Lecture Analysis3-30.11.2020

Last week

We concluded the lecture of last week with the D'Alembert Formula which gives a representation formula for the solution of the 1D wave equation with given initial conditions. We obtained such a formula by performing a suitable change of variables v= x+ct, w=x-ct and by solving the equation

$$\boxed{u_{vw} = 0 \quad \text{in } \mathbb{R}^2}$$

The point of having reduced the wave equation $u_{tt} - c^2 u_{xx} = 0$ to the form: $u_{vw} = 0$ was that the new expression preserved the original feature of the differential equations. The expression $u_{vw} = 0$ is called the normal form of the equation

In fact with an appropriate change of coordinates a second order linear PDE can be brought into a normal form

$$u_{vw} = F^*(v, w, u, u_v, u_w) \quad \text{(hyperbolic)}$$

$$u_{vv} = F^*(v, w, u, u_v, u_w) \quad \text{(parabolic)}$$

$$u_{vv} + u_{ww} = F^*(v, w, u, u_v, u_w) \quad \text{(elliptic)}$$

This year we will skip Theorem 4.10 in Iozzi's notes.

Today: we analyse in details the D'Alembert formula, method of characteristics, heat equation via Fourier series

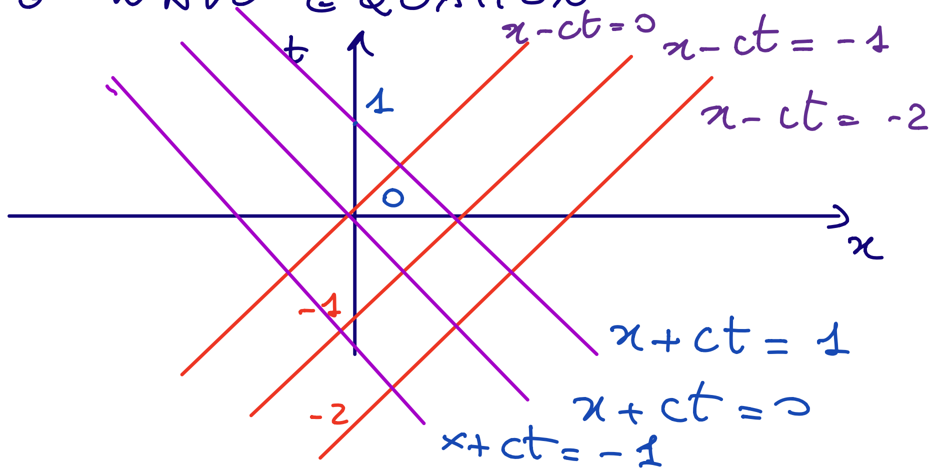
Characteristic lines

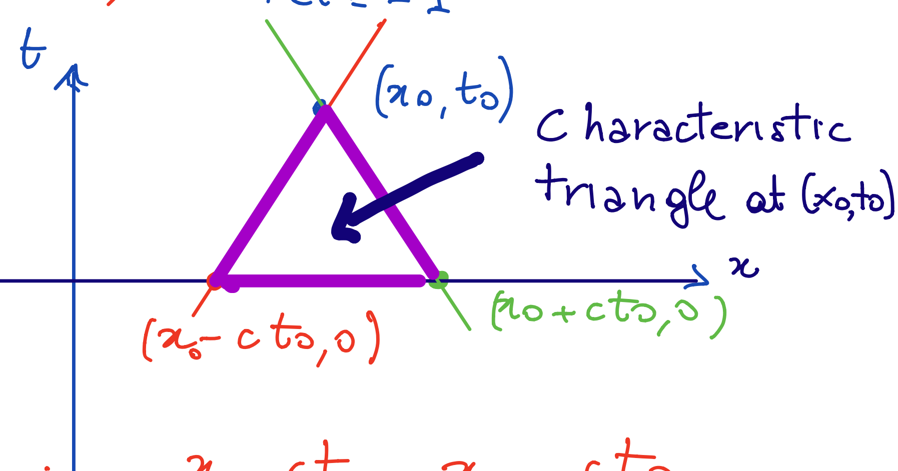
D'Alembert formula

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

REMARK

The bundle of lines in the (x,t)-plane $x + ct = k_1$, $x - ct = k_2$, $k_1, k_1 \in \mathbb{R}$, are called CHARACTERISTIC LINES ASSOCIATED TO THE WAVE EQUATION





