the sum of the remaining digits that are different from 1. Moreover we have $S \leq \sum_{i=N+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^N}$. Consequently also in this case we have $F_1(y) \geq F_1(x)$.
Case 3: we assume as above that $d_k(x) \neq 1$ for all $k \in \{1, \cdots, n\}$, but this time we let $d_n(x)$ be equal 1. Hence we obtain

$$F_1(y) = \sum_{i=1}^{N-1} \frac{d_i}{2} \frac{1}{2^i} + \frac{1}{2^N}$$

and

$$F_1(x) = \sum_{i=1}^{N-1} \frac{d_i}{2} \frac{1}{2^i} + 0 + S.$$

As above we have $S \leq \sum_{i=N+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^N}$. Hence $F_1(y) \geq F_1(x)$. Therefore F_1 is an increasing function. □

Claim 3.4.11. *The extended Cantor-Lebesgue function F_1 is constant on each interval $[0,1] \setminus C$.*

Proof. Take x, y in one of the open middle intervals that we have removed in the construction of the Cantor Triadic Set. Let $N \in \mathbb{N}$ be the smallest positive integer that satisfies $d_N(x) = 1 = d_N(y)$. Then we have that $d_i(x) = d_i(y)\ \forall i < N$, hence

$$F_1(x) = \sum_{i=1}^{N-1} \frac{d_i(x)}{2^{i+1}} + \frac{1}{2^N} = \sum_{i=1}^{N-1} \frac{d_i(y)}{2^{i+1}} + \frac{1}{2^N} = F_1(y).$$

□

Claim 3.4.12. *The extended Cantor-Lebesgue function F_1 is not absolutely continuous.*

Proof. Let $\epsilon = 1/2$. Then we consider the set of intervals C_n defined as in the construction of the Cantor Triadic Set. We explicitly give the first three in order to understand better.

$$
\begin{aligned}
C_1 &= \left\{ \left[0, \frac{1}{3}\right], \quad \left[\frac{2}{3}, 1\right] \right\}, \\
C_2 &= \left\{ \left[0, \frac{1}{9}\right], \quad \left[\frac{2}{9}, \frac{1}{3}\right], \quad \left[\frac{2}{3}, \frac{7}{9}\right], \quad \left[\frac{8}{9}, 1\right] \right\}, \\
C_3 &= \left\{ \left[0, \frac{1}{27}\right], \quad \left[\frac{2}{27}, \frac{1}{9}\right], \quad \left[\frac{2}{9}, \frac{7}{27}\right], \quad \left[\frac{8}{27}, \frac{1}{3}\right], \right. \\
&\qquad \left. \left[\frac{2}{3}, \frac{19}{27}\right], \quad \left[\frac{20}{27}, \frac{7}{9}\right], \quad \left[\frac{8}{9}, \frac{25}{27}\right], \quad \left[\frac{26}{27}, 1\right] \right\}.
\end{aligned}
$$

We have already seen that in order to obtain the next set of closed intervals we remove the open middle third of each closed interval in the previous set. We label the closed intervals $\{[c_i, d_i] \mid 1 \leq i \leq n\}$. Then we have $F_1(d_i) = F_1(c_{i+1})$, since the extended Cantor-Lebesgue function is constant on each interval of $[0,1] \setminus C$. Therefore have that

$$
\begin{aligned}
\sum_{i=1}^{n} (F_1(d_i) - F_1(c_i)) &= (F_1(d_1) - F_1(0)) + (F_1(d_2) - F_1(c_2)) + \cdots \\
&\qquad + (F_1(d_{n-1}) - F_1(c_{n-1})) + (F_1(1) - F_1(c_n)) \\
&= -F_1(0) + (F_1(d_1) - F_1(c_2)) + \cdots \\
&\qquad + (F_1(d_{n-1}) - F_1(c_n)) + F_1(1) \\
&= 0 + 0 + \cdots + 0 + 1 \\
&= 1.
\end{aligned}
$$

We also use that $F_1(0) = 0$ and $F_1(1) = 1$. We have already seen that each C_n contains 2^n closed intervals each one of length $\frac{1}{3^n}$. This implies that the total length of C_n is $\left(\frac{2}{3}\right)^n$ (it is simply the sum over the length of all intervals in C_n), but $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$. This implies that for a large enough n we can make the length of C_n as small as we want. Therefore for any δ there exists $n \in \mathbb{N}$ such that $|C_n| < \delta$. However, we have shown above that the sum of the variance of F_1 over those intervals is always 1 and $1 \not< \epsilon$. Therefore F_1 is not absolutely continuous. $\square$

3.5 The general Cantor set

We are now wondering if there are other Cantor sets or if our constructions (using ternary expansion or open/closed sets) of the Cantor set C are the only ones. This is clearly not true, because one can simply imagine that removing middle quarters instead of removing middle thirds or using any other fraction,

gives us very similar constructions. We could also think that we could vary the fraction from one step to the next one and so on. There are infinitely ways to make small variations to the original construction, we only have to pay attention to the fact that the lengths of the remaining intervals approach zero.
Another important fact is that all these constructions produce Cantor sets that are homeomorphic to the Cantor Triadic Set C and that all the properties we have described in Section 3.1 describe a space that is homeomorphic to the Cantor set C.

We characterise more precisely what we have already said.
As above we begin with the interval $I_1^{(0)} = [0,1]$. We remove from $I_1^{(0)}$ the open middle interval $\Omega_1^{(1)}$ with length λ_1. This creates two new intervals $I_1^{(1)}$ and $I_2^{(1)}$. Then we remove from both the new intervals the open middle interval $\Omega_l^{(2)}$ of length λ_2, where $l \in \{1,2\}$. We proceed with this strategy, which means that at the k-step we remove the open middle interval $\Omega_l^{(k)}$ of length λ_k from the remaining intervals $I_l^{(k-1)}$, where $1 \leq l \leq 2^{k-1}$ and $\lambda_k \leq 3^{-k}$ for each $k \in \mathbb{N}$. Let Ω be the union of all the open middle intervals we have removed from $I_1^{(0)}$, i.e.

$$\Omega := \bigcup_{k=1}^{\infty} \bigcup_{l=1}^{2^{k-1}} \Omega_l^{(k)}.$$

We define the general Cantor set as

$$\mathcal{C} := [0,1] \setminus \Omega.$$

This set is still closed, uncountable and it is nowhere dense, i.e it is totally disconnected. Moreover taking $\lambda_i = \frac{1}{3}$ $\forall i$ we obtain precisely the Cantor Triadic Set.

Claim 3.5.1. *The general Cantor set is not Jordan measurable. Indeed we have that*

$$\underline{\mu}(\mathcal{C}) = 0$$

and that

$$\overline{\mu}(\mathcal{C}) \geq 1 - \sum_{k=1}^{\infty} 2^{k-1}\lambda_k.$$

Proof. To determine the Jordan outer measure it is enough to consider the open set $G \supseteq \mathcal{C}$. We have that $\delta := \operatorname{dist}(\mathcal{C}, [0,1] \setminus G) > 0$. Moreover there is $k_0 \in \mathbb{N}$ such that $2^{-k_0} < \delta$. Therefore we have

$$\bigcup_{l=1}^{2^{k_0}} I_l^{(k_0)} \subseteq G.$$

Hence

$$\mu(G) \geq \mu\Big(\bigcup_{l=1}^{2^{k_0}} I_l^{(k_0)}\Big) = 1 - \sum_{l=1}^{2^{k_0}} 2^{k-1}\lambda_l \geq 1 - \sum_{l=1}^{\infty} 2^{k-1}\lambda_l.$$

Hence $\mathcal{C}$ is not Jordan measurable. □

The set $\Omega = [0,1] \setminus \mathcal{C}$ is not Jordan measurable as well. Since $\mathcal{C}$ is closed Ω is an open set. This means that there are open, not Jordan measurable subsets of $\mathbb{R}$.

