LECTURE ANALYSIS 3, 19.10.2020

1. f P-periodic, $\alpha \in \mathbb{R} \Rightarrow f(\alpha x)$ is $\frac{P}{|\alpha|}$ periodic

SOLUTION $\alpha > 0$ (the proof for $\alpha < 0$ is the same)

Let T be the period of $f(\alpha x)$

$\Rightarrow f(\alpha(x+T)) = f(\alpha x) \quad \forall x \in \mathbb{R}$

$f(\alpha(x+T)) = f(\alpha x + \alpha T) = f(\alpha x) \ \forall x$

$\Rightarrow \alpha T = P \Rightarrow T = \frac{P}{\alpha}$

2. f, g continuous of period $P_1, P_2 \Rightarrow f+g$ is periodic $\Leftrightarrow \frac{P_1}{P_2} \in \mathbb{Q}$

VERIFICATION

"$\Leftarrow$" $\quad \frac{P_1}{P_2} = \frac{m}{n} \qquad m, n \geq 0$

$n P_1 = m P_2 = T$

CLAIM: $f+g$ is T-periodic

$(f+g)(x+T) = f(x+T) - g(x+T)$

$= f(x + m P) + g(x + m P_2)$

$= f(x) + g(x)$

$\blacksquare$

EXERCISE

(1) $\sin(x) + \sin(\pi x)$ is NOT periodic

$f(x) = \sin(x)$, $\quad g(x) = \sin(\pi x)$

$P_1 = 2\pi \qquad P_2 = \frac{2\pi}{\pi} = 2$

$$\frac{P_1}{P_2} = \frac{2\pi}{2} = \pi \notin \mathbb{Q}$$

(2) $$\left(\sin(x) + \sin(\pi x)\right)'' = \left(\cos(x) + \pi\cos(\pi x)\right)'$$
$$= -\sin(x) - \pi^2\sin(\pi x)$$

Assume by contradiction that (1) is periodic of period T

(3) $\sin(x+T) + \sin(\pi(x+T)) = \sin(x) + \sin(\pi x)$

(4) $\sin(x+T) + \pi^2\sin(\pi(x+T)) = \sin(x) + \pi^2\sin(\pi x)$

$$\cancel{\sin(x+T)} + \sin(\pi(x+T)) - \cancel{\sin(x+T)} - \pi^2\sin(\pi(x+T))$$
$$= \cancel{\sin(x)} + \sin(\pi x) - \sin(x) - \pi^2\sin(\pi x)$$

$\Rightarrow \cancel{(1-\pi^2)}\sin(\pi(x+T)) = (1-\pi^2)\sin(\pi x)$

$\Rightarrow \pi T = 2m\pi \Rightarrow T = 2m \qquad m > 0$

(3) $\sin(x+T) + \sin\left(\cancel{\pi(x+T)}\right) = \sin(x) + \cancel{\sin(\pi x)}$

$\Rightarrow \sin(x+T) = \sin(x) \quad \forall x \in \mathbb{R}$

$T = 2m\pi$

$\Rightarrow$ CONTRADICTION

AIM APPROXIMATE 2L PERIODIC FUNCTIONS by "SIMPLE" 2L-PERIODIC FUNCTIONS

$$A = \left\{ \sin\left(\frac{n\pi}{L}x\right), \cos\left(\frac{n\pi}{L}x\right), n \geq 0 \right\}$$

A IS CALLED A TRIGONOMETRIC SYSTEM.

The period of each function in A is

$$P_n = \frac{2\cancel{\pi}}{n\frac{\cancel{\pi}}{L}} = \frac{2L}{n}$$

FROM SERIE 1

1) $$\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \\ 2L & n = m = 0 \end{cases}$$

2) $$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \end{cases}$$

3) $\displaystyle\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = 0 \quad \forall n, m \geq 0$

DEF A TRIGONOMETRIC POLYNOMIAL IS A FUNCTION OF THE FORM

$$a_0 + \sum_{n=1}^{N}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right] \quad N \geq 1$$

DEF A TRIGONOMETRIC SERIES IS A FUNCTION OF THE FORM

$$a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right]$$

a_n, b_n are called COEFFICIENTS

$$s_N(x) = a_0 + \sum_{n=1}^{N}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right]$$

REMARK If $f(x)$ is represented by a Fourier series:

$$(*)\quad f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right]$$

$\Rightarrow$ $f(x)$ has to be $2L$-periodic as well.