Red line : $x - ct = x_0 - ct_0$

reen line : $x + ct = x_0 + ct_0$

The triangle with vertices (x_0, t_0), $(x_0 - ct_0, 0)$, $(x_0 + ct_0, 0)$ is called CHARACTERISTIC TRIANGLE AT (x_0, t_0)

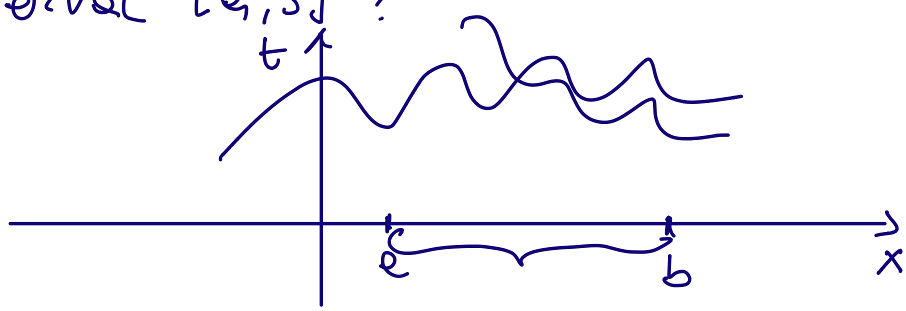
$$u(x_0,t_0) = \frac{1}{2}\left[f(x_0+ct_0)+f(x_0-ct_0)\right] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(s)\,ds$$

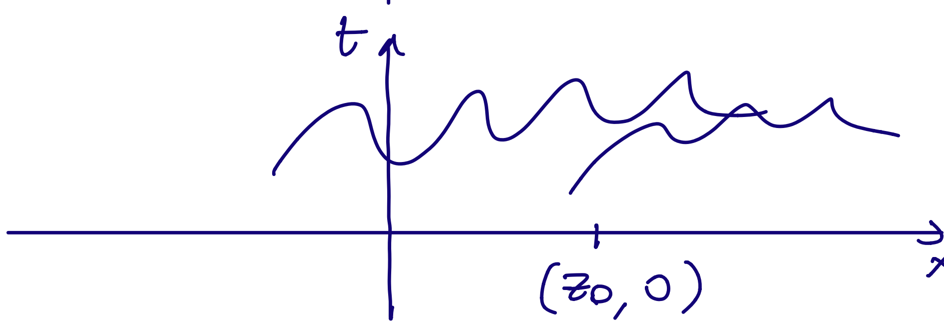
u depends on the values of the initial data between $[x_0-ct_0,\ x_0+ct_0]$

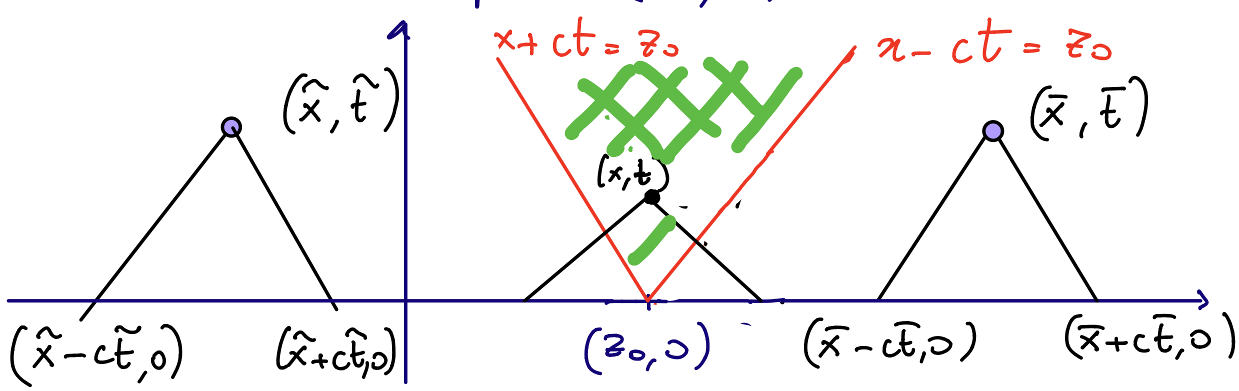
This INTERVAL is CALLED **DOMAIN OF DEPENDENCE** of u at (x_0,t_0).

