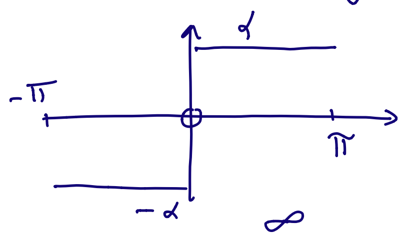
Lecture Analysis 3, 26.10.2020

Square wave

$$f(x) = \begin{cases} -\alpha & x \in (-\pi, 0) \\ +\alpha & x \in (0, \pi) \end{cases}$$



$$f(x) \sim \sum_{m=0}^{\infty} \frac{4\alpha}{\pi(2m+1)} \sin[(2m+1)x]$$

REPRESENTATION OF A FUNCTION by FOURIER SERIES

Let f be $2L$-periodic function, piecewise continuous, f has left and right derivative at each point in each interval of length $2L$

$$\left(\lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h} = f'_+(x_0), \right.$$
$$\left. \lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0)}{h} = f'_-(x_0) \right)$$

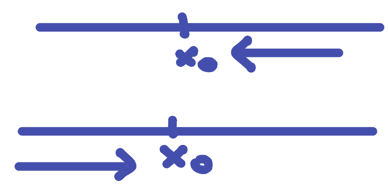
Then a) if x_0 is a point of continuity of f $\Rightarrow$ the FS converges to $f(x_0)$

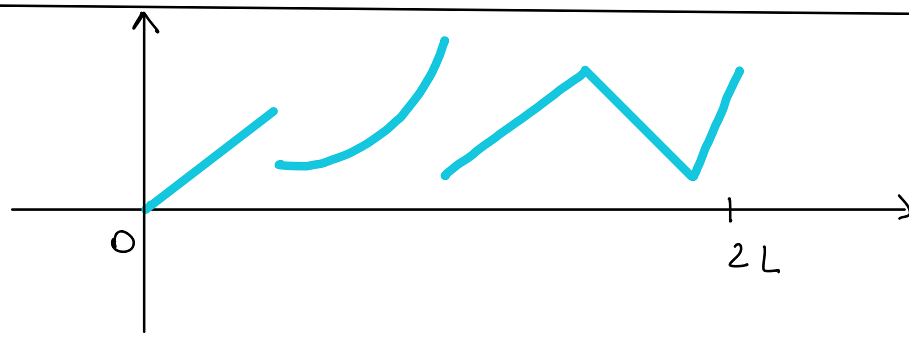
b) if x_0 is a point of discontinuity of f $\Rightarrow$ the FS converges to

$$\frac{f(x_0^+) + f(x_0^-)}{2}$$

where $f(x_0^+) = \lim\limits_{x \to x_0^+} f(x)$

$f(x_0^-) = \lim\limits_{x \to x_0^-} f(x)$

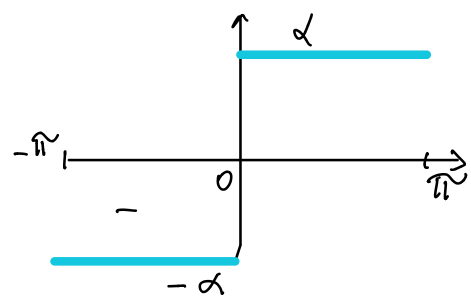




$$f(x) = \sum_{n=0}^{\infty} \frac{4\alpha}{\pi(2n+1)} \sin[(2n+1)x]$$

$f(x) = \alpha \qquad x \in (0;\pi)$

$f(x) = -\alpha \qquad x \in (-\pi, 0)$



$\forall x \in (0, \pi)$

$$\alpha = \sum_{n=0}^{\infty} \frac{4\alpha}{\pi(2n+1)} \sin[(2n+1)x]$$

$\forall x \in (-\pi, 0)$

$$-\alpha = \sum_{n=0}^{\infty} \frac{4\alpha}{\pi(2n+1)} \sin[(2n+1)x]$$

If "$x = 0$" $\Rightarrow \sin[(2n+1).0] = 0 = \frac{\alpha - \alpha}{2}$

$x = \frac{\pi}{2} \Rightarrow f(x) = \alpha$

$$\forall n : \sin\left((2n+1)\frac{\pi}{2}\right) = \underbrace{\sin\left(\frac{\pi}{2} + n\pi\right)}_{=(-1)^n} = \begin{cases} 1 & n = 2k \\ -1 & n = 2k+1 \end{cases}$$