Another important result we can demonstrate with the help of the general Cantor set is that the space of the Riemann integrable function is not complete.

We consider now the space of the piecewise continuous function C^0_{pw} with the norm

$$\|f\|_1 = \int_0^1 |f(t)|dt.$$

Let $(f_k)_k = \Big(\sum_{l=1}^{2^k} \chi_{I_l^{(k)}}\Big)_k$ be the sequence of functions obtained in the construction of the general Cantor set. We have $\big\|f_j - f_k\big\|_{L^1} \to 0$ and $f_k \to \chi_{\mathcal{C}}$ for $j, k \to \infty$.

3.6 The Cantor Dust

In this Section we closely follow the lecture notes of Da Lio [1]. In addition in this Section we follow the results of Sheet 5 [9]. We first recall briefly the important definitions of the Hausdorff measure.

3.6.1 Hausdorff measure

We recall that the ball with radius r and centre x is defined as $B(x,r) = \{y \in \mathbb{R}^n \mid |y - x| < r\}$. Take $s \geq 0$, $\delta > 0$ and let A be a non empty subset of $\mathbb{R}^n$. Then we set

$$\mathcal{H}^s_\delta(A) = \inf\Big\{\sum_{k=1}^{\infty} r_k^s \;\mid\; A \subseteq \bigcup_{k=1}^{\infty} B(x_k, r_k),\ 0 < r_k < \delta\Big\}.$$

Moreover we set $0 = \mathcal{H}^s_\delta(\emptyset)$. $\mathcal{H}^s_\delta(A)$ is a measure on $\mathbb{R}^n$. In addition we have that $\mathcal{H}^s_\delta(A)$ is a non-increasing function on δ, i.e

$$\mathcal{H}^s_{\delta_1}(A) \leq \mathcal{H}^s_{\delta_2}(A) \text{ if } \delta_2 \leq \delta_1.$$

Thus there exists the limit

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

Definition 3.6.1. We call $\mathcal{H}^s$ to be the *s-dimensional Hausdorff measure* on $\mathbb{R}^n$.

Moreover we have that for any $s > 0$, $\lambda > 0$ and for any subset A of $\mathbb{R}^n$

$$\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A).$$

Definition 3.6.2. The *Hausdorff dimension* of a subset A of $\mathbb{R}^n$ is

$$\dim_{\mathcal{H}}(A) := \{s \geq 0 \mid \mathcal{H}^s(A) = 0\}.$$

All these definitions are important to state another important property of the Cantor Triadic Set. The Cantor Triadic Set is an example of non-integer Hausdorff dimension.

Claim 3.6.3. *We have*

$$\dim_{\mathcal{H}}(C) = \frac{\ln 2}{\ln 3} =: s$$

and that

$$2^{-(s-1)} \leq \mathcal{H}^s(C) \leq 2^{-s}.$$

Proof. We first prove the inequality $2^{-(s-1)} \leq \mathcal{H}^s(C) \leq 2^{-s}$, then $\dim_{\mathcal{H}}(C) = s$ follows directly from this fact. If these inequalities hold there exists s such that $\mathcal{H}^s(C) \in (0, \infty)$. Then since C is the union of two copies $\frac{C}{3}$ we have

$$\mathcal{H}^s(C) = 2\mathcal{H}^s(\frac{C}{3}) = \frac{2}{3^s}\mathcal{H}^s(C)$$

hence we obtain

$$\frac{2}{3^s} = 1 \iff 3^s = 2 \iff s = \log_3(2) = \frac{\ln 2}{\ln 3}.$$

Therefore it remains only to estimate the s-Hausdorff measure of the Cantor Triadic set. We recall that

$$C = \bigcap_{k=1}^{\infty} \bigcup_{l=1}^{2^k} I_l^{(k)},$$

where $I_l^{(k)}$ are the closed intervals of length 3^{-k} that compose C. We try to find a cover for these intervals. This is possible taking just intervals a bit larger than $I_l^{(k)}$ but with the same midpoint. Let the radius or these new intervals be $r_k = \frac{\lambda}{2} \cdot 3^{-k}$ with $1 < \lambda < 2$. We set $\delta = 3^{-k}$, therefore it holds that $0 < r_k < \delta$ and we find

$$\mathcal{H}^s_\delta(C) \leq \sum_{l=1}^{2^k} r_k^s = \sum_{l=1}^{2^k} (\frac{\lambda}{2} \cdot 3^{-k})^s = 2^k \lambda^s 2^{-s} 3^{-ks} = 2^k \lambda^s 2^{-s} 2^{-k} = 2^{-s}\lambda^s,$$

where we used $3^s = 2$. We now let λ go to 1 and we obtain $\mathcal{H}^s_\delta(C) \leq 2^{-s}$. Letting k go to ∞ we prove the second inequality, i.e. $\mathcal{H}^s(C) \leq 2^{-s}$.
We want to show now the first inequality. We take a covering of C of open balls $\{B(x_k, r_k)\}_{k\in\mathbb{N}}$. But since C is compact we know there is a finite open cover of C. We assume without loss of generality that $B_1 := B(x_1, r_1), \ldots, B_N := B(x_N, r_N)$ form a finite covering of the Cantor set C. For each $j \in \{1, \ldots, N\}$ there is a $k \in \mathbb{N}$ such that

$$3^{k-1} \leq 2r_j \leq 3^{-k}.$$

This implies that each ball B_j intersects at most one interval $I_l^{(k)}$ for index k. Hence B_j intersects at most 2^{m-k} intervals of the form $I_l^{(m)}$ for any $m \geq k$. Therefore we have

$$2^{m-k} = 2^m \cdot 3^{-sk} = 2^m \cdot 3^s \cdot 3^{-s(k+1)} \leq 2^m \cdot 3^s \cdot (2r_j)^s.$$

Since our covering consists only of finitely many balls, there is m large enough such that $3^{-m-1} \leq 2r_j$ for all j. Since any interval is intersected by at least one ball we find out that 2^m intervals are intersected by the balls, so we obtain

$$2^m \leq \sum_{j=1}^{N} 2^m 3^s (2r_j)^s = 2^m 3^s 2^s \sum_{j=1}^{N} r_j^s$$

and hence

$$2^{-s-1} = 2^{-s} 3^{-s} \leq \sum_{j=1}^{N} r_j^s.$$

From the arbitrariness of the covering and by the definition of Hausdorff measure it follows that $\mathcal{H}^s \geq 2^{-s-1}$. □

3.6.2 The Cantor Dust

Let $A_0 = [0,1]^2 \subset \mathbb{R}^2$. We now divide A_0 into 16 squares of side length $\frac{1}{4}$ using a grip parallel to the axis. Then we consider the two straight lines $y = x \pm \frac{1}{2}$. With this construction we are ready to define the following element in the process to create the Cantor dust. A_1 is the union of the squares through which the two lines pass. Then we repeat the same construction for each square in A_1. Therefore $A_2 \subset A_1$ is the union of 4^2 squares of length 4^{-2}. The Cantor dust is defined as the intersection of every A_k

$$A := \bigcap_{k=1}^{\infty} A_k.$$

Claim 3.6.4. *It holds that* $\dim_{\mathcal{H}}(A) = 1$.