OPPOSITE QUESTION

What REGION of the (x,t) in the upper half plane is affected by the interval $[a,b]$?



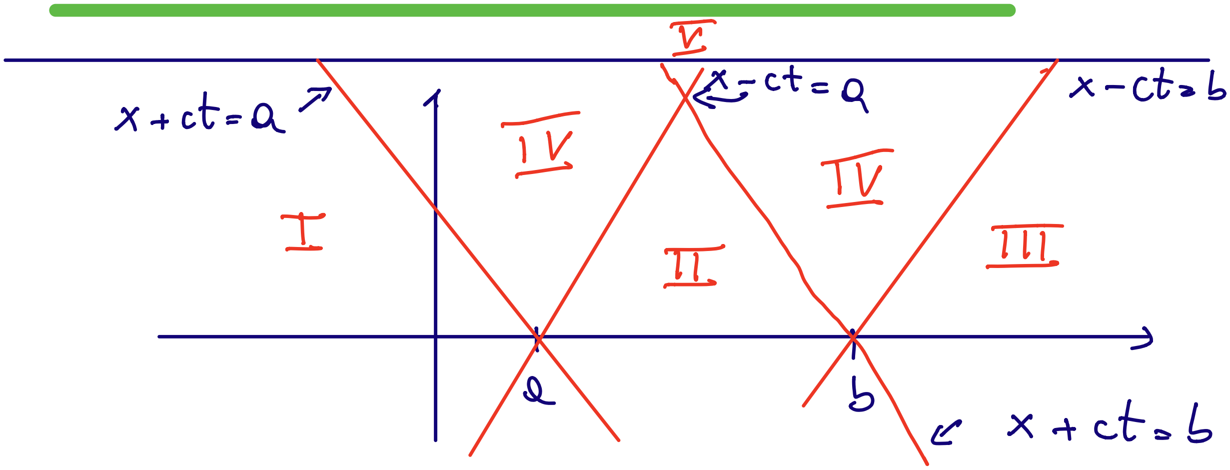


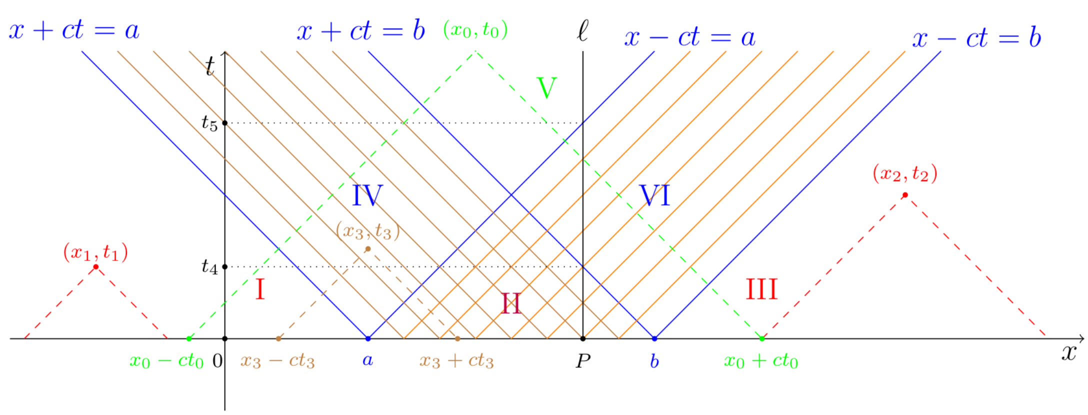


The values of f and g effect the points (x,t) which belong to the green region namely the region between the two charact. emanating from z_0.

$$z_0 - ct \leq x \leq z_0 + ct$$

The green region is called REGION OF INFLUENCE OF z_0.



