Lecture ANALYSIS 3-16.11.2020

LAST WEEK (lecture 9.11.2020)

- Fourier integral in complex form
- Fourier transform & inverse Fourier transform
  Properties of F.T.

Plan of today

- Fourier transform of the Gaussian
- Chapter 4: Introduction to PDEs (examples, classifications, method of separation of the variables

1) $f(x) = e^{-x^2} \quad x \in \mathbb{R}$

$$\mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2} e^{-i\omega x} dx =$$

Sol Method of completation the squares"

$$x^2 + i\omega x = x^2 + 2\frac{1}{2} i\omega x \pm \left(\frac{1}{2} i\omega\right)^2$$

$$= x^2 + 2\frac{1}{2} i\omega + \frac{(i\omega)^2}{4} - \frac{(i\omega)^2}{4}$$

$$\underset{i^2=-1}{=} \left(x + \frac{1}{2} i\omega\right)^2 + \frac{\omega^2}{4}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\left(x + \frac{1}{2} i\omega\right)^2 - \frac{\omega^2}{4}} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{4}} \int_{-\infty}^{+\infty} e^{-\left(x + \frac{1}{2} i\omega\right)^2} dx = (*)$$

<u>RECALL</u>: $\forall \alpha \in \mathbb{C}$ $(\alpha = \alpha_1 + i\alpha_2,\ \alpha_1, \alpha_2 \in \mathbb{R})$

$$\int_{-\infty}^{+\infty} e^{-(x+\alpha)^2}\,dx = \int_{-\infty}^{+\infty} e^{-x^2}\,dx.$$

This inequality can be justified by complex analysis tools.

$$(*) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{w^2}{4}} \int_{-\infty}^{+\infty} e^{-x^2}\,dx = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{w^2}{4}} \sqrt{\pi} = \frac{\sqrt{2}}{2}\, e^{-\frac{w^2}{2}}$$

See Ex 6 Serie 8

2) $\mathcal{F}\left(e^{-ax^2}\right)(w) = \mathcal{F}\left(e^{-(\sqrt{a}x)^2}\right) = (*)$

$a > 0$

apply the dilation property and the value of $\mathcal{F}\left(e^{-x^2}\right)$

$$\mathcal{F}\big(f(\lambda x)\big)(w) = \frac{1}{\lambda}\, \mathcal{F}\big(f(x)\big)\left(\frac{w}{\lambda}\right) \quad \lambda > 0$$

$\lambda = \sqrt{a}$

$$(*) = \frac{1}{\sqrt{a}}\, \mathcal{F}\left(e^{-x^2}\right)\left(\frac{w}{\sqrt{a}}\right) = \frac{1}{\sqrt{a}} \left(e^{-\frac{w^2}{4}} \frac{1}{\sqrt{2}} \right)_{\frac{w}{\sqrt{a}}} =$$

$$= \frac{1}{\sqrt{a}} \frac{1}{\sqrt{2}}\, e^{-\frac{1}{4}\frac{w^2}{a}} = \frac{1}{\sqrt{2a}}\, e^{-\frac{w^2}{4a}}$$

<u>EXERCISE</u> Show

$$\mathcal{F}^{-1}\left(\frac{1}{\sqrt{2a}}\, e^{-\frac{w^2}{4a}}\right)(x) = e^{-ax^2}$$

CONVOLUTION

$$(f*g)(x) = \int_{-\infty}^{+\infty} f(y)\,g(x-y)\,dy$$
$$= \int_{-\infty}^{+\infty} f(x-y)\,g(y)\,dy = (g*f)(x)$$

$$\mathcal{F}(f*g)(\omega) = \sqrt{2\pi}\,\mathcal{F}(f)(\omega)\,\mathcal{F}(g)(\omega)$$

$$(f*g)(t) = \int_{0}^{t} f(s)\,g(t-s)\,ds$$
$$= \int_{-\infty}^{+\infty} f(s)\,g(t-s)\,ds$$

because for LT we assumed that the functions were zero when $t<0$.