Proof. We want to show that $\frac{1}{2} \leq \mathcal{H}^1(A) \leq \frac{\sqrt{2}}{2}$ We have already seen above that each A_k consists of 4^k squares Q_l of length 4^{-k}. We want to find something that covers the Cantor dust A. To this end we can take the 4^k balls of radius $r = 4^{-k}\frac{\sqrt{2}}{2}$ that cover the squares Q_l. These cover A_k for every k, and hence also A. Thus we have $\forall k \geq k_0 : r_{k_0} = 4^{-k_0}\frac{\sqrt{2}}{2} < \delta$

$$\mathcal{H}^1_\delta(A) \leq \mathcal{H}^1_\delta(A_k) \leq \sum_{j=1}^{4^k} r_j^1 = \sum_{j=1}^{4^k} 4^{-k}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}.$$

Taking $\delta \to 0$ we find $\mathcal{H}^1(A) \leq \frac{\sqrt{2}}{2}$.
We now take the projection along the x axis $\pi : \mathbb{R}^2 \to \mathbb{R}, \quad \pi(x, y) = x$. We claim it holds that $\pi(A) = [0, 1]$.
To prove this we use the fact that the sequence A_k is decreasing and that the interval $[0, 1]$ is compact. We have $\pi(A) \subseteq \pi(A_k)$ for all k. Hence we have $\pi(A) \subseteq \bigcap_{k=1}^\infty \pi(A_k) = [0, 1]$. For the reverse direction we need to prove that $\bigcap_{k=1}^\infty \pi(A_k) \subseteq \pi(A)$. Let $x \in \bigcap_{k=1}^\infty \pi(A_k)$, i.e there exists $a_k \in A_k$ such that $\pi(a_k) = x$ for all k. The sequence $(a_k)_k$ converges to a. Since the sequence of A_k is non-increasing it holds $a \in A$. By continuity of the projection π we have that $\pi(a_k)$ converges to $\pi(a) = x$. Hence $x \in \pi(A)$.

Let $A \subseteq \bigcup_{k=1}^\infty B(z_k, r_k)$ with $z_k = (x_k, y_k)$ and $r_k < \delta$ $\forall k \in \mathbb{N}$. We get

$$[0, 1] = \pi(A) \subseteq \bigcup_{k=1}^\infty \pi(B(x_k, r_k)) = \bigcup_{k=1}^\infty (x_k - r_k, x_k + r_k).$$

Thus

$$1 = \mathcal{L}^1([0, 1]) \leq \sum_{k=1}^\infty \mathcal{L}^1((x_k - r_k, x_k + r_k)) = 2\sum_{k=1}^\infty r_k.$$

Hence

$$\sum_{k=1}^\infty r_k \geq \frac{1}{2}.$$

Taking the infimum we find $\mathcal{H}^1_\delta(A) \geq \frac{1}{2}$ and letting $\delta \to 0$ we have $\mathcal{H}^1(A) \geq \frac{1}{2}$ as wanted. □

4 The Vitali set

In this Section we follow the lectures notes of Da Lio [1].

4.1 Zermelo's Axiom

In order to construct the Vitali set we first need to state the Zermelo's Axiom, also called Axiom of Choice.

Theorem 4.1.1 (Zermelo's Axiom)**.** *Let $\mathcal{F}$ be a collection of arbitrary non-empty disjoint sets, i.e $\mathcal{F} = \{E_i |\ i \in I\}$, where I represents an index set. Then there is a set consisting of exactly one element from each E_i, for all $i \in I$.*

The sum modulo 1 of elements x, y in the interval $[0, 1)$ is defined as

$$x \oplus y = \begin{cases} x + y, & \text{if } x + y < 1 \\ x + y - 1, & \text{if } x + y \geq 1. \end{cases}$$

Let $E \subseteq [0, 1)$ be a $\mathcal{L}^1$-measurable set.

Claim 4.1.2. *$E \oplus x \subseteq [0, 1)$ is $\mathcal{L}^1$-measurable.*

Proof. We have $E \oplus x = E_1 \cup E_2$, where we define

$$E_1 := E \cap [0, 1 - x) \oplus x = E \cap [0, 1 - x) + x,$$

$$E_2 := E \cap [1 - x, 1) \oplus x = E \cap [1 - x, 1) + (x - 1).$$

We have that E_1 and E_2 are $\mathcal{L}^1$-measurable, since the intersection of measurable set is still measurable and both $[0, 1 - x)$ and $[1 - x, 1)$ are measurable. Then we have

$$\mathcal{L}^1(E_1) = \mathcal{L}^1(E \cap [0, 1 - x) + x) = \mathcal{L}^1(E \cap [0, 1 - x))$$

$$\mathcal{L}^1(E_2) = \mathcal{L}^1(E \cap [1 - x, 1) + (x - 1)) = \mathcal{L}^1(E \cap [1 - x, 1))$$

since we have seen in Section 2.1 that the Lebesgue measure is invariant under isometries. Moreover it is easy to see that $E_1 \cap E_2 = \emptyset$ and therefore E_1, E_2 are disjoint sets. Hence $E \oplus x$ is $\mathcal{L}^1$-measurable. □

Claim 4.1.3. $\mathcal{L}^1(E \oplus x) = \mathcal{L}^1(E)$.

Proof. By the above construction in the proof of Claim 4.1.2 we have the following

$$\begin{aligned} \mathcal{L}^1(E \oplus x) &= \mathcal{L}^1(E_1 \cup E_2) = \mathcal{L}^1(E_1) + \mathcal{L}^1(E_2) \\ &= \mathcal{L}^1(E \cap [0, 1 - x)) + \mathcal{L}^1(E \cap [1 - x, 1)) \\ &= \mathcal{L}^1(E \cap [0, 1)) = \mathcal{L}^1(E) \end{aligned}$$

□

4.2 The Vitali set

To construct the Vitali set $\mathcal{V}$ we first need to state the following equivalence relation. Set $x, y \in [0,1)$, then we have: $x \sim y$ if $x - y \in \mathbb{Q}$.

Definition 4.2.1. Let $\sim$ be an equivalence relation on a set X. Let $y \in X$. Then we call the set $[y] = \{x \in X \mid x \sim y\}$ an *equivalence class.*

Then Zermelo's Axiom proves that there exists a set $\mathcal{V} \subseteq [0,1)$ that is composed by all the representative points from the equivalence classes given by the above equivalence relation. We call this set the Vitali set. We can apply the Axiom of Choice because clearly the equivalent classes must be disjoint, otherwise we have that they are equal by definition.