$$\Rightarrow \cancel{\alpha} = \sum_{n=0}^{\infty} \frac{4\alpha}{\pi(2n+1)} \sin\left((2n+1)\frac{\pi}{2}\right) = \sum_{n=0}^{\infty} \frac{4\cancel{\alpha}(-1)^n}{\pi(2n+1)}$$

$$\Rightarrow \frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)}$$

$\boxed{\text{EVEN AND ODD FUNCTIONS}}$

$\boxed{\text{DEFINITION}}$ $f : \mathbb{R} \to \mathbb{R}$

1) f is EVEN if $f(-x) = f(x)$
(graph SYMMETRIC W.R.T Y-AXIS)

- $\cos(x)$, x^2, $\sin(x^2)$, $|\sin(x)|$, $e^{-|x|}$

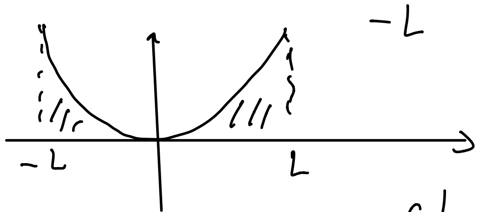
2) f is ODD if $f(-x) = -f(x)$
(graph SYMMETRIC W.R.T THE ORIGIN)

- x, x^3, $\sin(x)$, $\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\frac{x}{1+x^2}$

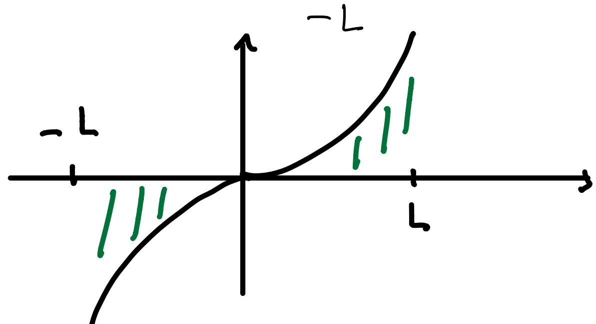
Properties

a) Product of two even/odd function is even ($+ \times + = +$, $- \times - = +$), product of an even function with a odd function is odd.

b) f even $\Rightarrow \int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx$



c) f odd $\Rightarrow \int_{-L}^{L} f(x)\,dx = 0$



THEOREM (THM 3.10 in Iozzi's NOTES)

f $2L$ periodic and assume f representable by its FS.

1) f even $\Rightarrow$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx = 0$$

2) f odd $\Rightarrow$

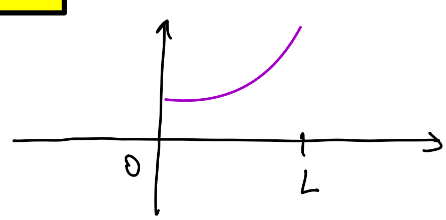
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

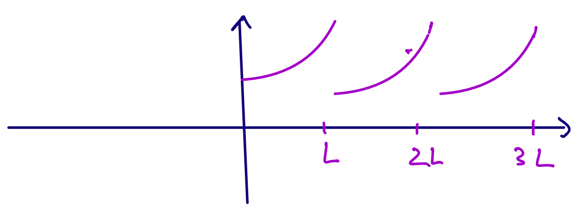
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 0$$

Half-range expansions

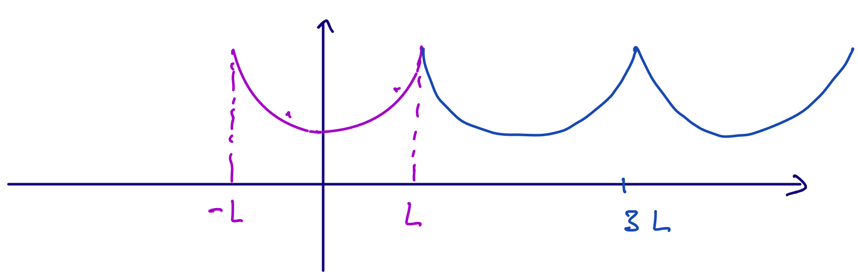
IDEA $\qquad f : [0, L] \to \mathbb{R}$



① EXTEND f as a L-periodic function, f_p $\to$ develop the extended function into its FS

