Lecture Analysis 3-23.11.2020

Last time:

- **Computation of Fourier Transform of the Gaussian by completion of the squares**
- **Introduction of method of separation of the variables**

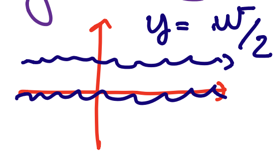
REMARK

- $\mathcal{F}(e^{-x^2})(w) = e^{-\frac{w^2}{4}} \frac{\sqrt{2}}{2} \int_{-\infty}^{+\infty} e^{-(x+\frac{iw}{2})^2} dx$

Then the fact that

$$\int_{-\infty}^{+\infty} e^{-(x+\frac{iw}{2})^2} dx = \int_{-\infty}^{+\infty} e^{-x^2} dx$$

namely it is the same to integrate e^{-z^2} along $\{y=0\}$ and along the line $\{y = \frac{w}{2}\}$.



- We considered:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } (0,L)\times\{t>0\} \\ \left.\begin{array}{l} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{array}\right\} & (IC) \\ u(0,t) = u(L,t) = 0 & (BC) \end{cases}$$

  - We looked for product solutions

$$u(x,t) = F(x)\,G(t)$$
$$u_{tt}(x,t) = F(x)\,G''(t)$$
$$u_{xx}(x,t) = F''(x)\,G(t)$$

$$0 = u_{tt} - c^2 u_{xx} = F(x)\,G''(t) - c^2 F''(x)\,G(t)$$

$$\Rightarrow \quad c^2 F''(x)\,G(t) = F(x)\,G''(t) \quad (3) \quad x \in [0,L],\ t > 0$$

Divide (3) by $c^2 F(x)\,G(t)$ $(\neq 0)$

$$\Rightarrow \quad \frac{F''(x)}{F(x)} = \frac{G''(t)}{c^2 G(t)} =: k \in \mathbb{R}$$

$$\Rightarrow \quad F''(x) = k\,F(x)$$
$$G''(t) = c^2 k\,G(t)$$

a) $u(0,t) = 0 \qquad F(0)\,G(t) = 0 \qquad \forall\, t > 0$

$\Rightarrow \quad F(0) = 0$

b) $u(L,t) = 0 \qquad F(L)\,G(t) = 0 \qquad \forall\, t > 0$

$\Rightarrow \quad F(L) = 0$

<u>Step 2</u>

$$(P_F)\begin{cases} F''(x) - k\,F(x) = 0 & x \in (0,L) \\ F(0) = 0 \\ F(L) = 0 \end{cases}$$

$(P_G)\ G''(t) - kc^2 G(t) = 0$

To solve (P_F) we have to separate the cases $k=0$, $k>0$, $k<0$.

$\boxed{\text{NOTE}}$

The determination of k is part of the problem!

PLAN OF TODAY:

- **We conclude the solution of the initial-boundary value problem for the 1D wave equation in a bounded interval**
- **We introduce the D` Alembert Formula which gives the solution to the wave equation in the real line.**

Three cases

"$k=0$"
$$\begin{cases} F''(x) = 0 \\ F(0) = F(L) = 0 \end{cases}$$

$F(x) = Ax + B$ $\quad A, B \in \mathbb{R}$

$F(0) = 0 \Rightarrow B = 0$

$F(L) = 0 \Rightarrow AL = 0 \Rightarrow A = 0$

NON INTERESTING CASE

"$k>0$" $\Rightarrow F(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}$ $\quad A, B \in \mathbb{R}$

$F(0) = 0 \Rightarrow$
$F(L) = 0 \Rightarrow$
$$\begin{cases} A + B = 0 \\ Ae^{\sqrt{k}L} + Be^{-\sqrt{k}L} = 0 \end{cases}$$

$A = -B$

$$A e^{\sqrt{k}L} - A e^{-\sqrt{k}L} = A(\underbrace{e^{\sqrt{k}L} - e^{-\sqrt{k}L}}_{\neq 0 \quad (L \neq 0)}) = 0 \Rightarrow A = 0 \Rightarrow B = 0$$