Now let $\mathbb{Q} \cap [0,1) = \{q_j\}_{j\in\mathbb{N}}$ be an enumeration of $\mathbb{Q} \cap [0,1)$, where we set $q_0 = 0$. Then we define

$$\mathcal{V}_j = \mathcal{V} \oplus q_j, \quad \forall j \in \mathbb{N}.$$

Claim 4.2.2. $\{\mathcal{V}_j\}_j$ *is a disjoint family.*

Proof. Suppose by contradiction that these sets are not disjoint. Then there is $x \in \mathcal{V}_j \cap \mathcal{V}_i$. Therefore by the above construction of $\mathcal{V}_j$ we have $x = v_j \oplus q_j = v_i \oplus q_i,\ q_i, q_j \in \mathbb{Q}$. This implies $v_j - v_i \in \mathbb{Q}$. But this means $v_j \sim v_i$ and therefore both v_j, v_i are in the same equivalence class. However, by construction of Zermelo's Axiom we take only one representative element for each equivalent class. Therefore $v_j = v_i$ and also $q_j = q_i$. Indeed we have $\mathcal{V}_j \equiv \mathcal{V}_i$. □

Claim 4.2.3. *It holds that*

$$[0,1) = \bigcup_{j=0}^{\infty} \mathcal{V}_j.$$

Proof. Since $\mathcal{V}_j \subseteq [0,1)$ for all j we also have that $\bigcup_{j=0}^{\infty} \mathcal{V}_j \subseteq [0,1)$.
For the other direction we set x to be an element in $[0,1)$ and $[x]$ to be the correspondent equivalence class. Then by construction of the Vitali set there exists a unique $v \in \mathcal{V}$ that is equivalent to x. We have three cases:

1. $v = x$, then $x \in \mathcal{V}_0 = \mathcal{V}$.
2. $v > x$, then $x = v + q_j - 1 = v \oplus q_j$ for some q_j, therefore $x \in \mathcal{V}_j$.
3. $v < x$, then $x = v + q_i = v \oplus q_i$ for some q_i, therefore $x \in \mathcal{V}_i$.

Therefore it follows that $[0,1) \subseteq \bigcup_{j=0}^{\infty} \mathcal{V}_j$. Hence we have proved equality. □

Proposition 4.2.4. *The Vitali set* $\mathcal{V}$ *is not Lebesgue measurable.*

Proof. We assume by contradiction that $\mathcal{V}$ is Lebesgue measurable. Then using the results of Section 4.1 we have that also $\mathcal{V}_j$ is Lebesgue measurable. Moreover we obtain

$$1 = \mathcal{L}^1([0,1)) = \mathcal{L}^1\Big(\bigcup_{j=0}^{\infty} \mathcal{V}_j\Big) = \sum_{i=0}^{\infty} \mathcal{L}^1(\mathcal{V}_j) = \sum_{i=0}^{\infty} \mathcal{L}^1(\mathcal{V} \oplus q_j) = \sum_{i=0}^{\infty} \mathcal{L}^1(\mathcal{V}).$$

However, this leads to a contradiction since we have only the following two possibilities

1. $\mathcal{L}^1(\mathcal{V}) = 0$ but then also the sum is equal zero and this is a contradiction.

2. $\mathcal{L}^1(\mathcal{V}) > 0$ but then the sum is infinity and this is also a contradiction.

Therefore $\mathcal{V}$ is not Lebesgue measurable. □

4.3 Important properties about non measurable sets

In addition in this Section we follow the results of Sheet 4 [8].

The non measurability of $\mathcal{V}$ implies that there exists a set $B \subseteq \mathbb{R}$ such that

$$\mathcal{L}^1(B) < \mathcal{L}^1(B \setminus \mathcal{V}) + \mathcal{L}^1(B \cap \mathcal{V}).$$

Where $B \cap \mathcal{V}$ and $B \setminus \mathcal{V}$ are two disjoint sets for which the additivity of measure does not hold.

Proposition 4.3.1. *Let E be a Lebesgue measurable set and let $E \subset \mathcal{V}$. Then we have $\mathcal{L}^1(E) = 0$.*

Proof. We repeat the same procedure as for the construction of the Vitali set. Let $E_i = E \oplus q_i$. Then by Section 4.1 we have that also E_i is Lebesgue measurable and we have that $\mathcal{L}^1(E_i) = \mathcal{L}^1(E)$. Then we set $\bigcup_{i=1}^{\infty} E_i = F \subset [0,1)$. Then F is Lebesgue measurable since it is the infinite union of Lebesgue measurable sets, and we know that the Lebesgue measurable sets form a σ-algebra. Therefore by monotonicity we have

$$1 = \mathcal{L}^1([0,1)) \geq \mathcal{L}^1(F) = \mathcal{L}^1\Big(\bigcup_{j=0}^{\infty} E_j\Big) = \sum_{i=1}^{\infty} \mathcal{L}^1(E).$$

If $\mathcal{L}^1(E) > 0$ then we obtain $\sum_{i=1}^{\infty} \mathcal{L}^1(E) = \infty$ that is a contradiction. Therefore $\mathcal{L}^1(E) = 0$. □

Proposition 4.3.2. *Let $A \subset \mathbb{R}$ be a Lebesgue measurable set, with $\mathcal{L}^1(A) > 0$. Then there exists a set $B \subset A$ which is not Lebesgue measurable.*

Proof. Without loss of generality we suppose that $A \subseteq [0, 1)$. This is possible since $\mathcal{L}^1(A) > 0$ and therefore there is some n such that $\mathcal{L}^1(A \cap [n, n+1)) > 0$. Let $\bar{A} := (A \cap [n, n+1)) - n$. We know that the Lebesgue measure is invariant under translation, therefore also $\mathcal{L}^1(\bar{A}) > 0$ and $\bar{A} \subseteq [0, 1)$.
Then let $B \subset A$ be the intersection of A with the Vitali set, i.e. $B = A \cap \mathcal{V}$, and set $B_i = A \cap \mathcal{V}_i$. Since the sets $\mathcal{V}_i, \mathcal{V}_j$ for all i, j are disjoint also the sets B_i, B_j for all i, j are disjoint. Now we suppose by contradiction that B_i is Lebesgue measurable. We have $B_i \subseteq \mathcal{V}_i = \mathcal{V} \oplus q_i$ and therefore $B_i - q_i \subset \mathcal{V}$. Since B_i is measurable, so it is $B_i - q_i$. Then by Proposition 4.3.1 we obtain $\mathcal{L}^1(B_i - q_i) = 0$. Hence $\mathcal{L}^1(B_i) = \mathcal{L}^1(B_i - q_i) = 0$. Therefore $\sum_{i=1}^{\infty} \mathcal{L}^1(B_i) = 0$. We know that

$$A = A \cap [0, 1) = A \cap \bigcup_{j=0}^{\infty} \mathcal{V}_j = \bigcup_{j=0}^{\infty} (A \cap \mathcal{V}_1) = \bigcup_{j=0}^{\infty} B_i.$$

But then $\mathcal{L}^1(A) = \sum_{i=1}^{\infty} \mathcal{L}^1(B_i) = 0$ and this is a contradiction. Therefore B_i is not Lebesgue measurable and hence B is not Lebesgue measurable. □

Proposition 4.3.3. *Every countable subset of* $\mathbb{R}$ *is a Borel set and has Lebesgue measure* 0.

