LECTURE 21.09.2020
ANALYSIS 3 D-MAVT,D-MATL

# CHAPTER 1: LAPLACE TRANSFORM

## 1.1 DEFINITION AND BASIC PROPERTIES OF LAPLACE TRANSFORM

DEFINITION 1. $f: \mathbb{R} \to \mathbb{R}$. The Laplace transform of $f$ is the function defined by:

$$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^{+\infty} f(t)\, e^{-st}\, dt$$ ■

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REMARK

a) For the original functions → lower letters
For the corresponding LT → capital letters

b) Control systems: domain of original functions → TIME DOMAIN $(t)$
domain of LT → FREQUENCY DOMAIN $(s)$

c) The def of LT involves only the values of $f$ for $t \geq 0$ ⇒ we can always suppose w.l.o.g that $f(t) = 0$ for $t < 0$

d) Computing indefinite integrals

$$\int_0^{+\infty} a(t)\, dt = \lim_{T \to +\infty} A(t)\Big|_0^T = \lim_{T \to +\infty} \left[A(T) - A(0)\right]$$

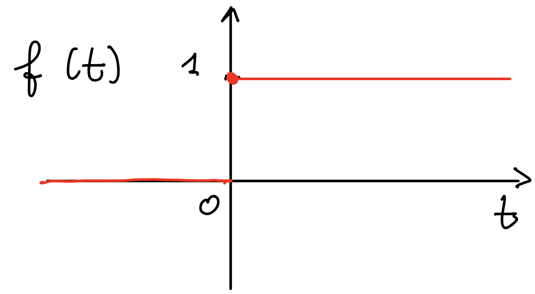
$A'(t) = a(t)$

$A(t)$ is an antiderivative of $a(t)$

## EXAMPLES

### 1) Heaviside Function

$$f(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$




→ signal that switches on at $t = 0$

$$\mathcal{L}\{f(t)\}(s) = \int_0^{+\infty} e^{-st}\, 1 \, dt = \lim_{T \to +\infty} \left( \frac{e^{-st}}{-s} \right) \Bigg|_0^T$$

$$= \lim_{T \to +\infty} \left( \frac{e^{-sT} - 1}{-s} \right) = \begin{cases} +\infty & \text{if } s \leq 0 \\ \frac{1}{s} & \text{if } s > 0 \end{cases}$$

- If $s = 0 \Rightarrow \int_0^{+\infty} dt = \lim_{T \to +\infty} T = +\infty$
- If $s < 0 \Rightarrow -sT > 0 \Rightarrow \lim_{T \to +\infty} -sT = +\infty$

$$\Rightarrow \lim_{T \to +\infty} \left( \frac{e^{-sT} - 1}{-s} \right) = +\infty$$

- If $s > 0 \Rightarrow \lim_{T \to +\infty} -sT = -\infty \Rightarrow \lim_{T \to +\infty} e^{-sT} = 0$

$$\Rightarrow \lim_{T \to +\infty} \left( \frac{e^{-sT} - 1}{-s} \right) = \left( \frac{-1}{-s} \right) = \frac{1}{s} \quad \blacksquare$$

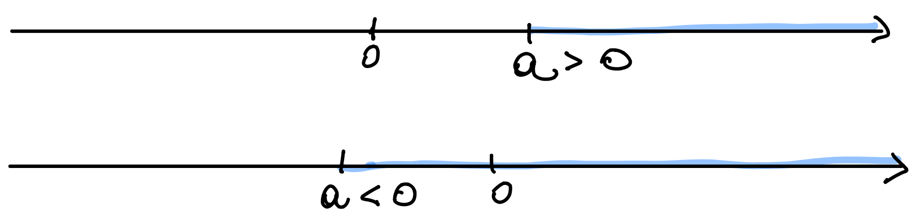
**2) Exponential Function**

$$f(t) = \begin{cases} e^{at} & a \in \mathbb{R} \quad t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}\{f(t)\}(s) = \int_0^{+\infty} \underbrace{e^{at} \cdot e^{-st}}_{e^{(a-s)t}} \, dt$$

$$= \left. \frac{e^{(a-s)t}}{(a-s)} \right|_0^{+\infty} = \lim_{T \to +\infty} \left( \frac{e^{(a-s)T} - 1}{a-s} \right)$$

$$= \begin{cases} +\infty & \text{if } s \leq a \\ \dfrac{1}{s-a} & \text{if } s > a \end{cases}$$




In the case $a < 0$ the domain of definition of the Laplace Transform contains the half line $[0, +\infty)$