Chapter 4: Partial Differential Equations

Notation:

- $u = u(x_1, \ldots, x_n)$ $\quad$ x_i are variables in $\mathbb{R}$

$$u_{x_i} = \frac{\partial u}{\partial x_i} = \partial_{x_i} u$$
$$u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j} = \partial_{x_i x_j} u$$

$u_{x_i x_j} = u_{x_j x_i}$

- $x_1, x_2 \longrightarrow x, y$ or x, t
- $x_1, x_2, x_3 \longrightarrow x, y, z$

Examples

1) WAVE EQUATION IN 1D

$$u_{tt} - c^2 u_{xx} = 0 \qquad x \in \mathbb{R},\ t > 0$$

2) Heat equation in 1D

$$u_t - k u_{xx} = 0 \qquad x \in \mathbb{R},\ t > 0 \qquad k > 0$$

3) LAPLACE EQUATION / POISSON EQUATION

$$u_{xx} + u_{yy} = 0 \qquad (x,y) \in \mathbb{R}^2$$

$$u_{xx} + u_{yy} + u_{zz} = 0 \qquad (x,y,z) \in \mathbb{R}^3$$

$$n = 2 \qquad \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}(x,y) = D^2 u\,(x,y)$$

NOTE $\quad u_{xx} + u_{yy} = \operatorname{trace}\left(D^2 u(x,y)\right)$

$$u_{xx} + u_{yy} = f(x,y)$$

Classification of PDEs

1) ORDER OF AN EQUATION: it is the order of the highest derivative of the PDE

2) LINEAR EQUATION

The PDE is linear if u and its partial derivatives enter in a linear way.

If it is NOT the case the PDE is said nonlinear.

Examples

i) $u_t - u_{xx} = 0$ $\quad 2^{nd}$ order, linear

ii) $u_t + u u_x = u_{xx}$ $\quad 2^{nd}$ order, nonlinear

iii) $u_{tt} - c^2 u_{xxxx} = 4$ $\quad 4^{th}$ order, linear

iv) $x^2 u_x + y\, u_y + \sin(u^2) = 0$ $\quad 1^{st}$ order, nonlinear.

3) LINEAR HOMOGENEOUS PDE

It is a LINEAR PDE where each term contains either u or its partial derivatives

EXAMPLES

i) $u_x + u_y = 0$ $\quad$ linear homogeneous

ii) $u_x + u_y = x^2$ $\quad$ linear nonhomogeneous

Second order linear PDEs in two variables

$$A u_{xx} + 2B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G \quad (1)$$

A, B, C, .. are given functions in (x,y).

For $(x,y) \in \mathbb{R}^2$

$$d(x,y) = A(x,y)C(x,y) - B^2(x,y)$$

$$\det \begin{pmatrix} A(x,y) & B(x,y) \\ B(x,y) & C(x,y) \end{pmatrix} = d(x,y)$$

(1) is PARABOLIC in (x,y) if $d(x,y)=0$

(1) is ELLIPTIC in (x,y) if $d(x,y)>0$

(1) is HYPERBOLIC in (x,y) if $d(x,y)<0$

$$au^2 + 2buv + cv^2 + du + ev + f = 0$$

$ac - b^2 = 0 \rightarrow$ parabola

$ac - b^2 > 0 \rightarrow$ ellipse

$ac - b^2 < 0 \rightarrow$ hyperbola

Examples

① WAVE EQUATION

$$u_{tt} - c^2 u_{xx} = 0$$

$x \rightarrow t$

$y \rightarrow x$ $\quad \det \begin{pmatrix} A=1 & 0 \\ 0 & C=-c^2 \end{pmatrix} = -c^2 < 0$

$\Rightarrow$ hyperbolic

② Heat equation

$$u_t - k\,u_{xx} = 0$$

$$\det\begin{pmatrix} 0 & 0 \\ 0 & -k \end{pmatrix} = 0$$

$\Rightarrow$ parabolic

③ LAPLACE EQUATION

$$u_{xx} + u_{yy} = 0$$

$$\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 > 0$$

$\Rightarrow$ ELLIPTIC

④ EULER-TRICOMI EQUATION

$$y\,u_{xx} + u_{yy} = 0$$

$$\det\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = y = \begin{cases} \text{elliptic} & y > 0 \\ \text{parabolic} & y = 0 \\ \text{hyperbolic} & y < 0 \end{cases}$$

REMARK

Consider the equation (1) in the case $G \equiv 0$. If u_1, u_2 are solutions of

$$A u_{xx} + 2B u_{xy} + C u_{yy} + D u_x + E u_y + F u = 0 \quad (2)$$

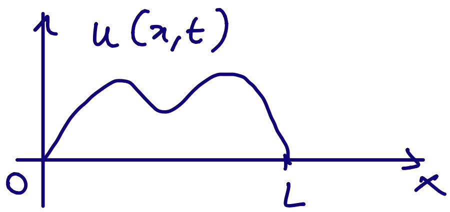
Then $\alpha u_1 + \beta u_2$ is a solutions of (2).
$\forall \alpha, \beta \in \mathbb{R}$

$\Rightarrow$ SO-CALLED SUPERPOSITION PRINCIPLE.