Proof. Let $x \in \mathbb{R}$ and $\epsilon > 0$. Then $\{x\} \subseteq [x - \epsilon, x + \epsilon]$. Then

$$\mathcal{L}^1(\{x\}) \leq \mathcal{L}^1([x - \epsilon, x + \epsilon]) = 2\epsilon.$$

Since we can take ϵ arbitrarily small we conclude that $\mathcal{L}^1(\{x\}) = 0$. Now we take a countable set $A \subset \mathbb{R}$. Let $A = \{a_1, a_2, \cdots\} = \bigcup_{n=1}^{\infty} \{a_n\}$. Since singletons in $\mathbb{R}$ are closed subsets we have that $A \in \mathcal{B}$, where $\mathcal{B}$ is the σ-algebra of Borel-subsets which is generated by all the open subsets. This is true since closed subsets, as complement of open subsets, also belong to $\mathcal{B}$. Then

$$\mathcal{L}^1(A) = \mathcal{L}^1\Big(\bigcup_{n=1}^{\infty} \{a_n\}\Big) \leq \sum_{n=1}^{\infty} \mathcal{L}^1(\{a_n\}) = 0.$$

Therefore we have that $\mathcal{L}^1(A) = 0$. □

4.4 A Lebesgue measurable non-Borel set

In addition in this Section we follow the results of Sheet 7 [10].

We proceed trying to construct a measurable set that is non-Borel. To do this we define the following function

$$g : [0, 1] \to [0, 2], \quad x \longmapsto F_1(x) + x.$$

Lemma 4.4.1. *The function g is strictly monotone and is an homeomorphism.*

Proof. The function g is strictly monotone since we have already proved that the extended Cantor-Lebesgue function is a monotone increasing function and since $x \mapsto x$ is strictly increasing. It remains to check that g is a continuous bijection with a continuous inverse. This function is injective since g is monotone. Then, since g is composed by the sum of continuous functions, g is also continuous. To check surjectivity we recall that we have $F_1(0) = 0$ and $F_1(1) = 1$ and therefore $g(0) = 0$ and $g(1) = 2$. Moreover g is continuous and this implies by the intermediate value theorem that g attains values only between $g(0)$ and $g(1)$. Therefore we have proved that g is surjective and hence bijective. Therefore there is an inverse function of g. Let $h := g^{-1}$. We need to check that h is continuous. Suppose $U \subseteq [0, 1]$ is open. Then $[0, 1] \setminus U$ is closed and bounded, therefore it is compact. Since g is continuous also $g([0, 1] \setminus U)$ is compact. But we also have that

$$g([0, 1] \setminus U) = h^{-1}([0, 1] \setminus U) = [0, 2] \setminus h^{-1}(U).$$

Therefore $[0, 2] \setminus h^{-1}(U)$ is compact, hence closed. This means $h^{-1}(U) \subseteq [0, 2]$ is open and therefore h is continuous. Hence g is a homeomorphism. □

Lemma 4.4.2. *We have that $\mathcal{L}^1(g(C)) = 1$, where we recall that C is the Cantor Triadic Set.*

Proof. We recall that the expanded Cantor-Lebesgue function is constant on each interval $[0, 1] \setminus C$. Therefore for any interval $(a, b) \subseteq [0, 1] \setminus C$ we have that

$$\mathcal{L}^1((g(a), g(b))) = g(b) - g(a) = F_1(b) + b - (F_1(a) + a) = b - a.$$

We now classify the removed collection of intervals at each stage n in the construction of the Cantor Set C. We set this to be $\{I_{n,k}\}_{k=1}^{2^{n-1}}$. Then

$$\begin{aligned}
\mathcal{L}^1([0, 2] \setminus C) &= \mathcal{L}^1(g([0, 1]) \setminus C) \\
&= \mathcal{L}^1\left(g\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^{n-1}} I_{n,k}\right)\right) \\
&= \mathcal{L}^1\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^{n-1}} g\left(I_{n,k}\right)\right) \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}} \mathcal{L}^1(g(I_{n,k})) \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}} \mathcal{L}^1(I_{n,k}) = 1.
\end{aligned}$$

We already know by Proposition 3.4.1 that the total measure of the removed intervals is 1, since the Cantor Triadic Set has measure 0. We then also have that $[0, 2] = g(C) \cup ([0, 2] \setminus g(C))$ and hence, since these two sets are disjoint, we have

$$2 = \mathcal{L}^1([0, 2]) = \mathcal{L}^1(g(C)) + \mathcal{L}^1([0, 2] \setminus g(C)) = \mathcal{L}^1(g(C)) + 1.$$

Therefore $\mathcal{L}^1(g(C)) = 1$. □

Since $\mathcal{L}^1(g(C)) > 0$ then there exists by Proposition 4.3.2 a non-measurable set $E \subset g(C)$. Let $A = g^{-1}(E)$. Since $A \subset \mathcal{C}$, then $\mathcal{L}^1(A) \leq \mathcal{L}^1(C) = 0$. Thus A has Lebesgue measure 0, hence it is measurable. But $g(A) = E$ is not Lebesgue measurable.

Claim 4.4.3. *A is not a Borel set.*

Proof. We suppose by contradiction that A is a Borel set. We know that $g^{-1} = h$ is continuous, hence measurable. We have that $h^{-1}(A) = g(A) = E$ is also measurable. But this is a contradiction. Therefore A is not a Borel set. □

5 The Hausdorff and Banach-Tarski paradoxes

In this Section we closely follow the article of French [11]. The pictures are also taken by the article of French [11].

5.1 Equivalence by finite decomposition

In order to study the Banach-Tarski theorem we need first to better understand some important geometric properties.

Definition 5.1.1. Two subsets of the plane are *congruent* if and only if we can convert one of them into the other (and viceversa) using only rotations and translations in the plane.

The most important property of two sets being congruent is that distance between points remains unmodified.
It is also important to not confuse this definition with the definition of *one to one correspondence*. The following example make the difference more clear.

Example 5.1.2. Let A be the set of even numbers and B the set of natural numbers. There is a one to one correspondence between A and B but they are not congruent, since there is no way to make these sets coincide.

However, an infinite set can be congruent to a subset of itself.

Example 5.1.3. We consider the infinite set A of natural numbers and the set $B := \{3, 4, 5, \dots\}$. These two sets are clearly congruent since we can simply shift toward infinity the first set by 3 units and we obtain exactly B.

Another important definition we need is the concept of equivalence by finite decomposition.

Definition 5.1.4. An object A is *equivalent by finite decomposition* to B if we can divide A into a finite number of disjoint parts and then put them together into a new object B, where A and B are congruent sets.

This type of equivalence is transitive. To better understand this new concept we consider the following claim.

Claim 5.1.5. *The set of natural numbers* $\mathbb{N}$ *is equivalent by finite decomposition with* $\mathbb{N} \setminus \{x\}$*, where* $x \in \mathbb{N}$*.*

Proof. Let $A := \{x, 2x, 3x, \dots\}$ be the set of multiples of x, and let $B := \mathbb{N} \setminus A$. By definition A and B are disjoint and their union is all $\mathbb{N}$. Then we shift A toward infinity by x units, this produces the congruent set $A' = \{2x, 3x, 4x, \dots\}$, since clearly B is congruent with itself we then have that $A \cup B$ is congruent to $A' \cup B$. Clearly the sets A' and B are disjoint and their union is $\mathbb{N} \setminus \{x\}$. Moreover we have that $A' \cup B$ is congruent to $\mathbb{N}$. Therefore the set of natural numbers $\mathbb{N}$ is equivalent by finite decomposition to $\mathbb{N} \setminus \{x\}$. □

Another important example of equivalence by finite decomposition is given by the following claim.

