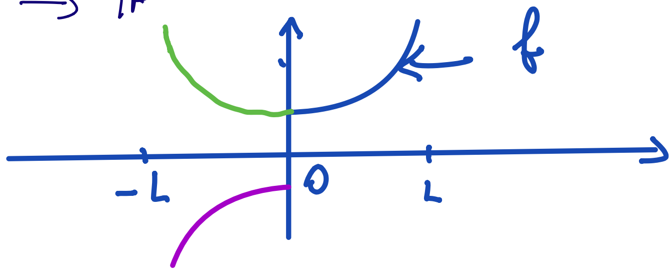
Lecture Analysis 3, 2 November 2020

LAST WEEK : Half - range expansions

$f : [0, L] \to \mathbb{R}$



- even $2L$ - periodic extension $\to$ FS contains ONLY cos terms
- odd $2L$ - periodic extension $\to$ FS contains ONLY sin terms

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COMPLEX FOURIER SERIES

f $2L$ - periodic

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)$$

Euler formulas

$\forall\ t \in \mathbb{R}$

① $e^{it} = \cos t + i \sin t$

② $e^{i(-t)} = e^{-it} = \cos(-t) + i \sin(-t)$
$= \cos t - i \sin t$

① + ② $\Rightarrow$ $\cos t = \dfrac{e^{it} + e^{-it}}{2}$

① − ② $\Rightarrow$ $\sin t = \dfrac{e^{it} - e^{-it}}{2i}$

$\forall n \geq 1 \quad t_n = \left(\frac{n\pi}{L} x\right)$

$a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)$

$= a_n \left(\dfrac{e^{it_n} + e^{-it_n}}{2}\right) + b_n \left(\dfrac{e^{it_n} - e^{-it_n}}{2i}\right)$

$= e^{it_n}\left(\frac{a_n}{2} + \frac{b_n}{2i}\right) + e^{-it_n}\left(\frac{a_n}{2} - \frac{b_n}{2i}\right)$

$= e^{it_n} \quad c_n \quad + \quad e^{-it_n} \quad k_n$

$c_n = \frac{1}{2}\left(a_n + \frac{b_n}{i}\right) = \frac{1}{2}\left(a_n + i\frac{b_n}{i^2}\right) = \frac{1}{2}\left(a_n - i b_n\right)$

$k_n = \frac{1}{2}\left(a_n - \frac{b_n}{i}\right) = \frac{1}{2}\left(a_n + i\, b_n\right)$

$c_n = \frac{1}{2}\frac{1}{L}\int_{-L}^{L} f(x)\left[\cos\left(\frac{n\pi}{L}x\right) - i \sin\left(\frac{n\pi}{L}x\right)\right] dx$

$= \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\frac{n\pi}{L}x}\, dx$

$k_n = \frac{1}{2}\frac{1}{L}\int_{-L}^{L} f(x)\left[\cos\left(\frac{n\pi}{L}x\right) + i \sin\left(\frac{n\pi}{L}x\right)\right] dx$

$= \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{i\frac{n\pi}{L}x}\, dx$

$c_0 := a_0$

$$f(x) \sim \underbrace{c_0}_{1} + \underbrace{\sum_{n=1}^{\infty} c_n e^{i\frac{n\pi}{L}x}}_{2} + \sum_{n=1}^{\infty} k_n e^{-i\frac{n\pi}{L}x}$$

We set $c_{-n} = k_n$

$$\sum_{n=1}^{\infty} k_n e^{-i\frac{n\pi}{L}x} = \sum_{n=1}^{\infty} c_{-n} e^{-i\frac{n\pi}{L}x} \underset{m=-n}{=} *$$

If $1 \leq n < +\infty \Rightarrow -1 \geq -n > -\infty$

$$* \; \sum_{-\infty}^{-1} c_m e^{i\frac{m\pi}{L}x} = \underbrace{\sum_{-\infty}^{-1} c_n e^{i\frac{n\pi}{L}x}}_{3}$$

$$f(x) \sim 1 + 2 + 3 = \sum_{n=-\infty}^{+\infty} c_n e^{i\frac{n\pi}{L}x}$$

If $n = 0$

$$c_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = a_0$$

If $n \geq 1$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\frac{n\pi}{L}x}\,dx$$