**3) Power Functions**

$$f(t) = \begin{cases} t^n & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad \boxed{n \geq 1}$$

$$\mathcal{L}\{t^n\}(s) = \int_0^{+\infty} t^n \underline{e^{-st}}\, dt = \underbrace{\frac{t^n e^{-st}}{-s}\Big|_0^{+\infty}}_{} + \int_0^{+\infty} n t^{n-1} \frac{e^{-st}}{(+s)}\, dt$$

$$= \underbrace{\lim_{T \to +\infty} \left( \frac{T^n e^{-sT} - 0}{-s} \right)}_{(*)} + \frac{n}{s} \underbrace{\int_0^{+\infty} t^{n-1} e^{-st}\, dt}_{(**)}$$

(*) exists finit $\Leftrightarrow$ $s > 0$ and (*) $= 0$

$$= \frac{n}{s} (**) = \frac{n}{s} \mathcal{L}\{t^{n-1}\}$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots \frac{\overbrace{n-(n-1)}^{=1}}{s} \mathcal{L}\{t^0\}$$

$$= \frac{n!}{s^n} \cdot \mathcal{L}\{1\} = \frac{n!}{s^n} \cdot \frac{1}{s} = \frac{n!}{s^{n+1}}$$

These computions hold

$\Leftrightarrow$ $s > 0$

## BASIC PROPERTIES

### a) Linearity

$f, g, \quad \alpha, \beta \in \mathbb{R}$

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}$$

"Proof"

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\}(s) = \int_0^{+\infty} (\alpha f(t) + \beta g(t)) e^{-st} dt$$

$$= \alpha \int_0^{+\infty} f(t) e^{-st} dt + \beta \int_0^{+\infty} g(t) e^{-st} dt$$

$$= \alpha \mathcal{L}\{f(t)\}(s) + \beta \mathcal{L}\{g(t)\}(s) \quad \blacksquare$$

EXAMPLES

1) $f(t) = 2t^2 - 8$

$$\mathcal{L}\{f(t)\}(s) = 2\mathcal{L}\{t^2\}(s) - 8\mathcal{L}\{1\}(s)$$

$$= 2 \frac{2!}{s^3} - 8 \frac{1}{s} = \frac{4}{s^3} - \frac{8}{s}$$

2) $f(t) = \sinh(t) = \frac{1}{2}(e^t - e^{-t})$

$$\mathcal{L}\{\sinh(t)\}(s) = \frac{1}{2} \underbrace{\mathcal{L}\{e^t\}}_{a=1}(s) - \frac{1}{2} \underbrace{\mathcal{L}\{e^{-t}\}}_{a=-1}(s)$$

$$= \frac{1}{2} \frac{1}{s-1} - \frac{1}{2} \frac{1}{s-(-1)} =$$

$$= \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1} = \frac{1}{2}\frac{\cancel{s}+1-\cancel{s}+1}{s^2-1} = \frac{1}{s^2-1}$$

NOTE : $\mathcal{L}\{e^t\}$ exists for $s>1$
$\mathcal{L}\{e^{-t}\}$ exists for $s>-1$

$\Rightarrow$ We have to take the intersection of the two sets : $\boxed{s>1}$

NOTE : $\mathcal{L}\{f(t)\cdot g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$

Take $f = g = 1$

$$\mathcal{L}\{1\} = \frac{1}{s} \neq \mathcal{L}\{1\}\cdot\mathcal{L}\{1\} = \frac{1}{s^2}$$

**b) EXISTENCE**

Conditions on a function $f(t)$ that guarantee that its LT exists for at least large values of $s$.

DEFINITION 2

a function $f(t)$ is called of exponential order (with constant $c \geq 0$) if

$$|f(t)| \leq M e^{ct} \quad \forall t \geq 0$$

EXPONENTIAL GROWTH CONDITION $\quad M > 0$

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**EXAMPLES**

1) Every bounded function is of exponential order
$f(t) = 1$, $f(t) = \cos(at)$, $f(t) = \sin(at)$
$a \in \mathbb{R}$.

EXERCISE $\mathcal{L}\{\cos(at)\}(s) = \dfrac{s}{s^2 + a^2}$

$$\mathcal{L}\{\sin(at)\}(s) = \frac{a}{s^2 + a^2}$$

2) $f(t) = t^n$ $n \geq 1$ is of exponential order

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \Rightarrow \text{if } t \geq 0 \Rightarrow \frac{t^n}{n!} \leq e^t$$

$$\Rightarrow t^n \leq n!\, e^t \quad t \geq 0 \quad (c = 1,\ M = n!)$$

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**THEOREM** (see Remark 2.6 in Iozzi's NOTES)

If $f(t)$ is <u>piecewise continuous</u> and of exponential order with constant $c \geq 0$, then $F(s) = \mathcal{L}\{f(t)\}$ exists and it is finite for $\boxed{s > c}$

"Proof"

