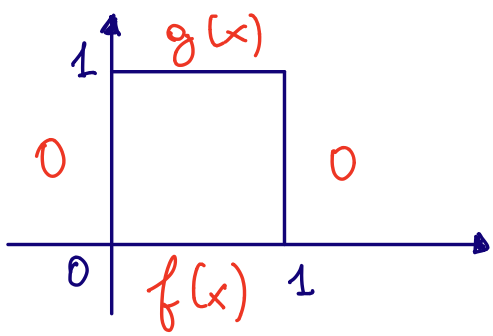
Lecture Analysis 3-14.12.2020

Last week

- Laplace Equation in a Rectangle
- Heat Equation on Rx {t>0} by two methods

- I PROPOSED YOU TO SOLVE THE FOLLOWING PROBLEM:



$$\begin{cases} u_{xx} + u_{yy} = 0 \\ \begin{cases} u(0,y) = 0 \\ u(1,y) = 0 \end{cases} & y \in [0,1] \\ \begin{cases} u(x,0) = f(x) \\ u(x,1) = g(x) \end{cases} & x \in [0,1] \end{cases}$$

Did you think about that?

- SOLUTION OF HEAT EQUATION BY TAKING THE FOURIER TRANSFORM WITH RESPECT TO THE SPACE VARIABLE x

$$\begin{cases} u_t - c^2 u_{xx} = 0 \\ u(x,0) = f(x) \end{cases} \xRightarrow{\mathcal{F}} \begin{cases} \frac{\partial}{\partial t} \hat{u}(\omega,t) + c^2 \hat{u}(\omega,t) = 0 \\ \hat{u}(\omega,0) = \hat{f}(\omega) \end{cases}$$

$$\Rightarrow \quad \hat{u}(\omega,t) = \hat{f}(\omega)\, e^{-c^2 t \omega^2}$$

$$\Rightarrow \quad u(x,t) = \mathcal{F}^{-1}\left(\hat{f}(\omega)\, e^{-c^2 t \omega^2}\right) \quad (*)$$

To compute $\boxed{*}$ we could have used the <u>convolution property</u>

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \underbrace{\mathcal{F}^{-1}\left(\hat{f}(w)\right)}_{= f(x) \text{ by definition}} * \mathcal{F}^{-1}\left(e^{-c^2 t w^2}\right)$$

RECALL

$$\mathcal{F}(f * g) = \sqrt{2\pi}\, \mathcal{F}(f)\, \mathcal{F}(g)$$

$$\mathcal{F}^{-1}\left(e^{-ax^2}\right) = \mathcal{F}\left(e^{-ax^2}\right) = \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}$$

By applying such a formula with $a = c^2 t$ we get

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2c^2 t}} \int_{-\infty}^{+\infty} f(w)\, e^{-\frac{(w-x)^2}{4c^2 t}}\, dw$$

Today

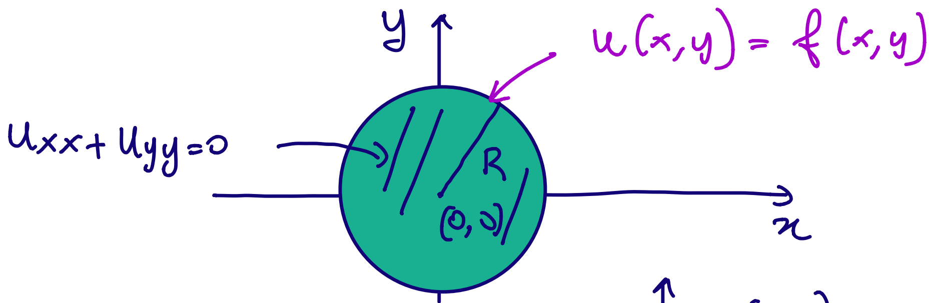
- **Laplace operator in polar coordinates**
- **Laplace equation on a disk**
- **Poisson Formula**
- **Mean value formula & Maximum/Minimum Principle**

Solve the following problem.