NON INTERESTING CASE

"$k<0$" $\Rightarrow$ $F(x) = A\cos(\sqrt{-k}\,x) + B\sin(\sqrt{-k}\,x)$

$F(0) = 0 \Rightarrow A = 0$

$F(L) = 0 \Rightarrow B\sin(\sqrt{-k}\,L) = 0$

$B = 0$ $\qquad$ $\sin(\sqrt{-k}\,L) = 0$

$\Leftrightarrow \sqrt{-k}\,L = n\pi \quad n > 0$

$\Leftrightarrow \sqrt{-k} = \frac{n\pi}{L} \Leftrightarrow k = -\left(\frac{n\pi}{L}\right)^2$

For every $n>0$ there is the NONTRIVIAL SOLUTION (up to constants):

$$F_n(x) = \sin\left(\frac{n\pi}{L}x\right)$$

For $k_n = -\left(\frac{n\pi}{L}\right)^2$ we look for the solution

$$G''(t) = -c^2\left(\frac{n\pi}{L}\right)^2 G(t)$$

$$G''(t) + \underbrace{c^2\left(\frac{n\pi}{L}\right)^2}_{=:\lambda_n > 0} G(t) = 0$$

We get

$$G_n(t) = B_n \cos\left(\underbrace{\frac{cn\pi}{L}}_{\sqrt{\lambda_n} =: \mu_n} t\right) + B_n^* \sin\left(\frac{cn\pi}{L}t\right)$$

For every $n>0$

$$u_n(x,t)=\left[B_n \cos(\mu_n t)+B_n^* \sin(\mu_n t)\right]\sin\left(\frac{n\pi}{L}x\right)$$

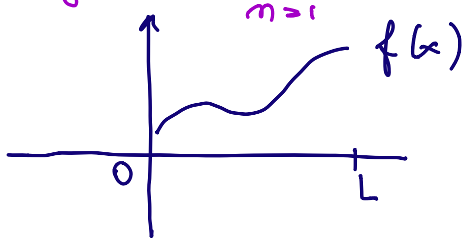
Step 3 : Use of Fourier series

$$u(x,0)=f(x),\quad u_t(x,0)=g(x)$$

$$u(x,t)=\sum_{n=1}^{\infty}\left[B_n\cos(\mu_n t)+B_n^*\sin(\mu_n t)\right]\sin\left(\frac{n\pi}{L}x\right)$$

$$u(x,0)=f(x)$$

$$\Rightarrow\ f(x)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi}{L}x\right)$$



$$B_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$\Rightarrow$ Look at half-range extensions.

$$u_t(x,0)=g(x)$$

$$g(x)=\sum_{n=1}^{\infty}\left(-B_n\sin(\mu_n t)\mu_n+B_n^*\cos(\mu_n t)\mu_n\right)\sin\left(\frac{n\pi}{L}x\right)\Big|_{t=0}$$

$$=\sum_{n=1}^{\infty}B_n^*\mu_n\sin\left(\frac{n\pi}{L}x\right)$$

$$B_n^*\mu_n=\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

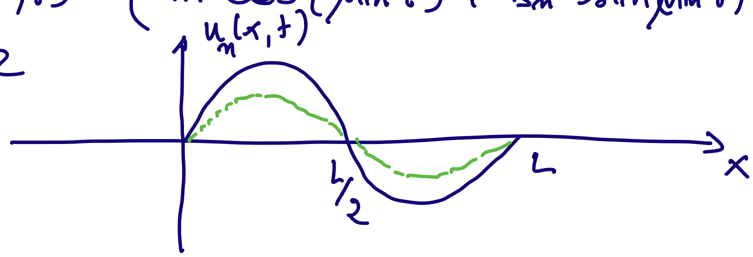
$$B_n^*=\frac{2}{L\mu_n}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Amplitude of u_n : $\sqrt{B_n^2 + B_n^{*2}}$

Frequency : $w_n = \frac{1}{T_n} = \frac{1}{\frac{2L}{nc}} = \frac{nc}{2L}$

$$u_n(x,t) = \left(B_n \cos(\mu_n t) + B_n^* \sin(\mu_n t)\right) \sin\left(\frac{n\pi}{L} x\right)$$

$n = 2$



REMARK

1) If $g = 0$ we get

$$u(x,t) = \frac{1}{2}\left(f^*(x-ct) + f^*(x+ct)\right)$$

f^* is the odd $2L$-periodic extension of f.