RELATION BETWEEN f, a_n, b_n

i) $\displaystyle a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

ii) $\displaystyle a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \quad n \geq 1$

iii) $b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi}{L}x\right)dx \quad m \geq 1$

VERIFICATION

i) Integrate both sides of * between $-L, L$

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} a_0\,dx + \int_{-L}^{L} \sum_{m=1}^{\infty}\left[a_m \cos\left(\frac{m\pi}{L}x\right) + b_m \sin\left(\frac{m\pi}{L}x\right)\right]dx$$

$$= a_0\, 2L + \sum_{m=1}^{\infty} a_m \underbrace{\int_{-L}^{L}\cos\left(\frac{m\pi}{L}x\right)dx}_{=0} + b_m \underbrace{\int_{-L}^{L}\sin\left(\frac{m\pi}{L}x\right)dx}_{=0}$$

$$\Rightarrow a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

ii) Multiply * by $\cos\left(\frac{m\pi}{L}x\right)$ AND integrate between $-L, L$

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right)dx = \underbrace{\cancel{\int_{-L}^{L} a_0 \cos\left(\frac{m\pi}{L}x\right)dx}}_{=0}$$

$$+\sum_{n=1}^{\infty}\left[a_n \int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx\right.$$

$$\left. + \; b_n \underbrace{\cancel{\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx}}_{=0}\right] =$$

$= a_m L$ (see ORTHOGONALITY RELATIONS)

$$\Rightarrow a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right)dx \qquad \forall m \geq 1$$

EX $\displaystyle\int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right)dx = 0$!

iii) EXERCISE multiply $(*)$ by $\sin\left(\frac{m\pi}{L}x\right)$ $(m \geq 1)$, INTEGRATE between $-L, L$ and USE ORTHOG. RELATIONS)

The coefficients a_0, a_m, b_m $(m \geq 1$ given by the above formulas are called FOURIER COEFFICIENT of $f(x)$

Given :
$f(x)$ 2L - periodic $\Rightarrow$ compute the coefficients a_0, a_n, b_n by the ABOVE FORMULAS (EULER FORMULAS)
Construct the TRIGONOMETRIC SERIES WITH SUCH COEFFICIENS:

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right) + b_n \sin\left(\frac{n\pi}{L} x\right)$$

FOURIER SERIES OF f.

Main question: can we write:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right) + b_n \sin\left(\frac{n\pi}{L} x\right)$$

Some considerations :

For every $n \geq 1$ we consider the following function:

$$f_n(x) = a_n \cos\left(\frac{n\pi}{L} x\right) + b_n \sin\left(\frac{n\pi}{L} x\right)$$

$$= \sqrt{a_n^2 + b_n^2} \left(\frac{a_n}{\sqrt{a_n^2 + b_n}} \cos\left(\frac{n\pi}{L} x\right) + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin\left(\frac{n\pi}{L} x\right) \right) = \boxed{*}$$

$A_n = \sqrt{a_n^2 + b_n^2}$, Φ_n angle such that

$$\sin \Phi_n = \frac{b_n}{A_n} \quad , \quad \cos \Phi_n = \frac{a_n}{A_n}$$

$$\tan \Phi_n = \frac{b_n}{a_n}$$

$\boxed{*}$

$$= A_n \left(\cos \Phi_n \cos\left(\frac{n\pi}{L} x\right) + \sin(\Phi_n) \sin\left(\frac{n\pi x}{L}\right) \right)$$

$$= A_n \cos\left(\frac{n\pi}{L} x - \Phi_n\right)$$

HARMONIC OSCILLATION with AMPLITUDE A_n and ANGULAR FREQUENCY $\omega_n = n\,\omega_1$ WHERE $\omega_1 = \frac{\pi}{L}$ IS THE FUNDAMENTAL FREQUENCY

$\{A_n\}$ DISCRETE SPECTRUM ASSOCIATED TO f.