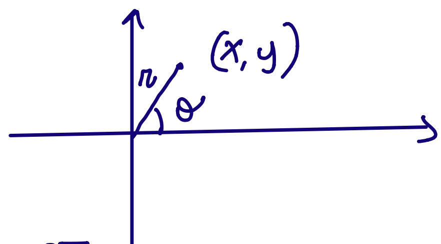
$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < R^2\} \\ u(x,y) = f(x,y) & \text{on } \{(x,y) \in \mathbb{R}^2\ x^2 + y^2 = R^2\} \end{cases}$$

$B((0,0), R)$ $\quad B(0,R)$ $\quad D_R$

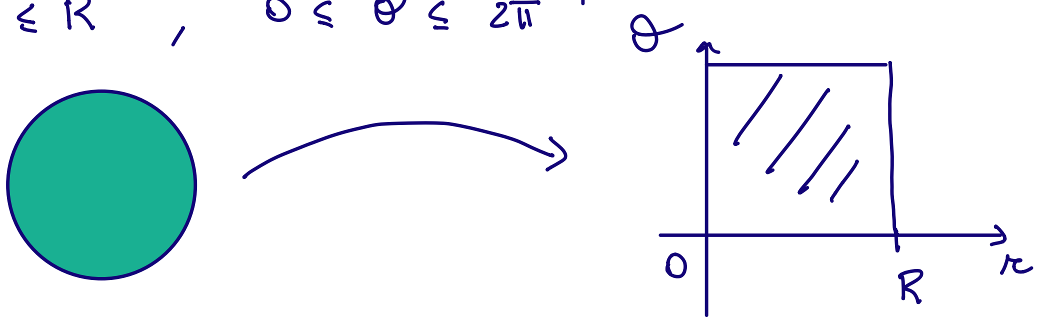
the boundary of such a ball: $\partial B((0,0),R)$, $\partial B(0,R)$, ∂D_R.



$$\begin{cases} x = r\cos(\theta) \\ y = r\sin\theta \end{cases}$$



$0 \le r \le R \quad , \quad 0 \le \theta \le 2\pi$



$$w(r,\theta) = u(\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y})$$

$$\begin{cases} w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta} = 0 & (0,R)\times(0,2\pi) \\ w(R,\theta) = f(R\cos\theta, R\sin\theta) = \tilde{f}(\theta) & 0 \le \theta \le 2\pi \end{cases}$$

■ $\lim_{r\to 0^+} w(r,\theta)$ is finite.

→ Apply the method of SEPARATION OF VARIABLES:

● $w(r,\theta) = R(r)\,\Theta(\theta)$

w_{rr} , w_r, $w_{\theta\theta}$

We get two ODES :

$$\begin{cases} r^2 R''(r) + r\, R(r) - K\, R(r) = 0 \\ \Theta''(\theta) + K\, \Theta(\theta) = 0 \end{cases}$$

NOTE : $\Theta(\theta)$ must be periodic of period $2\pi \Rightarrow \Theta(2\pi + \theta) = \Theta(\theta)$ and $\Theta'(2\pi + \theta) = \Theta'(\theta) \quad \forall\, \theta \in \mathbb{R}$.

$$\Rightarrow \quad \Theta(0) = \Theta(2\pi) \quad \& \quad \Theta'(0) = \Theta'(2\pi)$$

PROBLEM FOR Θ:

$$(P_\theta) \begin{cases} \Theta''(\theta) + K\, \Theta(\theta) = 0 \quad \text{in } (0, 2\pi) \\ \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases}$$

$K > 0$, $K = 0$, $K < 0$

GENERAL SOLUTION

$K > 0 \Rightarrow A \cos(\sqrt{K}\, \theta) + B \sin(\sqrt{K}\, \theta)$

$K = 0 \Rightarrow A\theta + B$

$K < 0 \Rightarrow A e^{\sqrt{-K}\, \theta} + B e^{-\sqrt{-K}\, \theta}$

LET US CONSIDER THE BOUNDARY CONDITIONS

$K=0$ $\quad \Theta(0)=\Theta(2\pi) \Rightarrow \begin{cases} A\cdot 0+B = A\cdot 2\pi + B \\ A = A \end{cases}$

$\quad\quad \Theta'(0)=\Theta'(2\pi) \Rightarrow$

$\Theta(\vartheta) = A\vartheta + B \Rightarrow \Theta'(\vartheta) = A$

$$\Rightarrow \begin{cases} \cancel{B} = 2\pi A + \cancel{B} \\ A = A \end{cases} \Rightarrow A = 0$$

$$\Rightarrow \Theta(\vartheta) = B \in \mathbb{R}$$

$K<0$ $\quad \Theta(0)=\Theta(2\pi) \Rightarrow \begin{cases} A+B = Ae^{2\pi\sqrt{-k}} + Be^{-2\pi\sqrt{-k}} \\ (A-B)\cancel{\sqrt{-k}} = \left(Ae^{2\pi\sqrt{-k}} - Be^{-2\pi\sqrt{-k}}\right)\cancel{\sqrt{-k}} \end{cases}$