By hyp. $|f(t)| \leq M e^{ct}$

$$|\mathcal{L}\{f(t)\}(s)| = \left|\int_0^{+\infty} f(t) e^{-st} dt\right|$$

$$\leq \int_0^{+\infty} |f(t) e^{-st}| dt$$

$$= \int_0^{+\infty} |f(t)| e^{-st} dt$$

$$\leq \int_0^{+\infty} M e^{ct} e^{-st} dt = \frac{M}{s-c}$$

$$\Leftrightarrow s > c$$

$$|\mathcal{L}\{f(t)\}(s)| \leq \frac{M}{s-c} \qquad s > c$$

$$\Rightarrow \lim_{s \to +\infty} |F(s)| = 0 \qquad \blacksquare$$

**Table of Laplace Transforms**

| | $f(t)=\mathfrak{L}^{-1}\{F(s)\}$ | $F(s)=\mathfrak{L}\{f(t)\}$ | | $f(t)=\mathfrak{L}^{-1}\{F(s)\}$ | $F(s)=\mathfrak{L}\{f(t)\}$ |
|---|---|---|---|---|---|
| 1. | $1$ | $\frac{1}{s}$ | 2. | $\mathbf{e}^{at}$ | $\frac{1}{s-a}$ |
| 3. | $t^n, \quad n=1,2,3,\ldots$ | $\frac{n!}{s^{n+1}}$ | 4. | $t^p, p > -1$ | $\frac{\Gamma(p+1)}{s^{p+1}}$ |
| 5. | $\sqrt{t}$ | $\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$ | 6. | $t^{n-\frac{1}{2}}, \quad n=1,2,3,\ldots$ | $\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$ |
| 7. | $\sin(at)$ | $\frac{a}{s^2+a^2}$ | 8. | $\cos(at)$ | $\frac{s}{s^2+a^2}$ |
| 9. | $t\sin(at)$ | $\frac{2as}{(s^2+a^2)^2}$ | 10. | $t\cos(at)$ | $\frac{s^2-a^2}{(s^2+a^2)^2}$ |
| 11. | $\sin(at)-at\cos(at)$ | $\frac{2a^3}{(s^2+a^2)^2}$ | 12. | $\sin(at)+at\cos(at)$ | $\frac{2as^2}{(s^2+a^2)^2}$ |
| 13. | $\cos(at)-at\sin(at)$ | $\frac{s(s^2-a^2)}{(s^2+a^2)^2}$ | 14. | $\cos(at)+at\sin(at)$ | $\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$ |
| 15. | $\sin(at+b)$ | $\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$ | 16. | $\cos(at+b)$ | $\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$ |
| 17. | $\sinh(at)$ | $\frac{a}{s^2-a^2}$ | 18. | $\cosh(at)$ | $\frac{s}{s^2-a^2}$ |
| 19. | $\mathbf{e}^{at}\sin(bt)$ | $\frac{b}{(s-a)^2+b^2}$ | 20. | $\mathbf{e}^{at}\cos(bt)$ | $\frac{s-a}{(s-a)^2+b^2}$ |
| 21. | $\mathbf{e}^{at}\sinh(bt)$ | $\frac{b}{(s-a)^2-b^2}$ | 22. | $\mathbf{e}^{at}\cosh(bt)$ | $\frac{s-a}{(s-a)^2-b^2}$ |
| 23. | $t^n\mathbf{e}^{at}, \quad n=1,2,3,\ldots$ | $\frac{n!}{(s-a)^{n+1}}$ | 24. | $f(ct)$ | $\frac{1}{c}F\left(\frac{s}{c}\right)$ |
| 25. | $u_c(t)=u(t-c)$ Heaviside Function | $\frac{\mathbf{e}^{-cs}}{s}$ | 26. | $\delta(t-c)$ Dirac Delta Function | $\mathbf{e}^{-cs}$ |
| 27. | $u_c(t)f(t-c)$ | $\mathbf{e}^{-cs}F(s)$ | 28. | $u_c(t)g(t)$ | $\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$ |
| 29. | $\mathbf{e}^{ct}f(t)$ | $F(s-c)$ | 30. | $t^n f(t), \quad n=1,2,3,\ldots$ | $(-1)^n F^{(n)}(s)$ |
| 31. | $\frac{1}{t}f(t)$ | $\int_s^\infty F(u)\,du$ | 32. | $\int_0^t f(v)\,dv$ | $\frac{F(s)}{s}$ |
| 33. | $\int_0^t f(t-\tau)g(\tau)\,d\tau$ | $F(s)G(s)$ | 34. | $f(t+T)=f(t)$ | $\frac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$ |
| 35. | $f'(t)$ | $sF(s)-f(0)$ | 36. | $f''(t)$ | $s^2F(s)-sf(0)-f'(0)$ |
| 37. | $f^{(n)}(t)$ | $s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$ | | | |