2) $f = 0$ we get

$$u(x,t) = \frac{1}{2L}\int_{x-ct}^{x+ct} g^*(s)\, ds$$

g^* is the odd $2L$-periodic extension of g.

$$\cos\alpha \sin\beta = \frac{1}{2}\left(\sin(\alpha-\beta) + \sin(\alpha+\beta)\right)$$

$\alpha = \mu_n t \qquad \beta = \frac{n\pi}{L} x$

Solution : $g \equiv 0$ Recalling that $\mu_n = \frac{cn\pi}{L}$ the solution becomes

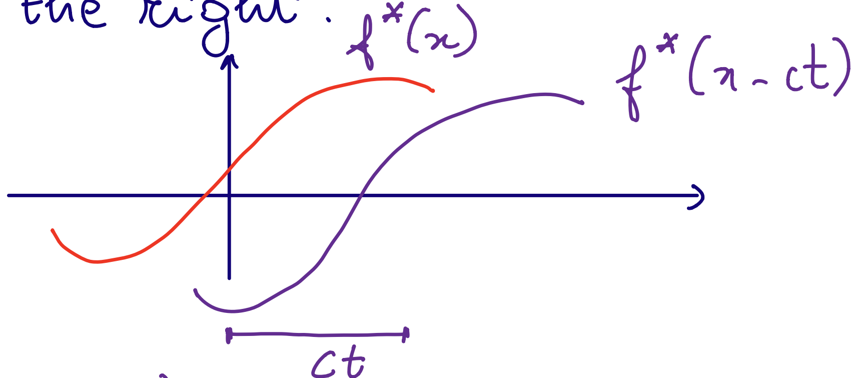
$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos(\mu_n t) \sin\left(\frac{n\pi}{L}x\right)$$

$$= \sum_{n=1}^{\infty} B_n \frac{1}{2}\left[\sin\left(\frac{n\pi}{L}x - \mu_n t\right) + \sin\left(\frac{n\pi}{L}x + \mu_n t\right)\right]$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}(x-ct)\right) + \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}(x+ct)\right)$$

$$+ \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct)\right]$$

The graph of $f^*(x-ct)$ is obtained from the graph of $f^*(x)$ by shifting the latter ct units to the right. $f^*(x-ct)$ is a "wave" that is traveling to the right.



$f^*(x+ct)$ is a "wave" traveling to the left.

4.4 D'Alembert solution of the wave equation in 1-D.

$$(P)\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } \mathbb{R} \times (0,+\infty) \\ u(x,0) = f(x) & \text{in } \mathbb{R} \\ u_t(x,0) = g(x) & \text{in } \mathbb{R} \end{cases}$$

THEOREM (D'ALEMBERT'S FORMULA)

The general solution of (P) is given by

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$$

PROOF

$$\begin{cases} v = x + ct \\ w = x - ct \end{cases} \qquad \begin{cases} x = \frac{w+v}{2} \\ t = \frac{v-w}{2c} \end{cases}$$

$$u(x,t) \longrightarrow u(v,w)$$

<u>NOTE</u> $v_x = 1 \quad v_t = c, \quad w_x = 1, \quad w_t = -c$

$$(x,t) \longrightarrow (\underbrace{x+ct}_{v}, \underbrace{x-ct}_{w}) \longrightarrow u(v,w) = u(x+ct, x-ct)$$

$$u_x = u_v \cdot \underbrace{v_x}_{=1} + u_w \cdot \underbrace{w_x}_{=1} = u_v + u_w$$

<u>Recall: Chain Rule!</u>

$$u_{xx} = (\underbrace{u_v + u_w}_{u_x})_x = (u_v + u_w)_v \underbrace{v_x}_{=1} + (u_v + u_w)_w \underbrace{w_x}_{=1}$$

$$= u_{vv} + \underbrace{u_{wv}} + \underbrace{u_{vw}} + u_{ww}$$

$$u_{vw} = u_{wv}$$

$$= u_{vv} + 2u_{vw} + u_{ww}.$$

$$u_{tt} = c^2 (u_{vv} - 2u_{vw} + u_{ww})$$

$$0 = u_{tt} - c^2 u_{xx} = c^2(\cancel{u_{vv}} - 2u_{vw} + \cancel{u_{ww}}) - c^2(\cancel{u_{vv}} + 2u_{vw} + \cancel{u_{ww}})$$

$$= -2u_{vw}$$

$$2\underbrace{\tfrac{1}{2}}_{B}(u_{vw}) = u_{vw} = 0 \rightsquigarrow \det\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} < 0$$