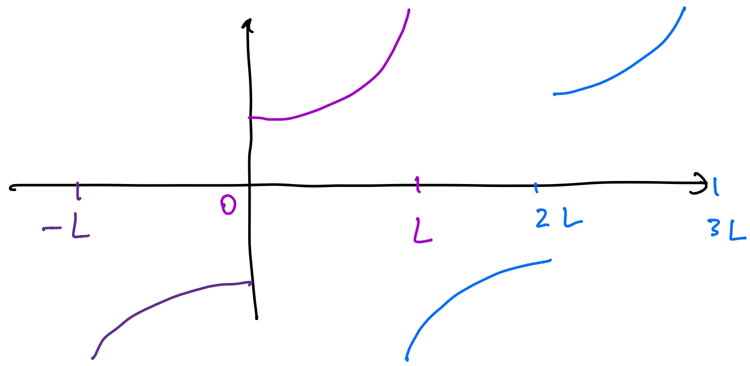


②



We consider the even $2L$-periodic extension of f, f_{ep} $\rightarrow$ the FS contains only cosine terms.

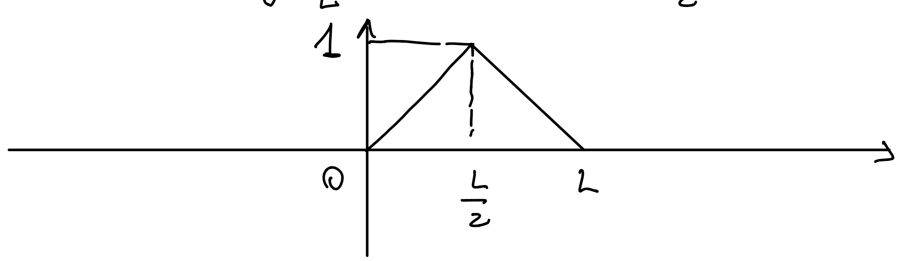
③



We consider the odd $2L$-periodic extension of f, f_{op} $\rightarrow$ the FS contains only sin terms.

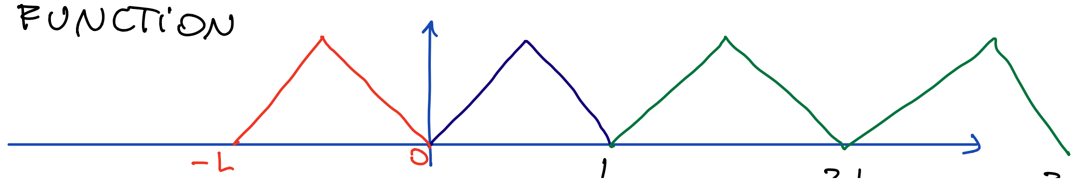
EXAMPLE

$$f(x) = \begin{cases} \frac{2}{L} x & x \in [0, \frac{L}{2}] \\ \frac{2}{L}(L - x) & x \in [\frac{L}{2}, L] \end{cases}$$



COMPUTE THE HALF-RANGE EXPANSIONS

1) EXTENSION AS AN EVEN $2L$-PERIODIC FUNCTION



NOTE: If you think f as a L-periodic function $\Rightarrow$ its FS contains cosine and sine terms and

$$w_n = n \cdot w_1 , \quad w_1 = \frac{L}{2\pi}$$

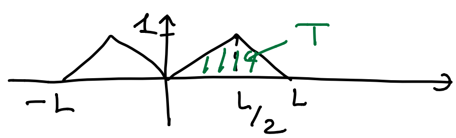
$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{L}{2\pi} n x\right) + b_n \sin\left(\frac{L n}{2\pi} x\right)$$

In the case we consider f an even $2L$-periodic function $\Rightarrow$ its FS contains only cos terms.

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(w_n x)$$

$$w_n = n \cdot w_1 \qquad w_1 = \frac{2L}{2\pi} = \frac{L}{\pi}$$

Solution



- $a_0 = \frac{1}{L} \int_0^L f(x)\,dx$

$$= \frac{1}{L}\left(\frac{L \times 1}{2}\right) = \frac{1}{2}$$

- $n \geq 1$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L} x\right) dx$$

$$= \frac{2}{L}\left[\underbrace{\int_0^{\frac{L}{2}} \frac{2}{L} x \cos\left(\frac{n\pi}{L} x\right) dx}_{1} + \underbrace{\int_{L/2}^{L} \frac{2}{L}(L-x) \cos\left(\frac{n\pi}{L} x\right) dx}_{2}\right]$$