$\quad\quad \Theta'(0)=\Theta'(2\pi) \Rightarrow$

$A=B=0 \Rightarrow$ ONLY TRIVIAL SOLUTIONS

$K>0$

$$\begin{cases} (1)\ AB = AB\cos(2\pi\sqrt{k}) + B^2\sin(2\pi\sqrt{k}) & \times B \\ (2)\ BA\cancel{\sqrt{k}} = \cancel{\sqrt{k}}\left(-A^2\sin(2\pi\sqrt{k}) + BA\cos(2\pi\sqrt{k})\right) & \times A \end{cases}$$

$(1)-(2) \Rightarrow 0 = (A^2+B^2)\sin(2\pi\sqrt{k})$

There are two possibilities:

either $A^2+B^2=0 \Rightarrow$ trivial solution!

or $\sin(2\pi\sqrt{k})=0 \Leftrightarrow 2\pi\sqrt{k} = n\pi$

$\Leftrightarrow \sqrt{k} = \frac{n}{2} \quad n\in\mathbb{N}\setminus\{0\} \Rightarrow K = \left(\frac{n}{2}\right)^2$

On the other hand if we multiply the first equation by A and the second equation by B and we sum the two, we get:

$$\begin{cases} A^2 = A^2\cos(2\pi\sqrt{k}) + AB\sin(2\pi\sqrt{k}) \\ B^2 = -AB\sin(2\pi\sqrt{k}) + B^2\cos(2\pi\sqrt{k}) \end{cases}$$

$$\Rightarrow A^2+B^2 = (A^2+B^2)\cos(2\pi\sqrt{k})$$

Since we exclude $A^2+B^2 \neq 0$ we should have $\cos(2\pi\sqrt{k}) = 1$

$$\Rightarrow 2\pi\sqrt{k} = 2\pi n \Rightarrow \sqrt{k} = n \Rightarrow k = n^2$$

Since we want that both

$$\begin{cases} \sin(2\pi\sqrt{k}) = 0 \\ \cos(2\pi\sqrt{k}) = 1 \end{cases}$$

are satisfied, we have to take $k = n^2$, $n > 0$.

We could have said that since we want that

$$A\cos(\sqrt{k}\theta) + B\sin(\sqrt{k}\theta)$$

is 2π-periodic, it should be that $2\pi\sqrt{k} = 2\pi n$ $\quad n > 0 \Rightarrow \sqrt{k} = n$

$$\Rightarrow \sqrt{K} = n \iff K = n^2$$

For $K = n^2 > 0$, we obtain

$$\Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)$$

For $K = 0$

$$\Theta_0(0) = A_0$$

$$\Rightarrow \forall n \geq 0 \Rightarrow \boxed{\Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)}$$

Problem for R :

$K = n^2$:

$$r^2 R''(r) + r R'(r) - n^2 R(r) = 0$$

(Euler Equation)

$n > 0$: $R(r) = r^\alpha$, $R'(r) = \alpha r^{\alpha-1}$

$R''(r) = \alpha(\alpha-1) r^{\alpha-2}$

$$r^2 \alpha(\alpha-1) r^{\alpha-2} + r \alpha r^{\alpha-1} - n^2 r^\alpha = 0$$

$$r^\alpha \alpha(\alpha-1) + r^\alpha \alpha - n^2 r^\alpha = 0$$

$$r^\alpha \left(\alpha(\alpha-1) + \alpha - n^2\right) = 0$$

$$\alpha^2 - \cancel{\alpha} + \cancel{\alpha} - n^2 = 0 \Rightarrow \alpha = \pm n$$