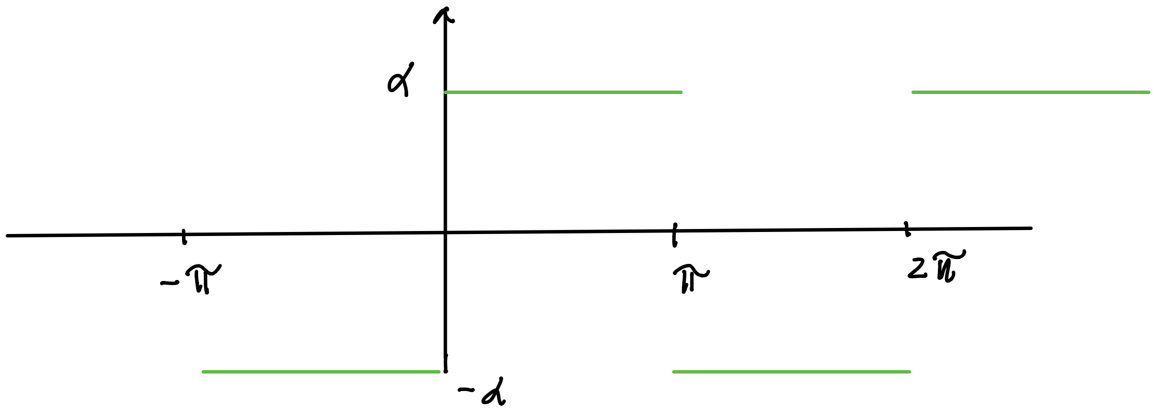
$$\lim_{n \to +\infty} A_n = 0$$

Basic example: Square Wave

$$f(x) = \begin{cases} -\alpha & x \in (-\pi, 0) \\ \alpha & x \in (0, \pi) \end{cases} \qquad \alpha > 0$$

$$f(x + 2\pi) = f(x)$$

Note: the value at x=0 do not affect the computations of the coefficients,so we can leave f(0) indefinite.



FOURIER COEFFICIENTS $(2L = 2\pi)$

i) $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} -\alpha\,dx + \frac{1}{2\pi}\int_{0}^{\pi} \alpha\,dx$

$= \frac{1}{2\pi}(-\alpha)\pi + \frac{1}{2\pi}\alpha.\pi = 0$

ii) $a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{0} -\alpha\cos(mx)\,dx + \frac{1}{\pi}\int_{0}^{\pi}\alpha\cos(mx)\,dx$

$= \frac{1}{\pi}(-\alpha)\left.\frac{\sin(mx)}{m}\right|_0^{\pi} + \frac{1}{\pi}\alpha\left.\frac{\sin(mx)}{m}\right|_0^{\pi} = 0$

iii) $b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{0}(-\alpha)\sin(mx)\,dx + \frac{1}{\pi}\int_{0}^{\pi}\alpha\sin(mx)\,dx$

$$= \frac{1}{\pi}(+\alpha)\left(+\frac{\cos(m x)}{m}\right)\Big|_{-\pi}^{0} + \frac{\alpha}{\pi}\left(-\frac{\cos(m x)}{m}\right)\Big|_{0}^{\pi}$$

$$= \frac{+\alpha}{\pi}\left[\frac{1-\cos(-m\pi)}{m}\right] + \frac{(-\alpha)}{\pi}\left[\frac{\cos(m\pi)-1}{m}\right]$$

$$= \frac{2\alpha}{\pi}\left[\frac{1-\cos(m\pi)}{m}\right]$$

Recall that:

$$\cos(m\pi) = \cos(-m\pi)$$

$$b_m = \frac{2\alpha}{\pi}\left[\frac{1-\cos(m\pi)}{m}\right] = \frac{2\alpha}{\pi}\left[\frac{1-(-1)^m}{m}\right]$$