If $n \leq -1$

$$c_n = k_{-n} = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\frac{n\pi}{L}x}\,dx$$

$$\Rightarrow \forall n \in (-\infty, +\infty) \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\frac{n\pi}{L}x}\,dx$$

REMARK

1) $c_0 = a_0$

$$\left.\begin{aligned} &\text{a)}\quad c_n = \frac{a_n - i b_n}{2} \\ &\text{b)}\quad c_{-n} = \frac{a_n + i b_n}{2} \end{aligned}\right\} \quad n > 0$$

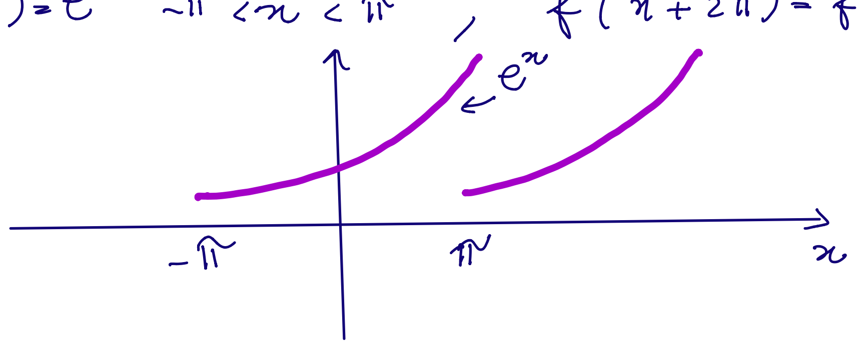
$$c_{-n} = \bar{c}_n$$

2)
$$a_n = c_n + c_{-n} = c_n + \bar{c}_n = 2\,\mathrm{Re}(c_n)$$
$$b_n = i\,(c_n - c_{-n}) = i\,(i2\,\mathrm{Im}(c_n)) = -2\,\mathrm{Im}(c_n)$$

EXAMPLE

(see EXAMPLE 3.14 in Iozzi's NOTES)

$f(x) = e^x \quad -\pi < x < \pi \quad , \quad f(x + 2\pi) = f(x)$



$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x} e^{-inx}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x(1-in)}\, dx$$

$$= \frac{1}{2\pi} \left. \frac{e^{x(1-in)}}{(1-in)} \right|_{-\pi}^{\pi} = \frac{1}{2\pi} \frac{1}{(1-in)} \underbrace{\left[e^{\pi(1-in)} - e^{-\pi(1-in)} \right]}$$

observe that

1) $e^{\pm i n\pi} = \underbrace{\cos(\pm n\pi)} + i\,\cancel{\sin(\pm n\pi)} = (-1)^n$

2) $\dfrac{1}{1-in}\cdot\dfrac{1+in}{1+in} = \dfrac{1+in}{1+n^2}$

$$c_n = \frac{(-1)^n}{\pi}\cdot\frac{1+in}{1+n^2}\left(\frac{e^{\pi}-e^{-\pi}}{2}\right) = \frac{(-1)^n}{\pi}\,\frac{1+in}{1+n^2}\sinh\pi$$

$$e^x \sim \frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{+\infty}(-1)^n\frac{1+in}{(1+n^2)}e^{inx}$$

$\forall x\in(-\pi,\pi)$ you can replace $\sim$ with $=$

$$n\geq 1\begin{cases} a_n = c_n+\overline{c_n} = 2\,\mathrm{Re}(c_n) = (-1)^n\dfrac{1}{(1+n^2)}\,\dfrac{2\sinh\pi}{\pi} \\ b_n = -2\,\mathrm{Im}(c_n) = (-1)\cdot(-1)^n\dfrac{n}{(1+n^2)}\,\dfrac{2\sinh\pi}{\pi} \end{cases}$$

$$= (-1)^{n+1}\frac{n}{(1+n^2)}\,\frac{2\sinh\pi}{\pi}$$

$$f(x)\sim\frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{1+n^2}\cos(nx) - \frac{n}{1+n^2}\sin(nx)\right)$$