Two solutions : $R_1(r) = r^n$, $R_2(r) = r^{-n}$

$$R_n(r) = \tilde{A}_n r^n + \tilde{B}_n r^{-n}$$

$n = 0$

$$\cancel{r^2}\, R_0''(r) + \cancel{r}\, R_0'(r) = 0$$

$$\Rightarrow \quad \frac{d}{dr}\left(r R_0'(r) \right) = 0$$

$$\Rightarrow \quad \cancel{r}\, R_0'(r) = \frac{\tilde{B}_0}{r} \quad \Rightarrow \quad R_0(r) = \tilde{B}_0 \log(r) + \tilde{A}_0$$

Since I want solutions $\lim_{r \to 0} R_n(r) < +\infty$

$$\Rightarrow \quad R_n(r) = r^n$$

$$R_0(r) = 1$$

$\Rightarrow \quad \forall n \geq 0$

$$w_n(r, \vartheta) = r^n \left(A_n \cos(n\vartheta) + B_n \sin(n\vartheta) \right)$$

$$w(R, \vartheta) = \tilde{f}(\vartheta)$$

$\Rightarrow$ 3rd STEP of METHOD OF SEPARATION OF VARIABLES:

$$w(r, \vartheta) = A_0 + \sum_{n=1}^{\infty} r^n \left(A_n \cos(n\vartheta) + B_n \sin(n\vartheta) \right)$$

$$w(R, \vartheta) = A_0 + \sum_{n=1}^{\infty} R^n \left(A_n \cos(n\vartheta) + B_n \sin(n\vartheta) \right)$$

$$= \tilde{f}(\vartheta) = f(R\cos\vartheta, R\sin\vartheta)$$

$$\Rightarrow \quad A_0 = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(\tilde{\vartheta})\, d\tilde{\vartheta}$$

$$A_n \cancel{R^n} = \frac{1}{\pi R^n}\int_0^{2\pi} \tilde{f}(\tilde{\vartheta})\cos(n\tilde{\vartheta})\,d\tilde{\vartheta}$$

$$B_n \cancel{R^n} = \frac{1}{\pi R^n}\int_0^{2\pi} \tilde{f}(\tilde{\vartheta})\sin(n\tilde{\vartheta})\,d\tilde{\vartheta}$$

POISSON FORMULA

$$w(r,\vartheta) = A_0 + \sum_{n=1}^{\infty} r^n\left(A_n\cos(n\vartheta) + B_n\sin(n\vartheta)\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\tilde{f}(\tilde{\vartheta})\,d\tilde{\vartheta} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{r^n}{R^n}\int_0^{2\pi}\tilde{f}(\tilde{\vartheta})\left[\cos(n\vartheta)\cos(n\tilde{\vartheta}) + \sin(n\vartheta)\sin(n\tilde{\vartheta})\right]d\tilde{\vartheta}$$

$$= \frac{1}{\pi}\int_0^{2\pi}\tilde{f}(\tilde{\vartheta})\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\cos\left(n(\vartheta-\tilde{\vartheta})\right)\right]d\tilde{\vartheta}$$

EXERCISE

$$\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\cos\left(n(\vartheta-\tilde{\vartheta})\right) = \frac{1}{2}\,\frac{1-\left(\frac{r}{R}\right)^2}{1-2\left(\frac{r}{R}\right)\cos(\vartheta-\tilde{\vartheta})+\left(\frac{r}{R}\right)^2}$$

$$= \frac{1}{2}\,\frac{R^2-r^2}{R^2-2rR\cos(\vartheta-\tilde{\vartheta})+r^2}. \quad \square$$

HINT: It follows from the fact:

$\forall \alpha \in \mathbb{R}$, $|t| < 1$

$$\sum_{n=1}^{\infty} t^n \cos(n\alpha) = \mathrm{Re}\Big(\sum_{n=1}^{\infty} \underbrace{t^n e^{in\alpha}}_{=\big(\underbrace{t e^{i\alpha}}_{q}\big)^n}\Big)$$

GEOMETRIC SERIES WITH RATIO $|q| < 1$.

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta-\tilde{\theta}) + r^2} \cdot \tilde{f}(\tilde{\theta})\,d\tilde{\theta} \qquad (*)$$

the function

$$K(r,\theta,R,\tilde{\theta}) = \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta-\tilde{\theta}) + r^2}$$

is called the POISSON INTEGRAL KERNEL.

