ANALYSIS 3-28.09.2020

$f : \mathbb{R} \to \mathbb{R}$, $\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^{+\infty} f(t) e^{-st} dt$

- $\mathcal{L}\{1\}(s) = \frac{1}{s}$ $s>0$
- $\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$ $s>0$, $n \geq 1$
- $\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}$ $s>a$

- LINEARITY
- EXISTENCE : f is piecewise CONTINUOUS AND $|f(t)| \leq M e^{ct}$ $t \geq 0$, $M, c \geq 0$

$\Rightarrow |\mathcal{L}\{f(t)\}(s)| \leq \frac{M}{s-c}$ $s>c$

$\Rightarrow \lim_{s \to +\infty} \mathcal{L}\{f\}(s) = 0$

USEFUL CRITERIUM

f CONTINUOUS FOR $t \geq 0$, $\lim_{t \to +\infty} \frac{f(t)}{e^{ct}}$ exists and it is finite. (for some $c \geq 0$)
THEN f is of EXPONENTIAL ORDER WITH CONSTANT c.

PROOF

$$L = \lim_{t \to +\infty} \frac{f(t)}{e^{ct}} \qquad (c \geq 0)$$

- BY DEFINITION WE CAN FIND $T>0$ SUCH THAT FOR $t \geq T$

$$\left| \frac{f(t)}{e^{ct}} \right| \leq |L|+1$$
$$\Downarrow$$
$$\longrightarrow |f(t)| \leq e^{ct}(|L|+1) \quad t \geq T$$

- f continuous $\Rightarrow$ it HAS A MAXIMUM VALUE ON THE INTERVAL $[0,T]$

$\Rightarrow \dfrac{f(t)}{e^{ct}}$ IS CONTINUOUS AS WELL

$$K = \max_{0 \leq t \leq T} \left| \frac{f(t)}{e^{ct}} \right|$$

$$\Rightarrow \longrightarrow |f(t)| \leq e^{ct} K \qquad t \in [0,T]$$

FINALLY $\quad |f(t)| \leq e^{ct} \underbrace{\max(K, |L|+1)}_{M} \quad t \geq 0$

$$f(t) = t^n \qquad \lim_{t \to +\infty} \frac{t^n}{e^t} = 0 \qquad t^n \leq e^t \quad t \geq T$$

$$t^n \leq T^n \quad \text{for } 0 \leq t \leq T$$

$$\Rightarrow \quad t^n \leq e^t \underbrace{(T^n+1)}_{M}$$

- EXAMPLE OF FUNCTION WHICH IS NOT OF EXPONENTIAL ORDER

$$f(t) = e^{t^2} : \nexists\, c \geq 0 \quad \lim_{t \to +\infty} \frac{e^{t^2}}{e^{ct}} < +\infty$$

DEFINITION f is CALLED THE INVERSE LAPLACE TRANSFORM of $F(s)$ AND WE DENOTE IT $\mathcal{L}^{-1}\{F(s)\}$ OR $\mathcal{L}^{-1}\{F(s)\}(t)$ if

$$\mathcal{L}\{f\}(t) = F(s)$$

LINEARITY

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha \mathcal{L}^{-1}\{F(s)\} + \beta \mathcal{L}^{-1}\{G(s)\}$$

$\alpha, \beta \in \mathbb{R}$

"PROOF"

F, G given, $f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$, $g(t) = \mathcal{L}^{-1}\{G(s)\}(t)$

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}$$
$$= \alpha F(s) + \beta G(s)$$

Take the INVERSE LT OF BOTH SIDES:

$$\mathcal{L}^{-1}\{\mathcal{L}\{\alpha f(t) + \beta g(t)\}\} = \mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\}$$

$$\alpha f(t) + \beta g(t) = \mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\}$$

$$\alpha \mathcal{L}^{-1}\{F(s)\} + \beta \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} \quad \blacksquare$$