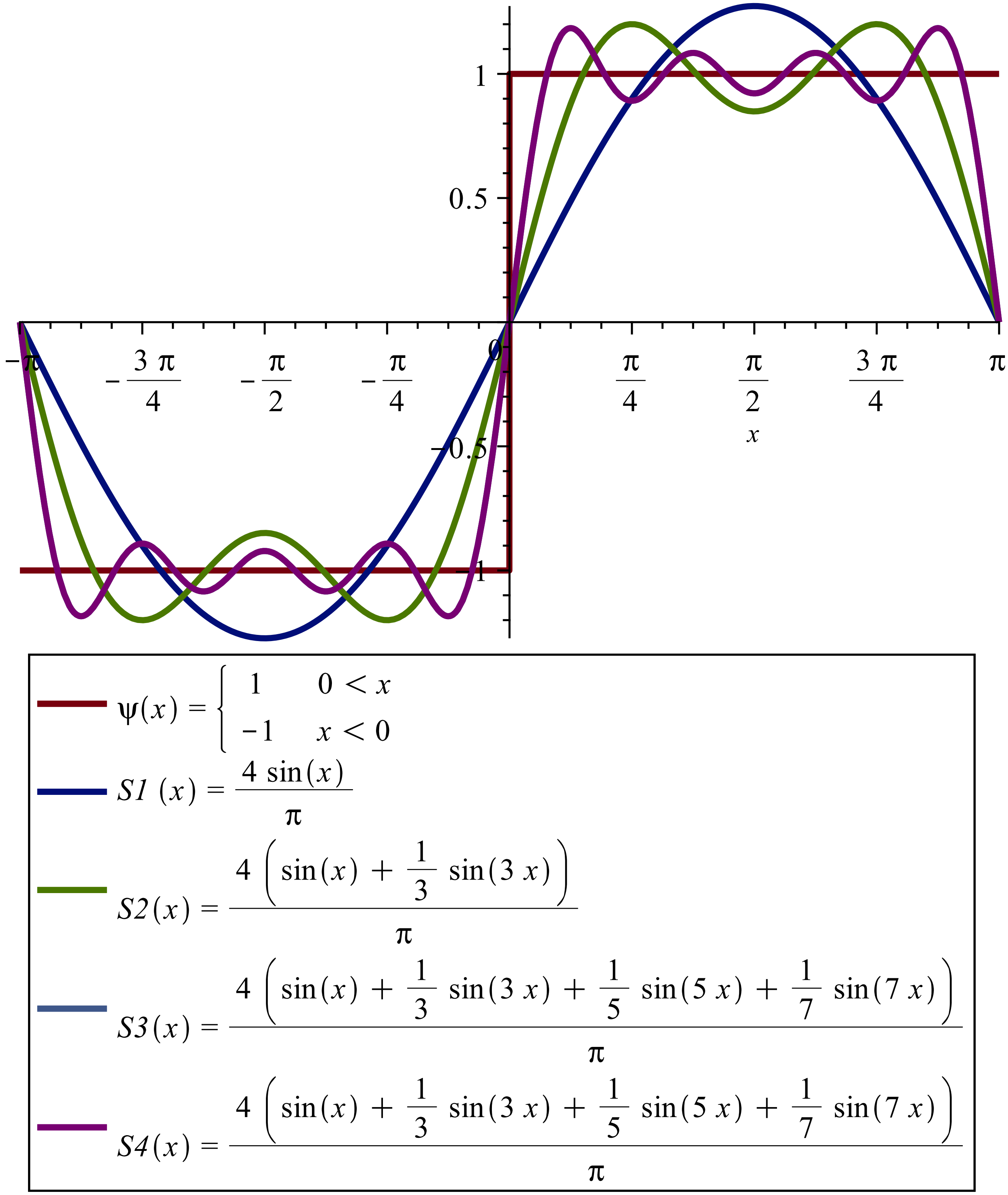
$$\cos(m\pi) = \begin{cases} 1 & \text{if } m = 2k \quad k \geq 0 \\ -1 & \text{if } m = 2k+1 \quad k \geq 0 \end{cases}$$

$$\Rightarrow \cos(m\pi) = (-1)^m$$

$$\Rightarrow b_m = \begin{cases} 0 & m \text{ is even} \\ \frac{2\alpha}{\pi}\left[\frac{1-(-1)}{m}\right] = \frac{4\alpha}{\pi m} & m \text{ is odd} \end{cases}$$

The Fourier series associated to $f(x)$ is given by

$$f(x) \sim \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4\alpha}{\pi n}\sin(n x) = \sum_{m=0}^{\infty} \frac{4\alpha}{\pi(2m+1)}\sin(2m+1)x$$



1
0.5
−0.5
−1
$-\pi$
$-\frac{3\pi}{4}$
$-\frac{\pi}{2}$
$-\frac{\pi}{4}$
0
$\frac{\pi}{4}$
$\frac{\pi}{2}$
$\frac{3\pi}{4}$
π
x
$\psi(x) = \begin{cases} 1 & 0 < x \\ -1 & x < 0 \end{cases}$
$S1\,(x) = \frac{4\sin(x)}{\pi}$
$S2(x) = \frac{4\left(\sin(x) + \frac{1}{3}\sin(3x)\right)}{\pi}$
$S3(x) = \frac{4\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x)\right)}{\pi}$
$S4(x) = \frac{4\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x)\right)}{\pi}$