THE NEW EQUATION IS STILL HYPERBOLIC !

$$u_{vw} = 0 \qquad (4)$$

- Integrate (4) w.r.t w:

$$u_v(v,w) = \underbrace{\int (u_v)_w \, dw}_{=0} + h(v)$$

$$= h(v)$$

$$\left(u_{vw}(v,w) = \partial_w (h(v)) = 0 \right)$$

$$u_v(v,w) = h(v) \qquad (5)$$

- Integrate (5) w.r.t v:

$$u(v,w) = \int h(v)\,dv + \psi(w)$$

$$\left(u_v(v,w) = h(v) + 0 \right)$$

Hence

$$u(v,w) = \int h(v)\,dv + \psi(w)$$

$$= \Phi(v) + \psi(w)$$

We come back to (x,t),

$v = x+ct$, $w = x-ct$

$$u(x,t) = \Phi(x+ct) + \psi(x-ct)$$

- FIRST (IC) $u(x,0) = f(x)$

$$\Leftrightarrow \quad \Phi(x) + \psi(x) = f(x)$$

- SECOND (IC) $u_t(x,0) = g(x)$

$$\Leftrightarrow \quad c\,\Phi'(x) - c\,\psi'(x) = g(x)$$

$$\begin{cases} \Phi(x) + \psi(x) = f(x) & a) \\ \Phi'(x) - \psi'(x) = \frac{1}{c} g(x) & b) \end{cases}$$

Integrate b) between x_0 and x where x_0 is fixed.

$$\underbrace{\int_{x_0}^{x} \Phi'(s)\,ds - \int_{x_0}^{x} \psi'(s)\,ds}_{} = \frac{1}{c}\int_{x_0}^{x} g(s)\,ds$$

$$\Phi(x) - \Phi(x_0) - [\psi(x) - \psi(x_0)] = \frac{1}{c}\int_{x_0}^{x} g(s)\,ds$$

$$\begin{cases} \Phi(x) - \psi(x) = \dfrac{1}{c}\displaystyle\int_{x_0}^{x} g(s)\,ds + \underbrace{\Phi(x_0) - \psi(x_0)}_{K} & \tilde{b}) \\ \Phi(x) + \psi(x) = f(x) & a) \end{cases}$$

a) + $\tilde{b}$):

$$\cancel{2}\,\Phi(x) = \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds + \frac{1}{2}f(x) + \frac{K}{2}$$

a) − $\tilde{b}$)

$$\cancel{2}\,\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds - \frac{K}{2}$$

$$u(x,t) = \Phi(x+ct) + \psi(x-ct)$$
$$= \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)\,ds + \cancel{\frac{K}{2}}$$
$$+ \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} g(s)\,ds - \cancel{\frac{K}{2}}$$

$$= \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)\,ds$$

$$+ \frac{1}{2c}\int_{x-ct}^{x_0} g(s)\,ds$$

$$= \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds \qquad \blacksquare$$

LOOK AT WEBPAGE LUIS SILVESTRE :