HOW TO COMPUTE THE $\mathcal{L}^{-1}$

1) $F(s) = \dfrac{1}{s+4}$ Q: $\mathcal{L}^{-1}\{F(s)\} = ?$

$$\mathcal{L}\{e^{at}\} \underset{s>a}{=} \frac{1}{s-a} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$\Longrightarrow \quad a = -4 \quad \Longrightarrow \quad \mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} = e^{-4t}$$

2) $\mathcal{L}^{-1}\left\{\dfrac{8}{s^2+4}\right\} = ?$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$$

$$\frac{8}{s^2+4} = \frac{4\cdot 2}{s^2+2^2} = 4\cdot\frac{2}{s^2+a^2} = 4\cdot\mathcal{L}\{\sin(2t)\}$$

$$\mathcal{L}^{-1}\left\{\frac{8}{s^2+4}\right\} = \mathcal{L}^{-1}\{4\cdot\mathcal{L}\{\sin(2t)\}\}$$

$$= 4\,\mathcal{L}^{-1}\{\mathcal{L}\{\sin(2t)\}\} =$$

$$= 4\sin(2t)$$

3) $F(s) = \dfrac{s^2}{s^2+1}$ Q: $\mathcal{L}^{-1}\{F(s)\} = ?$ (EXISTS)

EXERCISE: Justify the FACT THAT $\mathcal{L}^{-1}\{F(s)\}$ DOES NOT EXIST, PRECISELY $F(s)$ CANNOT BE THE LT OF $f(t)$ OF EXPONENTIAL ORDER.

RATIONAL FUNCTIONS $\Longrightarrow$ DECOMPOSITION OF SUCH FUNCTIONS IN SIMPLE FRACTIONS

- $F(s) = \dfrac{s^2+s+1}{s^3+s} \overset{=}{\circledast} \dfrac{1}{s} + \dfrac{1}{s^2+1}$

$$s^3 + s = s\,(s^2+1)$$

$$\underbrace{\frac{s^2+s+1}{s^3+s}} = \underbrace{\frac{A}{s} + \frac{Bs+C}{s^2+1}} = \underbrace{\frac{A(s^2+1)+s(Bs+C)}{s(s^2+1)}}$$

$$\begin{array}{lcccl} s^2: & 1 & = & A+B & \\ s: & 1 & = & C & \Rightarrow \begin{cases} A = C = 1 \\ B = 0 \end{cases} \\ s^0: & 1 & = & A & \end{array}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{1}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= 1 + \sin(t)$$

$\square$

S- SHIFTING PROPERTY

$$\mathcal{L}\{e^{-at} f(t)\}(s) = F(s+a)$$

$$F(s) = \mathcal{L}\{f(t)\}$$

$$\mathcal{L}^{-1}\{F(s+a)\}(t) = e^{-at} f(t)$$

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1) $g(t) = e^{2t}\sin(4t)$ $\qquad \mathcal{L}\{g(t)\}(s) = ?$

SOL $\mathcal{L}\{\sin(4t)\} = \dfrac{4}{s^2+16} = F(s)$

$$\mathcal{L}\{e^{2t}\sin(4t)\} \underset{\downarrow}{=} F(s-2) = \frac{4}{(s-2)^2+16}$$

S-SHIFTING PROPERTY $a = -2$

2) $G(s) = \dfrac{1}{s^2+4s+8}$ Q: $\mathcal{L}^{-1}\{G(s)\}$

$$s^2+4s+8=0$$

$$\triangle = 4^2 - 4\cdot 8 < 0 \Rightarrow \text{NO real roots!}$$

$\Rightarrow$ write it as sum of two squares

$$\underbrace{s^2+2\cdot 2s+4}_{(s+2)^2}+4 = (s+2)^2+2^2$$

$$G(s) = \frac{1}{(s+2)^2+2^2} = F(s+2)$$

$$F(s) = \frac{1}{s^2+2^2} = \frac{1}{2}\,\underbrace{\frac{2}{s^2+2^2}}\,\overset{\swarrow}{=} \frac{1}{2}\mathcal{L}\{\sin(2t)\}$$