Claim 5.1.6. *The circle and the circle without a point are equivalent by finite decomposition.*

Proof. Let C be a circle with radius 1. Take an arbitrary point on the circumference and numerate it with 0. Then we proceed numerating points along the circle as follows: start at point 0 and move along the circle counterclockwise, at distance 1 stop and numerate the corresponding point with 1 then go ahead at the same way, numerate the point at distance 1 from the point 1 with the number 2 and so on. We define a new set A that contains all the points we marked on the circumference, i.e $A := \{0, 1, 2, 3, \dots\}$. Then we define the set $B := C \setminus A$. Then we use the same idea as in Claim 5.1.5. We take the set A and we shift it toward infinity by one unit. The new set $A' = \{1, 2, 3, \dots\}$ is clearly congruent to A. We can write $C = A \cup B$ and $C' = C \setminus \{0\} = A' \cup B$. Clearly C and C' are congruent and therefore they are equivalent by finite decomposition. □

Before introducing the Hausdorff paradox we make another important example.

Example 5.1.7. We take a closed square with side length 1. We want to transform the closed square in a closed isosceles triangle whose altitude is exactly 1.
Apparently this problem seems easy to solve. We could think that the solution is given by cutting the square along the diagonal an them putting the two so obtained isosceles triangles together as illustrated in Figure 1.

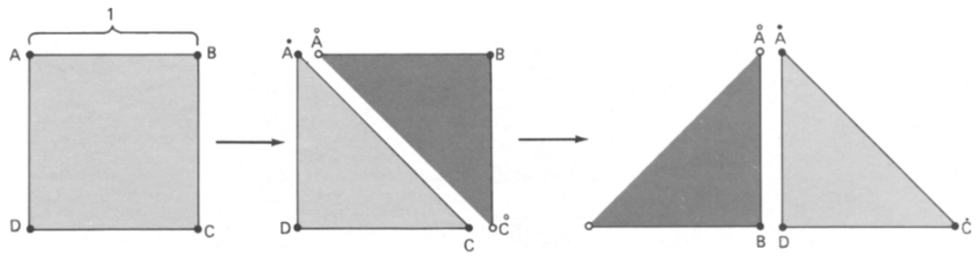

Figure 1: How to transform a square into an isosceles triangle.

Unfortunately this solution does not work. In fact cutting the square along the diagonal would generate two different triangles, since the diagonal can be used to form only one hypotenuse of one of the triangles, not both. Moreover we have seen above in the definition of equivalence by finite decomposition that the parts in the decomposition must be disjoint, and this clearly means that points cannot belong to both parts. Another problem is that when we put together the triangle, we have two candidates for the altitude.
The first idea to solve both problems is to take one altitude of the triangle and paste it along the hypotenuse of the other triangle, which has no points along the hypotenuse. But the hypotenuse has length $\sqrt{2}$ and we paste a side of length 1. Therefore it remains a hole of length $\sqrt{2}-1$ without any points as we can see in Figure 2.

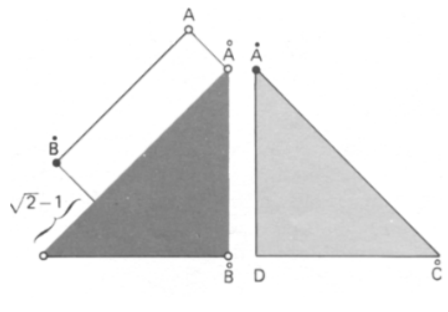

Figure 2: How create an isosceles triangle.

Claim 5.1.8. *The square and the square minus a segment are equivalent by finite decomposition. More precisely the segment has length $\sqrt{2}-1$ and it has only one endpoint.*

Proof. We proceed the same way as in Claim 5.1.6, but this time we do not remove a point from a circle but we remove a line segment from a disk. To do this we inscribe the unit circle C in the square. We repeat the same procedure as in Claim 5.1.6 to numerate points along the circle C and in order to create the set A. Let D be the closed disk whose boundary is given by the circle C. To each point in the set A on the circle we attach a segment of length $\sqrt{2}-1$ pointing towards the centre. We call the segments $L(0), L(1), \ldots$. See Figure 3 to understand better. Let $B := \{L(0), L(1), \ldots\}$ be the set of all the line segments. We shift this set toward infinity by one element. We obtain $B' := \{L(1), L(2), \ldots\}$ which is congruent to B. Let $C := D \setminus B$. We have $D = B \cup C$ and $D' = B' \cup C$. D, D' are congruent and therefore equivalent by finite decomposition. Then, since the removed segment is contained in the disk and does not affect any part of the square outside the disk, we can deduce that also the square and the square minus a segment are equivalent by finite decomposition. □

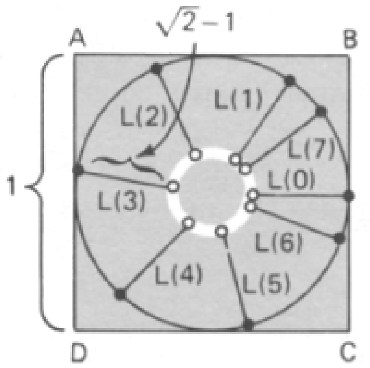

Figure 3: The square is equivalent by finite decomposition to the square minus a line segment of length $\sqrt{2-1}$.

With this assumption we can now finish our example simply inserting $L(0)$ in the hole on the hypotenuse of the corresponding triangle. Therefore we have obtained that the closed isosceles triangle is equivalent by finite decomposition to the closed square.

5.2 The Hausdorff paradox

The content of this Section follow the Sections 'Hausdorff's Paradox', 'A Full Iterative Machine' and the beginning of Section 'Two Spheres from One' of the article of French [11].

The Hausdorff paradox says that we can remove a countable subset from the two-dimensional sphere S with radius $r \in \mathbb{R}_{>0}$ and then divide the remaining part into three disjoint subsets X, Y, Z such that X, Y, Z and $Y \cup Z$ are all pairwise congruent. This means that the set X is congruent to the disjoint union of two copies of itself. But more important is the consequence of this statement, something that seems absurd, i.e putting back together (not casually) the sets X, Y, Z and the countable subset we can construct two spheres, each of which is equivalent by finite decomposition to the the original one.

Sketch of proof. Let $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid {x_1}^2 + {x_2}^2 + {x_3}^2 = r\}$ be a sphere of radius $r \in \mathbb{R}_{>0}$. We fix two axes on the sphere and we call them F and G. Let ϕ be the acute angle formed by the intersection of the axes at the centre of the sphere. We set $\phi = 45°$. Why we have imposed $\phi = 45°$ will be clear later. Now we construct two elementary transformations of the sphere about these axes. Let f be a clockwise rotation by 180° about the F axis and let g be a clockwise rotation by 120° about the G axis.