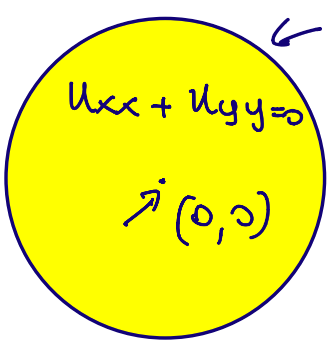
$\boxed{\text{CONSEQUENCES}}$

1) MEAN VALUE PROPERTY

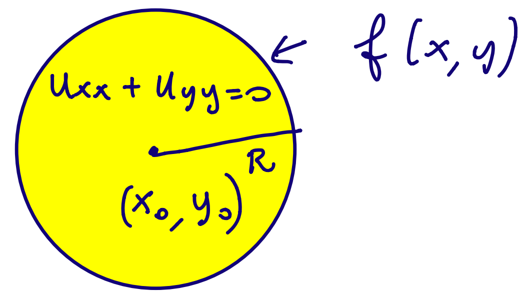
take in $(*)$, $r = 0$

$$\Rightarrow u(0,\theta) = u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{f}(\tilde{\theta})\,d\tilde{\theta}$$

$f(x,y)$



This property holds in general balls $B((x_0,y_0),R)$:

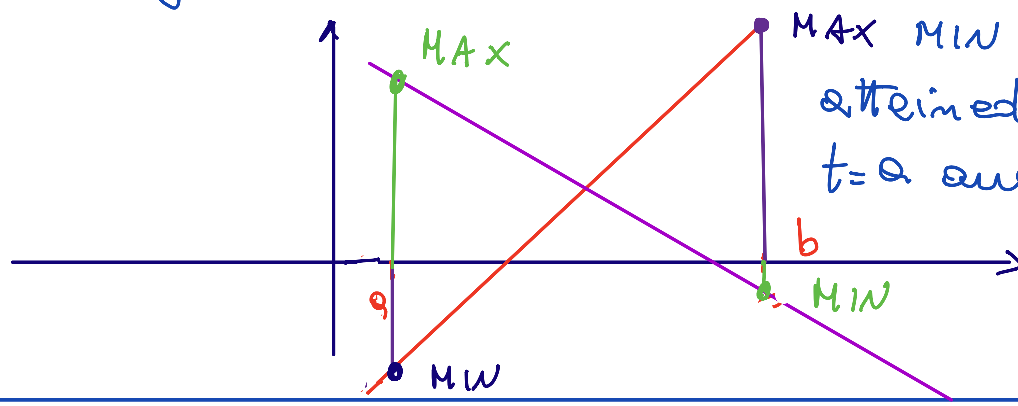


$$\Rightarrow u(x_0,y_0)=\frac{1}{2\pi}\int_0^{2\pi} f(x_0+R\cos\tilde{\vartheta},\ y_0+R\sin\tilde{\vartheta})\,d\tilde{\vartheta}$$

2) MAXIMUM / MINIMUM PRINCIPLE

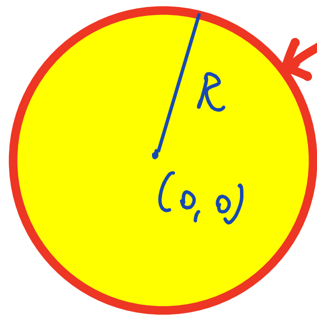
NOTE If $u''(x)=0$ then $u(x)=Ax+B \quad A,B\in\mathbb{R}$

For every $[a,b]\subseteq\mathbb{R}$ the MAX and the MAX MIN are attained at $t=a$ and $t=b$!



Suppose $u_{xx}+u_{yy}=0$ in $B(0,R)$ and u is continuous in $\overline{B}(0,R)$ $=\{(x,y):\ x^2+y^2\le R^2\}$

Then the MAX and the MIN values of u are attained on $\{(x,y)\in\mathbb{R}^2:\ x^2+y^2=R^2\}$



the MAX and the MIN are attained in the red part!

$$\min_{\partial B(0,R)} u < u(x,y) < \max_{\partial B(0,R)} u(x,y)$$

PROPERTIES OF POISSON KERNEL (EXERCISE)