$$\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{F(s+2)\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\{\mathcal{L}\{\sin(2t)\}(s+2)\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\{\mathcal{L}\{e^{-2t}\sin(2t)\}(s)\}$$

$$= \frac{1}{2}e^{-2t}\sin(2t) \quad \square$$

LAPLACE TRANSFORM OF DERIVATIVES

- f continuous, differentiable (it exists the f'), of EXPONENTIAL ORDER WITH CONSTANT $c \geq 0$, f' piecewise CONTINUOUS

$$\mathcal{L}\{f'(t)\}(s) = \int_0^{+\infty} e^{-st} f'(t)\, dt$$

$$= f(t)e^{-st}\Big|_0^{+\infty} + \int_0^{+\infty} f(t)(+s)e^{-st}\, dt$$

$$= \underbrace{\lim_{T\to+\infty} f(T)e^{-sT}}_{=0 \text{ for } s>c} - f(0) + s\mathcal{L}\{f\}(t)$$

For $s > c$

$$\mathcal{L}\{f'(t)\} = -f(0) + s\mathcal{L}\{f(t)\}$$

$f(t) = \sin(e^{t^2})$, $f'(t) = 2te^{t^2}\sin(e^{t^2})$

NOTE $\mathcal{L}\{2te^{t^2}\sin(e^{t^2})\}$ exists even if such a function is NOT OF EXPONENTIAL ORDER

- f, f' are CONTINUOUS, OF EXPONENTIAL ORDER OF CONSTANT $c \geq 0$, f'' PIECEWISE CONT. THEN

$$\mathcal{L}\{f''\}(s) = -f'(0) - sf(0) + s^2\mathcal{L}\{f\}(s) \quad \boxed{s>c}$$

IN GENERAL

$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\} - \sum_{j=0}^{n-1} s^{n-1-j} f^{(j)}(0) \quad s>c$$

($f, f', \dots, f^{(n-1)}$ are continuous and of EXPONENTIAL ORDER $c \geq 0$, $f^{(n)}$ is PIECEWISE CONTINUOUS)

EXERCISE COMPUTE $\mathcal{L}\{\cos(2t)\}$, $\mathcal{L}\{\sin(10t)\}$ by USING THE FORMULA ON THE LT OF DERIVATIVES

HINT:

$$\cos'(2t) = -2\sin(2t)$$
$$\cos''(2t) = -4\cos(2t)$$
$$\underbrace{\mathcal{L}\{\cos''(2t)\}} = \mathcal{L}\{-4\cos(2t)\}$$

.....

EXAMPLES

1)
$$\begin{cases} f'(t) = f(t) & t > 0 \\ f(0) = 1 \end{cases}$$

SOL

$$\mathcal{L}\{f'(t)\}(s) = \mathcal{L}\{f(t)\}(s)$$

$$-\underbrace{f(0)}_{=1} + s\underbrace{\mathcal{L}\{f(t)\}(s)}_{F(s)} = \underbrace{\mathcal{L}\{f(t)\}(s)}_{F(s)}$$

$$F(s)(s-1) = 1 \underset{s\neq 1}{\Longleftrightarrow} F(s) = \frac{1}{s-1} \underset{s>1}{=} \mathcal{L}\{e^t\}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t \qquad t > 0$$

2) $\begin{cases} f''(t) + f(t) = \cos(2t) & t > 0 \\ f(0) = 0 \\ f'(0) = 1 \end{cases}$

$$F(s) = \mathcal{L}\{f\}(s)$$

$$\mathcal{L}\{f''\} + \mathcal{L}\{f\} = \mathcal{L}\{\cos(2t)\}$$

$$-\underbrace{f'(0)}_{1} - \underbrace{s f(0)}_{=0} + s^2 \mathcal{L}\{f\} + \mathcal{L}\{f\} = \frac{s}{s^2+4}$$

$$(s^2+1)\, F(s) = 1 + \frac{s}{s^2+4}$$

$$F(s) = \frac{1}{s^2+1} + \frac{s}{(s^2+1)(s^2+4)}$$

We will continue next time...