Fourier Integrals

FS $\longrightarrow$ periodic functions

$f_L : \mathbb{R} \to \mathbb{R}$ $\quad 2L$-periodic

$$(3)\quad f_L(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(w_n x) + b_n \sin(w_n x)$$

$$w_n = n \cdot w_1 \qquad w_1 = \frac{\pi}{L}$$

Let $L \to +\infty$

$$f_L(x) = \underbrace{\frac{1}{2L}\int_{-L}^{L} f_L(y)\,dy}_{= a_0} + \sum_{n=1}^{\infty} \cos(w_n x)\,\frac{1}{L}\int_{-L}^{L} f(y)\cos(w_n y)\,dy$$

$$+ \sin(w_n x)\,\frac{1}{L}\int_{-L}^{L} f(y)\sin(w_n y)\,dy$$

$$\Delta w = w_{n+1} - w_n = (n+1)\frac{\pi}{L} - n\frac{\pi}{L} = \frac{\pi}{L}$$

$$\frac{1}{L} = \frac{\Delta w}{\pi}$$

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(y)\,dy + \frac{1}{\pi}\sum_{n=1}^{\infty} \cos(w_n x)\,\Delta w \int_{-L}^{L} f_L(y)\cos(w_n y)\,dy$$

$$+ \sin(w_n x)\,\Delta w \int_{-L}^{L} f_L(y)\sin(w_n y)\,dy$$

If $L \to +\infty \Rightarrow \Delta w \to 0$

Observe that the series can be "considered" heuristically the Riemann sum of:

$$w \mapsto \cos(wx)\int_{-L}^{L} f_L(y)\cos(wy)\,dy + \sin(wx)\int_{-L}^{L} f_L(y)\sin(wy)\,dy$$

We make the following ASSUMPTION.

$$\begin{cases} \lim\limits_{L\to+\infty} f_L(x) = f(x) \\ \lim\limits_{L\to+\infty} \left|\int_{-L}^{L} f_L(x)\,dx\right| \leq \int_{-\infty}^{+\infty} |f(x)|\,dx < +\infty \end{cases}$$

$$\Rightarrow \underbrace{\lim_{L\to+\infty}\left|\frac{1}{L}\int_{-L}^{L} f_L(x)\,dx\right|}_{=0} \leq \lim_{L\to+\infty}\frac{1}{L}\underbrace{\int_{-\infty}^{+\infty}|f(x)|\,dx}_{<+\infty}$$

THEREFORE when $L\to+\infty$

$$f(x) = \frac{1}{\pi}\int_0^{+\infty}\cos(wx)\left(\int_{-\infty}^{+\infty} f(y)\cos wy\,dy\right)dx$$
$$+\frac{1}{\pi}\int_0^{+\infty}\sin(wx)\left(\int_{-\infty}^{+\infty} f(y)\sin(wy)\,dy\right)dx$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(y)\cos(wy)\,dy$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(y)\sin(wy)\,dy$$

$$\Rightarrow f(x) \overset{(?)}{=} \int_0^{+\infty}\left[A(w)\cos(wx) + B(w)\sin(wx)\right]dw$$

Fourier integral representation of a given function f

$f: \mathbb{R} \to \mathbb{R}$ piecewise continuous in every bounded interval $I \subseteq \mathbb{R}$, f has right-hand derivative and left-hand derivative at every point $x \in \mathbb{R}$ and if $\int_{-\infty}^{+\infty} |f(x)|\, dx < +\infty$

(f is absolutely integrable)

Then

$$f(x) = \int_0^{+\infty} [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]\, d\omega$$

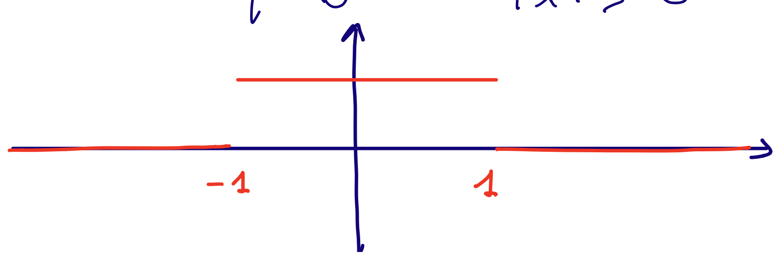
$\forall x \in \mathbb{R}$ where f is continuous.
Otherwise if x is a point of discontinuity for f, we have

$$\frac{f(x^+) + f(x^-)}{2} = \int_0^{+\infty} A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\, d\omega$$