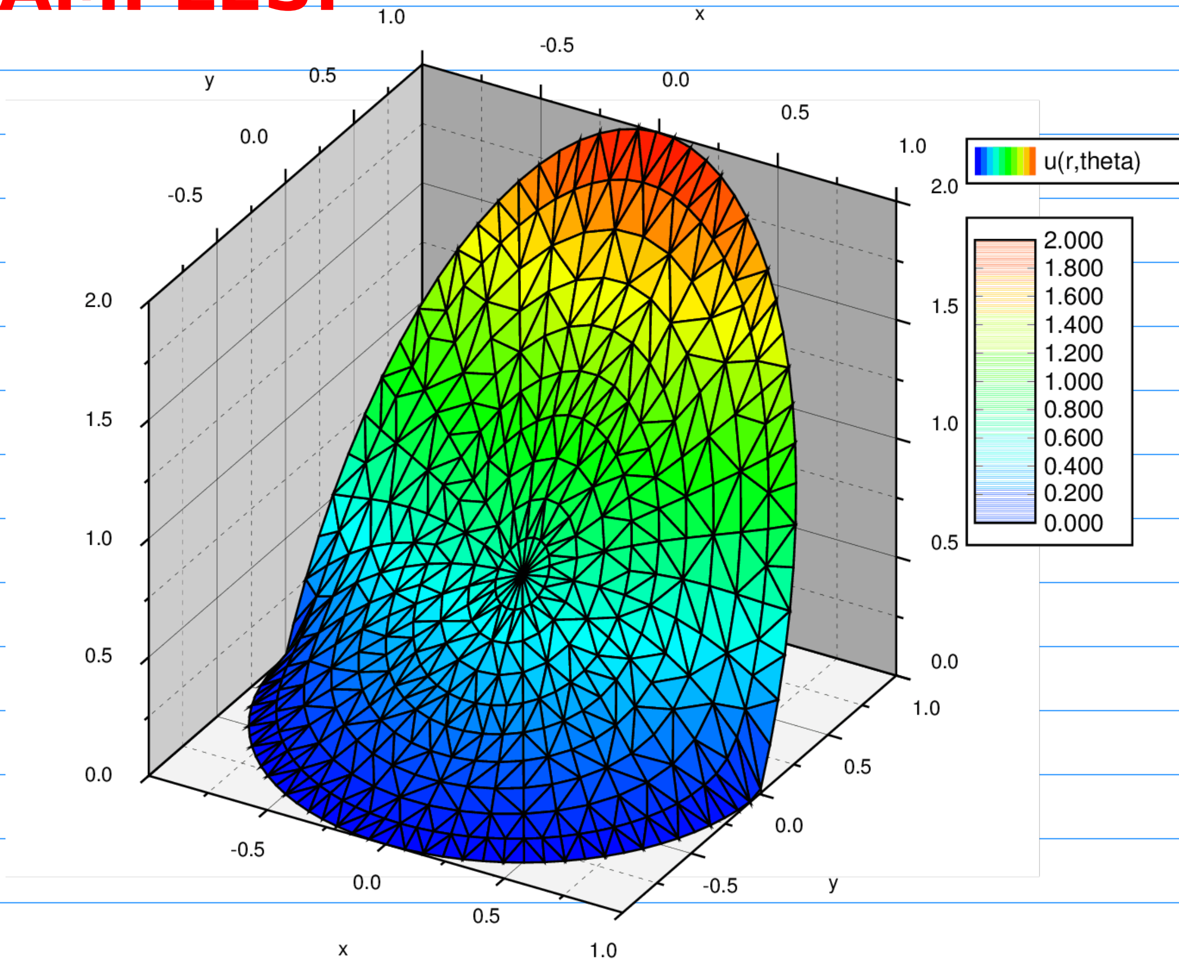
a) $K(r, 0, R, \vartheta) > 0 \quad \forall\, r < R$

b) $\frac{1}{2\pi} \int_0^{2\pi} K(r, 0, R, \vartheta)\, d\vartheta = 1$

c) $K(r, 0, R, \vartheta)$ is harmonic function inside the ball $B((0,0), R)$:

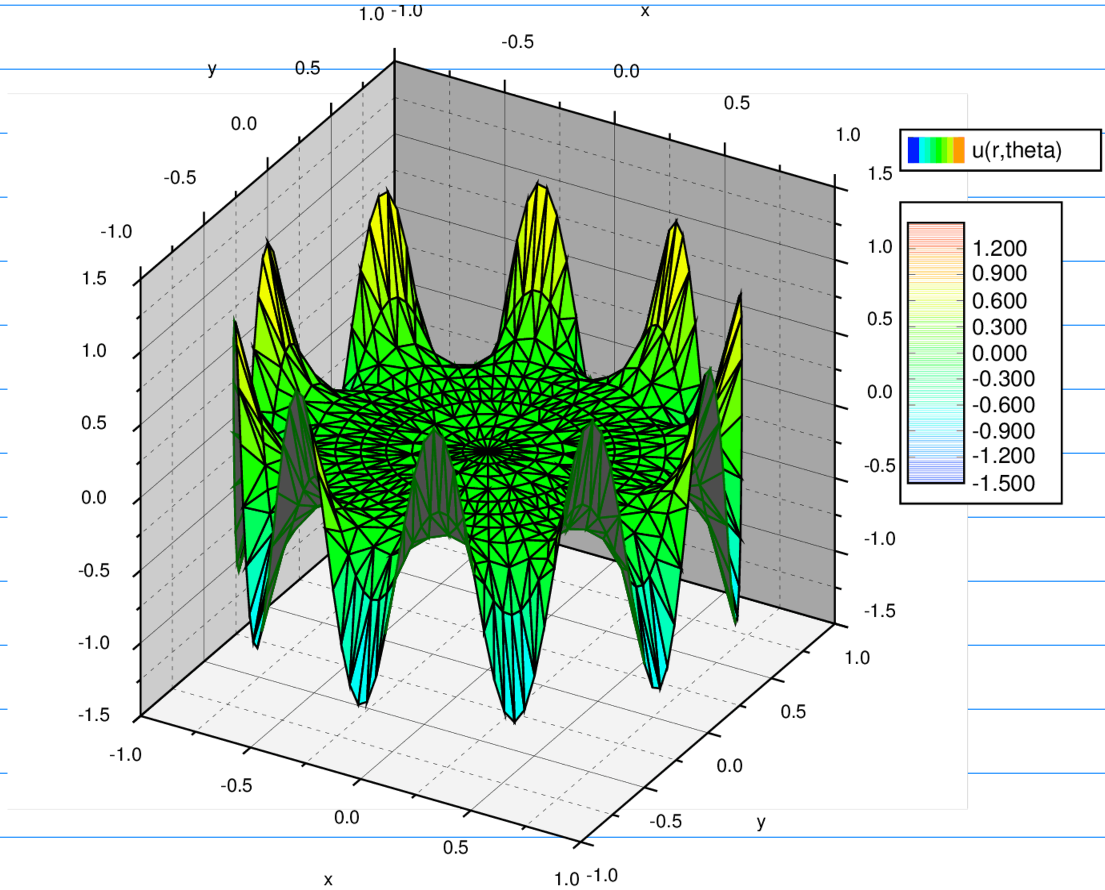
$$\partial_{rr} K + \frac{1}{r} \partial_r K + \frac{1}{r^2} \partial_{\vartheta\vartheta} K = 0$$

EXAMPLES:



Solution of the Laplace equation on the unit disk with boundary data f(x,y)=2y if y>0 and f(x,y)=0 if y<0.

(EXERCISE: COMPUTE THE SOLUTION)



The solution of the Dirichlet problem in the disc with cos(10θ) as boundary data.

(EXERCISE: COMPUTE THE SOLUTION)

EXERCISES (REVIEW)

Ex 1

Look at the following Dirichlet's problem on a disk centered at (0,0) with radius R in polar coordinates

$$\Delta u \begin{cases} \nabla^2 u = 0 & u_{xx} + u_{yy} \\ u(R,\theta) = |\theta| & 0 \leq \theta < 2\pi \end{cases}$$

a) Determine $u(0,0)$ without calculating the whole solution of the Dirichlet problem

b) Find the solution in polar coordinates $u(r,\theta)$

Solution

a) The value in the center is given by the mean value on the boundary $r = R$

$$u(0,0) = \frac{1}{2\pi} \int_0^{2\pi} u(R,\theta)\,d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \theta \, d\theta = \pi$$

b) The equation in polar coordinates is given by

$$u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r = 0$$

Last week you have seen that the general solution is given by

$$u(r,\vartheta) = \sum_{n=0}^{\infty} r^n \left(A_n \cos(n\vartheta) + B_n \sin(n\vartheta)\right)$$

at the boundary the solution must satisfy the following condition

$$u(R,\vartheta) = |\vartheta| = \sum_{n=0}^{\infty} R^n \left(A_n \cos(n\vartheta) + B_n \sin(n\vartheta)\right)$$