$$① = \frac{2}{L} x \sin\left(\frac{n\pi}{L}x\right) \frac{1}{\left(\frac{n\pi}{L}\right)} \Bigg|_0^{\frac{L}{2}} - \frac{2}{\cancel{L}} \frac{\cancel{L}}{n\pi} \int_0^{L/2} \sin\left(\frac{n\pi}{L}x\right) dx$$

$$= \frac{\cancel{2}}{\cancel{L}} \frac{\cancel{L}}{\cancel{2}} \cdot \frac{L}{n\pi} \sin\left(\frac{n\pi}{L} \cdot \frac{L}{2}\right) + \frac{2}{n\pi} \cos\left(\frac{n\pi}{L}x\right) \Bigg|_0^{L/2} \cdot \frac{1}{\left(\frac{n\pi}{L}\right)}$$

$$= \frac{L}{n\pi} \sin\left(\frac{n\pi}{2}\right) + \frac{2L}{n^2\pi^2} \left[\cos\left(\frac{n\pi}{L} \cdot \frac{L}{2}\right) - 1 \right]$$

$$② = \frac{2}{L}(L-x) \sin\left(\frac{n\pi}{L}x\right) \cdot \frac{1}{\frac{n\pi}{L}} \Bigg|_{L/2}^{L} + \frac{\cancel{L}}{n\pi} \int_{L/2}^{L} \frac{2}{\cancel{L}} \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= - \frac{\cancel{2}}{\cancel{L}} \left(\frac{\cancel{L}}{\cancel{2}}\right) \sin\left(\frac{n\pi}{L} \cdot \frac{L}{2}\right) \cdot \frac{L}{n\pi} - \frac{2}{n\pi} \cos\left(\frac{n\pi}{L}x\right) \Bigg|_{L/2}^{L} \frac{1}{\left(\frac{n\pi}{L}\right)}$$

$$= - \frac{L}{n\pi} \sin\left(\frac{n\pi}{2}\right) - \frac{2L}{n^2\pi^2} \cos\left(\frac{n\pi}{L} \cdot L\right) + \frac{2L}{(n^2\pi^2)} \cdot \cos\left(\frac{n\pi}{2}\right)$$

$a_n = \frac{2}{L} \left[① + ② \right] =$ to check ....

$$= \frac{4}{n^2\pi^2} \left[2 \cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1 \right]$$

EXPRESS a_n in a "better" way

$$\cos(n\pi) = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases} \Rightarrow$$

$$\cos(n\pi) = (-1)^n$$

$$\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^k & \text{if } n \text{ is even } n = 2k \end{cases}$$

$$\cos\left(\frac{\cancel{2}k\pi}{\cancel{2}}\right) = \cos(k\pi) = (-1)^k$$

$$\begin{cases} a_m = \dfrac{4}{m^2\pi^2}\left[2\cos\left(\dfrac{m\pi}{2}\right) - \cos(m\pi) - 1\right] & m \geq 1. \\ a_0 = \dfrac{1}{2} \end{cases}$$

If $m \geq 1$ we have

$$a_m = \frac{4}{m^2\pi^2}\begin{cases} \left[0 - (-1)^{2k+1} - 1\right] = 0 & m = 2k+1 \\ \left[2(-1)^k - (-1)^{2k} - 1\right] = \begin{cases} 0 & k = 2J \\ -4 & k = 2J+1 \end{cases} \end{cases}$$

We can write

$$f(x) = \frac{1}{2} + \sum_{\substack{m=1 \\ m=2k \\ k=2J+1}}^{\infty} \frac{-16}{m^2\pi^2}\cos\left(\frac{m\pi}{L}x\right)$$

$$= \frac{1}{2} + \sum_{J=0}^{\infty} \frac{-16}{(4J+2)^2\pi^2}\cos\left(\frac{(4J+2)}{L}\pi x\right)$$

Exercise

Compute the Fourier coefficients of the 2L periodic odd extension of the triangular wave

$$f_{op}(x) = \frac{8}{\pi^2}\sum_{J=0}^{\infty}\frac{(-1)^J}{(2J+1)^2}\sin\left(\frac{(2J+1)}{2}\pi x\right)$$

$$b_m = \frac{8}{\pi^2 m^2}\sin\left(\frac{m\pi}{2}\right) = \begin{cases} 0 & m \text{ even} & m = 2k \\ (-1)^k & m \text{ odd} & m = 2k+1 \end{cases}$$