$$f(x^+) = \lim_{h \to 0^+} f(x+h)\ ; \quad f(x^-) = \lim_{h \to 0^-} f(x+h)$$

EXAMPLE

$$f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 0 \end{cases}$$



$$\int_{-\infty}^{+\infty} |f(x)|\, dx = 2 < +\infty$$

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(y) \cos(wy)\, dy$$

$$= \frac{1}{\pi} \int_{-1}^{1} \cos(wy)\, dy = \frac{1}{\pi} \frac{\sin(wy)}{w} \Big]_{-1}^{1}$$

$$= \frac{2}{\pi} \frac{\sin w}{w}$$

$$B(w) = \frac{1}{\pi} \int_{-1}^{1} \sin(wy)\, dy = 0$$

It follows that $\forall x \neq -1, 1$

$$f(x) = \frac{2}{\pi} \underbrace{\int_0^{+\infty} \frac{\sin(w)}{w} \cdot \cos(wx)\, dw} = \begin{cases} 1 & |x|<1 \\ 0 & |x|>1 \\ \frac{1}{2} & |x|=1 \end{cases}$$

$$\underbrace{\int_0^{+\infty} \frac{\sin(w)}{w} \cos(wx)\, dw} = \begin{cases} \frac{\pi}{2} & |x|<1 \\ 0 & |x|>1 \\ \frac{\pi}{4} & |x|=1 \end{cases}$$

Dirichlet's DISCONTINUOUS FACTOR

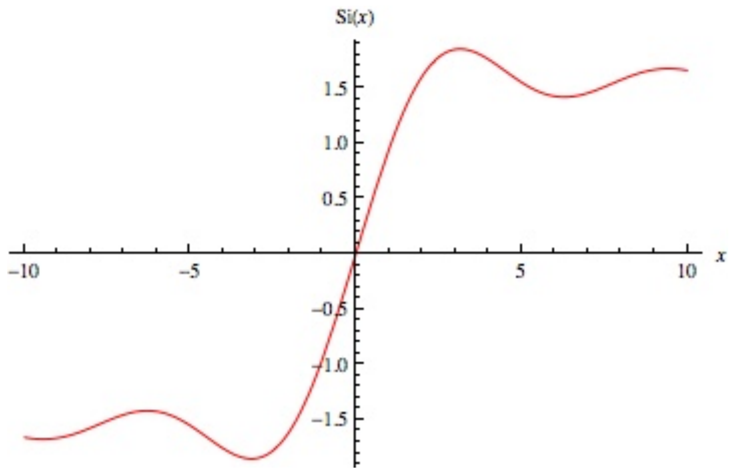
Take $x = 0$

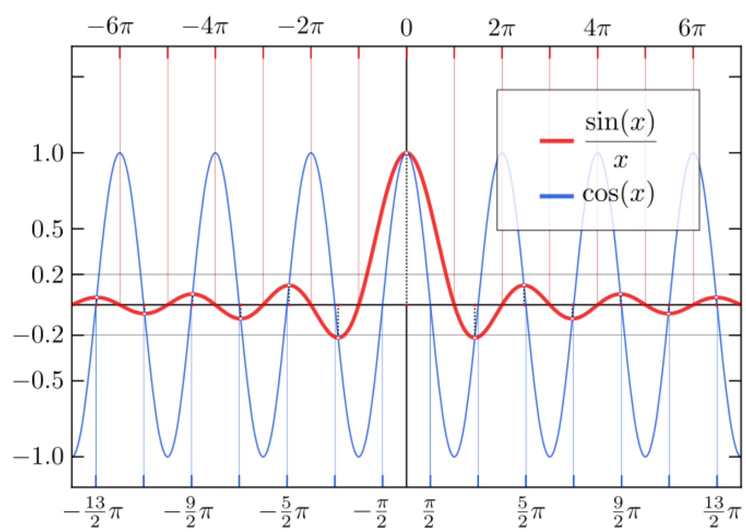
$$\underbrace{\int_0^{+\infty} \frac{\sin(w)}{w}\, dw}_{4} = \frac{\pi}{2}$$

4 can be seen as the limit of the so-called SINE INTEGRAL

$$Si(u) = \int_0^u \frac{\sin(w)}{w}\, dw$$

as $u \to +\infty$





EXERCISE

a) Derive the Fourier integral of $f(x) = e^{-k|x|}$ $x \in \mathbb{R}$, $k > 0$.

b) Deduce that

$$\int_0^{+\infty} \frac{\cos(wx)}{k^2 + w^2}\, dw = \frac{\pi}{2k} e^{-kx} \quad x > 0$$