Using only the axes we just created we can generate new rotations of the sphere. In order to do this we compose randomly and infinitely many times f and g. For example $g^2 f$ consist of first rotating the sphere about the F axis of 180° follow by two rotations of 120° about the G axis.

We can compose as many rotations as we want but at the end we can always define an axis on the sphere that allows us to go with just one rotation to the initial position directly to the final position. This can be made for any elementary transformations we are talking about.

Now we have all the instruments to understand why we set $\phi = 45°$ in the beginning. We choose ϕ such that $fg \neq gf$, i.e implementing first the rotation g and then f leaves the sphere in a different position than performing first the rotation f and then g.

We define id to be the identity transformation, i.e the transformation that leaves the sphere unchanged. Clearly we have $g^3 = id$ and $f^2 = id$, since three rotations of 120° about the G axis are a rotation of 360° about the G axis, and a rotation of 360° about any axes leaves the sphere exactly in the initial position, so this is equal as doing nothing. The same hold composing two rotations of 180°.

To make the notation simpler we set $\bar{g} := g^2$, where $\bar{g}$ represent a clockwise rotation of 240° or a rotation of 120° counterclockwise (there exist two different points of view).

With these properties it is possible to write all the composed transformations in their reduced form, i.e a transformation with as few element as possible, which is equal to the original one. For example $g^5 f^3$ can be reduced as follows

$$g^5 f^3 = (g^3)(g^2)(f^2)(f) = id\ \bar{g}\ id\ f = \bar{g}f.$$

Some elaborate transformations cannot be simplified, since they are already in their reduced form, see for example $gfgf\bar{g}$.

We are now ready to describe an iterative machine that creates a set Q that contains all the transformations we need. The machine work as follows. We put first the identity transformation into the hopper of the machine, then the machine works following three simple rules.

1. When the identity transformation in the only element present in the machine then the machine produces the new elementary transformations f, g and $\bar{g}$.

2. When the transformation ends (left end) with the rotation f the machine creates two new elements, the first adding an additional rotation g and the second adding an additional rotation $\bar{g}$ to the considered transformation.
   For example if fgf enters in the machine, the machine creates as result $gfgf$ and $\bar{g}fgf$.

3. When the transformation ends with the rotation g or $\bar{g}$ the machine produces a new transformation by adding an additional f to the transformation.
   For example if $gf\bar{g}$ comes into the rule box $fgf\bar{g}$ will come out.

We repeat briefly how this machine works. Initially we put the identity transformation into the hopper, the transformation enters in the rule box, it is analysed and then the machine applies the right rule and gives back as result the new transformations. Then the machine produces a copy of all these results. The copies of the results are sent back to the hopper in order to be processed, while the results are put in a box, which represents the set Q. It is important to produce a copy of all the results of the process, because in this way the machine continues working and never stops. This is fundamental in order to create a collection bag that contains all the possible rotations of the sphere. Therefore the set Q has the following form $Q = \{id, f, g, \bar{g}, gf, \bar{g}f, fg, f\bar{g}, \dots\}$. The set Q contains the identity and all the transformations produced by the machine. Figure 4 help us to figure out how this machine works.

This is an ingenious method to create a set Q that contains all the reduced transformations of the sphere. We have that each element of Q represents a transformations of the sphere. Moreover, since we have imposed $fg \neq gf$, each transformation in Q is unique in the sense that the final position of the sphere with respect to the (always equal) initial position is always different.

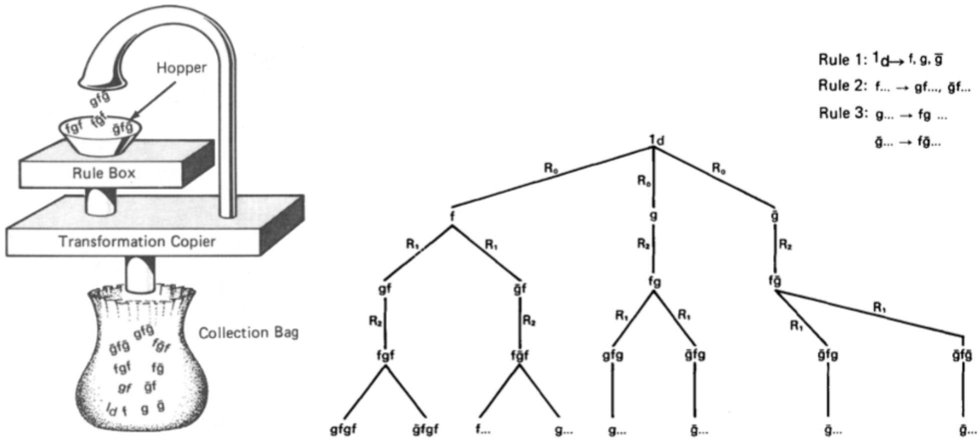

Figure 4: The machine produces the set of all the possible rotations of the sphere.

We now want to implement this machine in order to not only create all the transformations and gather them together in the set Q, but also to sort them into three disjoint subsets I, J and K, such that $Q = I \cup J \cup K$. Let

$$fI = J \cup K, \quad gI = J, \quad \bar{g}I = K.$$

This means that applying the elementary transformation f (a rotation by 180° about the F axis) to the set I we obtain $J \cup K$. This means that the sets I and $Y \cup K$ are congruent. Similarly I is congruent to J applying the transformation g (rotation of 120° about the G axis) and I is congruent to K applying the transformation $\bar{g}$ (rotation of 240° about the G axis). Figure 5 help us to understand better how the new machine works.

To create the three disjoints set I, J, K we connect three of the machines we described above. We work in sequence. First the input enters in machine 1, and the results are split in the transformation copiers of machine 2 and 3 following the sort method just explained. Then the results are copied and the process proceeds in both machine 2 and 3. Finally these results are split between the other two machines and copied there. We go on with this process since we return to the hopper of machine 1 and all this procedure restarts with new elements. Figure 6 help us to understand better how the sets I, J, K are created.

We said above that for each transformation (also composite transformation) we can determine the axis of rotation that allows us to go directly from the initial position to the final position of the sphere. Each axis cuts the sphere in two points, which we call poles. More precisely a pole is a point on the sphere that remains fixed during the rotation, i.e it does not move. We create a set D that contains all the poles associated to each axis of rotation of each