ONLINE PDE SOLVERS

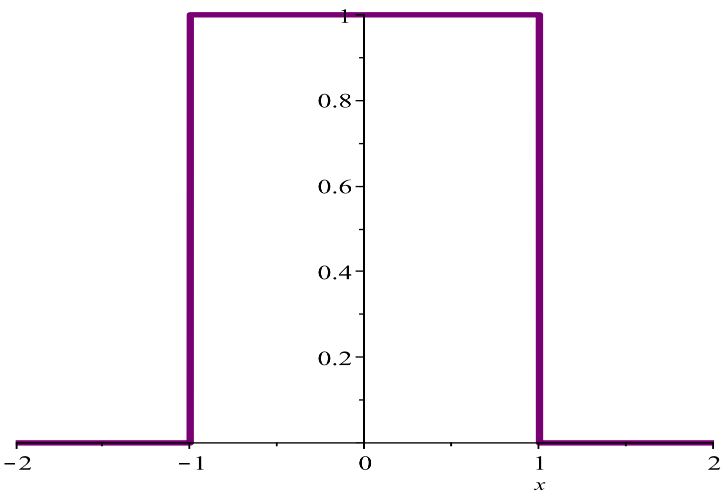
Online PDE solvers

The purpose of these pages is to help improve the student's (and professor's?) intuition on the behavior of the solutions to simple PDEs. I built them while teaching my undergraduate PDE class.

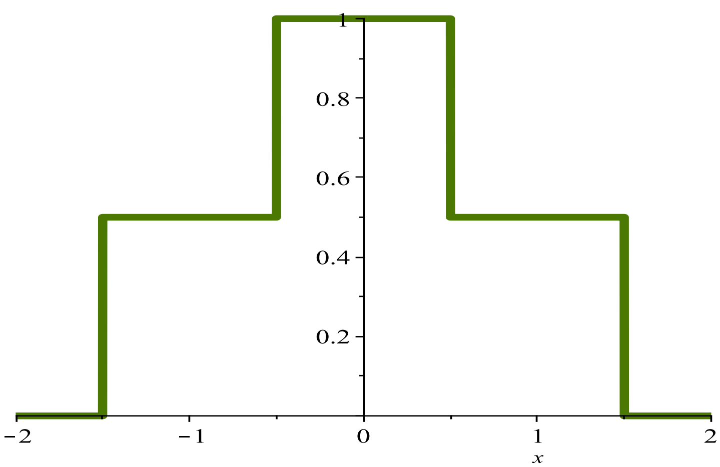
In all these pages the initial data can be drawn freely with the mouse, and then we press START to see how the PDE makes it evolve.

- **Heat equation solver.**
- **Wave equation solver.**
- **Generic solver of parabolic equations via finite difference schemes.**
  (after the last update it includes examples for the heat, drift-diffusion, transport, Eikonal, Hamilton-Jacobi, Burgers and Fisher-KPP equations)