Fourier series solution of the 1D wave equation

$$u_{tt} - c^2 u_{xx} = 0 \qquad x \in (0, L), \ t > 0$$

We assume that the string of length L is fastened at the ends $x = 0$, $x = L$



$$u(0, t) = 0 \quad , \quad u(L, t) = 0 \qquad \forall t$$

$$u(x, 0) = f(x) \qquad u_t(x, 0) = g(x)$$

$$(P) \begin{cases} u_{tt} - c^2 u_{xx} = 0 & x \in (0, L), \ t > 0 \\ u(0, t) = u(L, t) = 0 & t \geq 0 \quad \text{BOUNDARY CONDITIONS (BC)} \\ \left.\begin{array}{l} u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{array}\right\} & \text{INITIAL CONDITIONS (IC)} \end{cases}$$

Method of separation of the variables

Step 1 Look for "PRODUCT SOLUTIONS" of the form

$$u(x,t) = F(x)\,G(t)$$

$\Rightarrow$ we obtain from the PDE two ODEs one for F and one for G.

Step 2

We determine solutions of these ODEs that satisfy the (BC)

Step 3

By using Fourier series you compose the solutions found in Step 2 to obtain a solution of the PDE satisfying both (BC) and (IC).

Step 1

"Product solutions"

$$u(x,t) = F(x)\,G(t)$$

$$u_{tt}(x,t) = F(x)\,G''(t)$$

$$u_{xx}(x,t) = F''(x)\,G(t)$$

$$0 = u_{tt} - c^2 u_{xx} = F(x)\,G''(t) - c^2 F''(x)\,G(t)$$

$$\Rightarrow \quad c^2 F''(x)\,G(t) = F(x)\,G''(t) \qquad (3) \quad x \in [0,L],\ t>0$$

Divide (3) by $c^2 F(x) G(t)$ $(\neq 0)$

$$\Rightarrow \quad \frac{F''(x)}{F(x)} = \frac{G''(t)}{c^2 G(t)} = k \in \mathbb{R}$$

$$\Rightarrow \quad F''(x) = k\, F(x)$$
$$G''(t) = c^2 k G(t)$$

a) $u(0,t) = 0$ $\quad F(0) G(t) = 0 \quad \forall t > 0$

$\Rightarrow \quad F(0) = 0$

b) $u(L,t) = 0$ $\quad F(L) G(t) = 0 \quad \forall t > 0$

$\Rightarrow \quad F(L) = 0$

Step 2

$$(P_F) \begin{cases} F''(x) - k F(x) = 0 & x \in (0,L) \\ F(0) = 0 \\ F(L) = 0 \end{cases}$$

$$(P_G) \quad G''(t) - k c^2 G(t) = 0$$

To solve (P_F) we have to separate the cases $k = 0$, $k > 0$, $k < 0$.

Theorem 1 Let $u_0, u_1 \colon X \to I\!R$ and $v_0, v_1 \colon Y \to I\!R$ $(X, Y \subseteq I\!R)$ be such that u_0, v_0 are not identically zero. Then

$$u_0(x)v_1(t) = u_1(x)v_0(t), \quad \forall (x,t) \in X \times Y \tag{1}$$

if and only if there exists a unique constant $\lambda \in I\!R$ such that

$$u_1(x) = \lambda u_0(x), \quad \forall x \in X$$

and

$$v_1(t) = \lambda v_0(t), \quad \forall t \in Y.$$

Proof.

1. Suppose that (1) holds. Let $\bar{x} \in X$ be such that $u_0(\bar{x}) \neq 0$. Then if we set $\lambda = \frac{u_1(\bar{x})}{u_0(\bar{x})}$ then $v_1(t) = \lambda v_0(t), \quad \forall t \in Y$. Moreover if $v_0(\bar{t}) \neq 0$ then $\lambda = \frac{v_1(\bar{t})}{v_0(\bar{t})}$ and therefore from (1) it follows that $u_1(x) = \lambda u_0(x), \quad \forall x \in X$ as well.

2. On the other hand if $u_1(x) = \lambda u_0(x), \quad \forall x \in X$ and $v_1(t) = \lambda v_0(t), \quad \forall t \in Y$ then (1) trivially holds. $\Box$

This Theorem is related to the existence of the separation constant in the method of the separation of the variables. During the lecture we have applied it when we found the two ODEs associated to the wave equation

$$G''(t)F(x) = c^2 G(t)F''(x), \quad \forall x \in [0, L], t > 0.$$

In this case $X = (0, L)$, $Y = I\!R_+$**,**$u_0 = F$**,** $u_1 = F''$**,** $v_0 = G(t)$**,** $v_1 = G''(t)$ **.....**