the Fourier series of the function $|\vartheta|$ is

$$|\vartheta| = \sum_{n=0}^{\infty} a_n \cos(n\vartheta) + b_n \sin(n\vartheta)$$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} \varphi \, d\varphi = \pi$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \varphi \cos(n\varphi) \, d\varphi$$

$$= \frac{1}{\pi} \frac{\sin n\varphi}{n} \varphi \Big|_0^{2\pi} - \frac{1}{\pi} \int_0^{2\pi} \frac{\sin n\varphi}{n} d\varphi$$

$$= \frac{1}{\pi} \frac{1}{n^2} \left(\cos n\varphi \right) \Big|_0^{2\pi} = 0$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} \varphi \sin(n\varphi) d\varphi$$

$$= \frac{1}{\pi} \varphi \left(- \frac{\cos(n\varphi)}{n} \right) \Big|_0^{2\pi}$$

$$+ \frac{1}{\pi} \int_0^{2\pi} \frac{\cos(n\varphi)}{n} d\varphi$$

$$= \frac{1}{\pi} 2\pi \left(- \frac{\cos(n 2\pi)}{n} \right) = - \frac{2}{n}$$

Therefore $A_n = 0$ and

$$B_n = \frac{b_n}{R^n} = -\frac{2}{nR^n}$$

The solution is

$$u(r,\theta) = \pi - 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \sin(n\theta)$$

$\square$

$\boxed{\text{Ex 2}}$

$$\begin{cases} u_{xx} + u_{yy} = 0 & x^2+y^2<1 \\ u(x,y) = 1 + 2x^4 & x^2+y^2 = 1 \end{cases}$$

i) $u(0,0)$ by POISSON FORMULA

ii) Show that $u(x,y) > 1$

Solution

We express the boundary data in polar coordinates:

$$1+2x^4 = 1+2\cos^4\theta \qquad (R=1)$$

i) $u(0,0) = \frac{1}{2\pi}\int_0^{2\pi}\left(1+2\cos^4\theta\right)d\theta = \otimes$

<u>observe</u>

$$\cos^4\theta = \left(\cos^2\theta\right)^2 = \left(\frac{1+\cos(2\theta)}{2}\right)^2$$

<u>Recall</u> $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$$= 2\cos^2\theta - 1$$

$$= \frac{1}{4}\left(1+\cos^2(2\theta)+2\cos(2\theta)\right)$$

$$= \frac{1}{4}\left(1+\frac{\cos(4\theta)+1}{2}+2\cos(2\theta)\right)$$

$$= \frac{1}{4}+\frac{\cos 4\theta + 1}{8}+\frac{\cos 2\theta}{2}$$

$$\circledast = \frac{1}{2\pi}\int_0^{2\pi} 1 + 2\left(\frac{1}{4} + \frac{1+\cos(4\theta)}{8} + \frac{\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(1 + \frac{1}{2} + \frac{1}{4}\right) + \frac{\cos 4\theta}{4} + \cos 2\theta \, d\theta$$

$$= \frac{7}{4} + \underbrace{\frac{1}{2\pi}\left[\frac{\sin(4\theta)}{16} + \frac{\sin(2\theta)}{2}\right]\Bigg|_0^{2\pi}}_{=0}$$

$$= \frac{7}{4}$$

b) $u(x,y) > 1$

By "MAXIMUM and MINIMUM PRINCIPLE" the MAX and the MIN values are attained on $x^2+y^2=1$

Thus we have to study

$(x,y) \longmapsto 1 + 2x^4 \quad x^2+y^2=1$

We set $x = \cos\theta \Rightarrow \theta \mapsto 1 + 2\cos^4\theta$

$0 \leq \theta \leq 2\pi$

Observe $0 \leq \cos^4\theta \leq 1$

$\Rightarrow \quad 1 \leq 1 + 2\cos^4\theta \leq 1+2 = 3$

$\cos^4\theta = 0 \iff \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$\cos^4\theta = 1 \iff \theta = 0, \pi, 2\pi$

By MIN. principle we have

$$u(x,y) > \min_{x^2+y^2=1} u(x,y) = 1$$

and by MAX principle

$$u(x,y) < \max_{x^2+y^2=1} u(x,y) = 3$$

$\square$