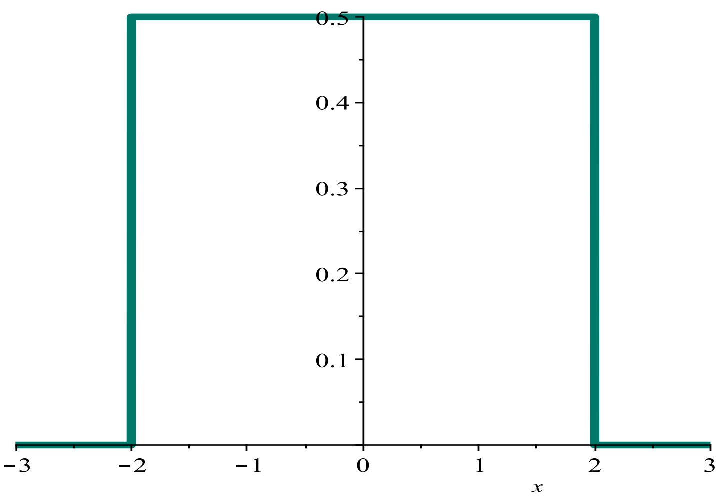
Back to Luis Silvestre's homepage



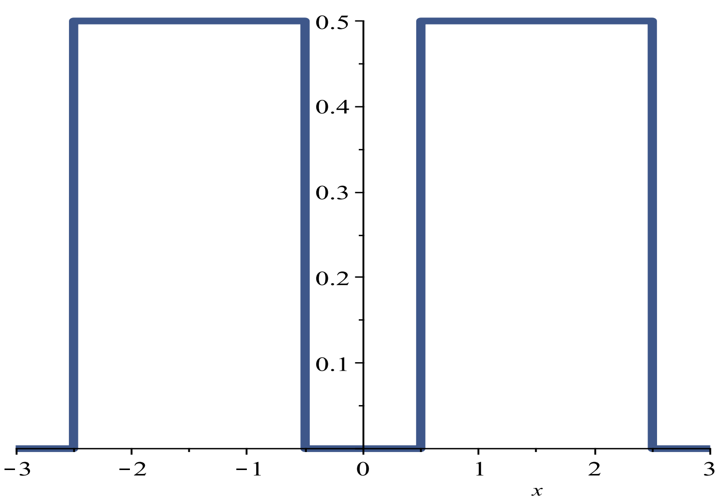
Initial Condition



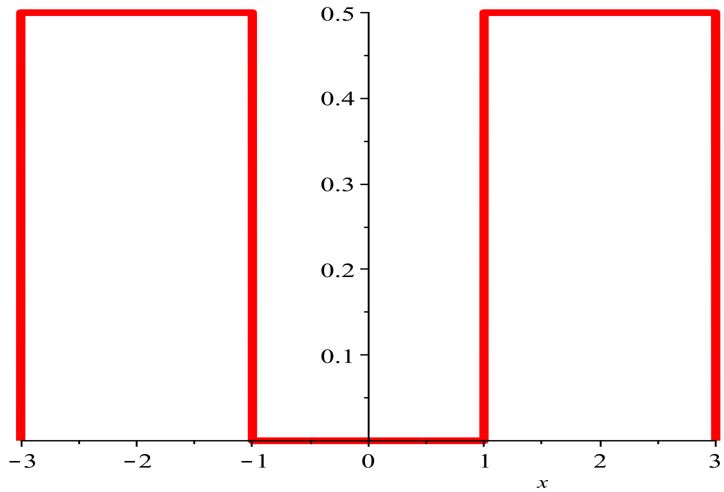
Solution at time t=1/2



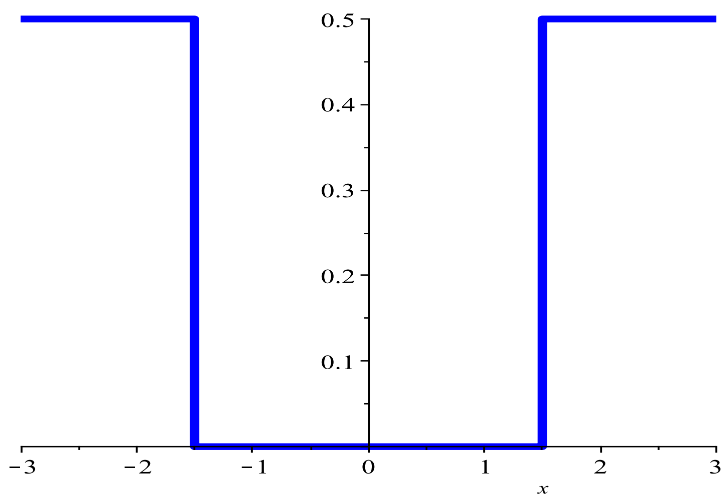
Solution at time t=1



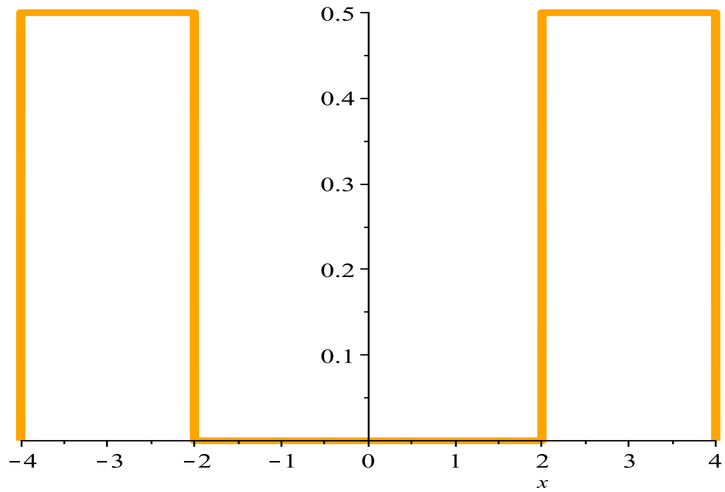
Solution at time t=3/2



Solution at time t=2



Solution at time t=5/2



Solution at t=3