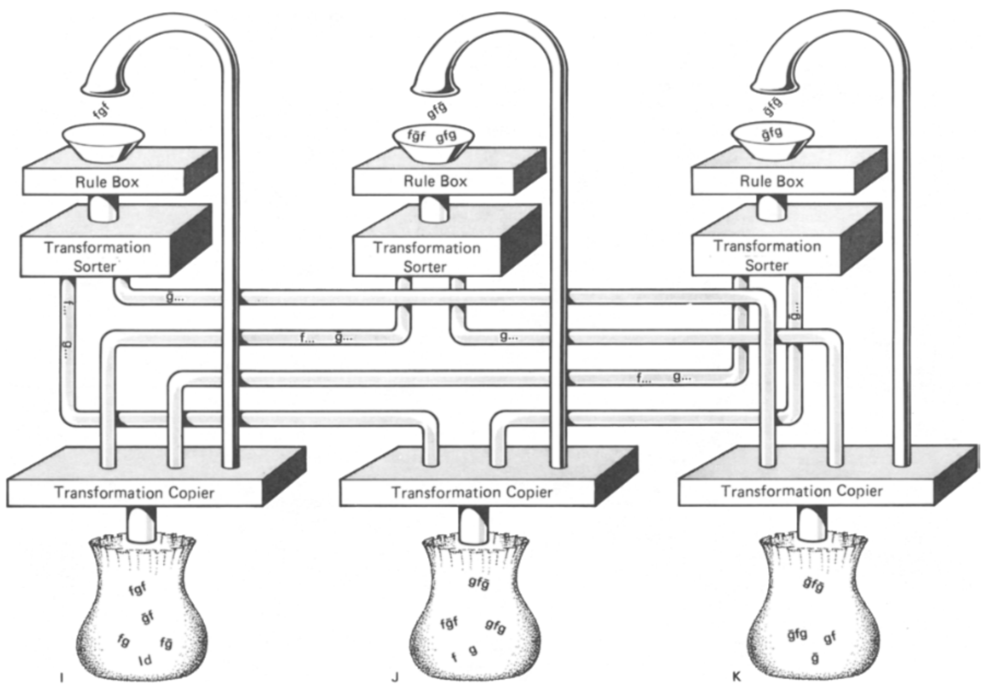

Figure 5: The implemented machine which produces and sorts all of the transformations.

transformation present in Q. This means that D contains all the points that remain fixed during a transformation $t \in Q$ of the sphere. Naturally D is a countable set and it is infinitesimally small compared to the entire sphere that contains uncountable many points. In addition all the points that do not belong to D move for every transformation t. We now define a new set C being the complement of the set D in the sphere, $C := S \setminus D$.

With this construction we find the countable set D. It remains to define the sets X, Y, Z that satisfy the desired property of congruence and such that $C = X \cup Y \cup Z$.

We take a point $p \in C$ and we create a new set $Q(p)$. We apply (individually) all the transformations present in Q to p and we put the results in $Q(p)$, therefore we obtain $Q(p) := \{p, f(p), g(p), \bar{g}(p), fg(p), \dots\}$. We repeat this process for all the points in C. Two distinct points p and p' generate two sets $Q(p)$ and $Q(p')$. These can be either equal or disjoint.
Now we want to use Zermelo's Axiom, we have $\mathcal{F} = \{Q(p) \mid p \in C\}$ that is a collection of non-empty disjoint sets. The Axiom of Choice permits us to create a new set A taking exactly one element from each $Q(p)$.

We can observe that C is equal to the set we obtain by applying all the transformations present in Q to the points present in A.

It remains to divide C into three disjoint subsets X, Y, Z such that X, Y, Z

	Cycle 0	Cycle 1	Cycle 2	Cycle 3
Contents of Bag I	1	1	1, fg, $\bar{g}$f, f$\bar{g}$	1, fg, $\bar{g}$f, f$\bar{g}$, fgf
Contents of Bag J	empty	f, g	f, g	f, g, f$\bar{g}$f, gfg, gf$\bar{g}$
Contents of Bag K	empty	$\bar{g}$	$\bar{g}$, gf	$\bar{g}$, gf, $\bar{g}$fg, $\bar{g}$f$\bar{g}$

Figure 6: The results of the first iterations of the transformation-producing and sorting machine.

and $Y \cup Z$ are all pairwise congruent. However, we have already created three disjoint sets, i.e the sets I, J, K and we want to use them. We define X to be the set of points resulting from the application of the transformations in I to the set A. Similarly we define Y, respectively Z, to be the set of points resulting from the application of the transformations in J, respectively in K, to the set A. This construction gives us the desired decomposition of C into three disjoint subsets. It remains only to prove that these sets we create are also pairwise congruent. We recall we have $fI = J \cup K$, therefore at the same way we have $f(X) = Y \cup Z$. Since f is a rotation of 180° about the F axis we have that X and $Y \cup Z$ are congruent. Similarly by $gI = J$ we obtain $g(X) = Y$ meaning that X and Y are congruent, and by $\bar{g}I = K$ we obtain $\bar{g}(X) = Z$ meaning that X is also congruent to Z. Since congruence is a transitive operation we find that X, Y, Z and $Y \cup Z$ are all pairwise congruent. □

5.3 The Banach-Tarski paradox

The content of this Section follow the section 'Two Spheres from One' of the article of French [11].

The Banach-Tarski paradox shows that the unit ball $B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid {x_1}^2 + {x_2}^2 + {x_3}^2 \leq 1\}$ can be decomposed into a finite number of disjoint pieces that can be used to form two new disjoint copies of B, which have the same size and volume of B. This paradox generalises the Hausdorff paradox.

Sketch of the proof. In Section 5.2 we have seen that starting from a sphere we can construct two new spheres identical to the original one. Now we ask ourselves if we can do the same starting with a solid ball.

The construction of the Hausdorff paradox can be used only for spheres but a sphere can also be seen as an hollow ball. This consideration helps us because we can imagine to apply the technique described above to a sphere with a thicker boundary (Hausdorff construction allow this). Therefore we apply the construction to a ball whose inside consists only of a single point (sphere with thicker boundary). Hence we obtain two new balls, each missing its central point and both being equivalent to the original ball.

However, this is still not enough to prove the Banach-Tarski theorem. We are not yet able to replicate a solid ball. With our basis we can only duplicate a solid ball without its centre.

We claim that a solid ball and a solid ball without its centre are equivalent by finite decomposition. This claim completes the proof of Banach-Tarski theorem, since a solid ball is equivalent by finite composition to a solid ball without its centre and this is equivalent by the above construction to two copies of itself.

We repeat more precisely the most important aspects of this proof. Given a solid ball, we take its surface, the sphere. By the Hausdorff paradox we already know that the sphere can be cut into four disjoint sets, i.e the countable set D and the sets X, Y, Z, such that X, Y, Z and $Y \cup Z$ are all pairwise congruent. We will use the set $Y \cup Z$ to produce the pairs of sets that will be reassembled into two separate spheres. Take the set X and cut out two sets X_1, X_2 that are congruent to Y and Z respectively. Since both Y and Z are congruent to X the decomposition of X into X_1 and X_2 is paradoxical. Then we decompose Y and Z in a similar way into Y_1, Y_2 and Z_1, Z_2. In other words we have

$$\begin{aligned} S &= X \cup Y \cup Z \cup D \\ &= (X_1 \cup X_2) \cup (Y_1 \cup Y_2) \cup (Z_1 \cup Z_2) \cup D \\ &= (X_1 \cup Y_1 \cup Z_1 \cup D) \cup (X_2 \cup Y_2 \cup Z_2). \end{aligned}$$

From $X_1 \cup Y_1 \cup Z_1 \cup D$ we construct the sphere S_1 which is equivalent by finite decomposition to the original sphere S. It remains to show that a second sphere can be constructed from $X_2 \cup Y_2 \cup Z_2$. Apparently it seems to miss the countable set D.

We claim that a sphere without the countable set D is equivalent by finite decomposition to the sphere. The proof is similar to the proof of Claim 5.1.6, that states that the circle is equivalent by finite decomposition to the circle without a point.

Thus S_2 and $S_2 \setminus D$ are equivalent by finite decomposition and therefore we find that S_1, S_2 are equivalent to S and this concludes the proof.

□

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