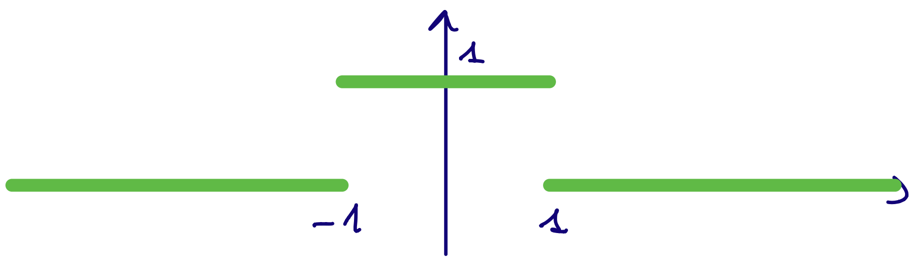


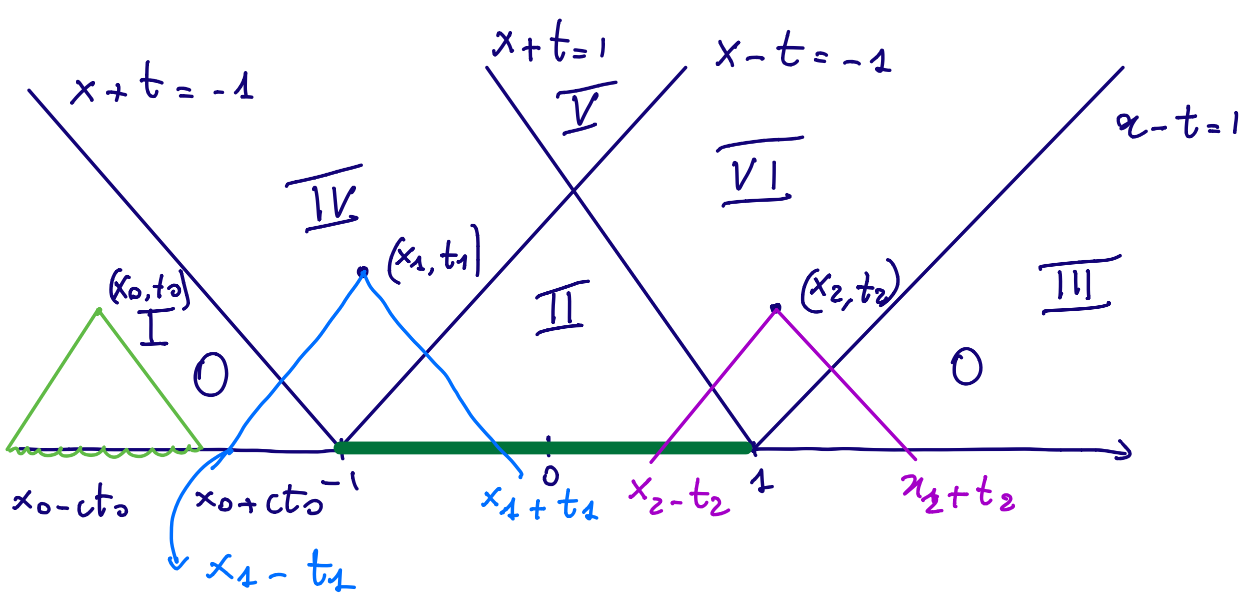
The points that are affected by the INITIAL CONDITIONS are exactly the POINTS (x,t) : $[x-ct, x+ct] \cap [a,b] = \emptyset$

APPLICATION

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,+\infty) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

$$f(x) = g(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$





- Region I and III

$$u(x_0,t_0) = \frac{1}{2}\left[f(x_0+ct_0) + f(x_0-ct_0)\right] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(s)\,ds$$

$u \equiv 0$

- Region IV :

$(x_1,t_1) \in \text{IV}$

$$u(x_1,t_1) = \frac{1}{2} f(x_1+t_1) + \frac{1}{2}\int_{-1}^{x_1+t_1} \underbrace{g(s)}_{=1}\,ds$$

$$= \frac{1}{2} + \frac{1}{2}(x_1+t_1+1) = 1 + \frac{x_1+t_1}{2}$$

- Region IV

$(x_2,t_2) \in (\text{IV})$

$$u(x_2,t_2) = \frac{1}{2} f(x_2-t_2) + \frac{1}{2}\int_{x_2-t_2}^{1} g(s)\,ds$$

$$= \frac{1}{2} + \frac{1}{2}(1-x_2+t_2)$$

$$= 1 + \frac{t_2-x_2}{2}$$

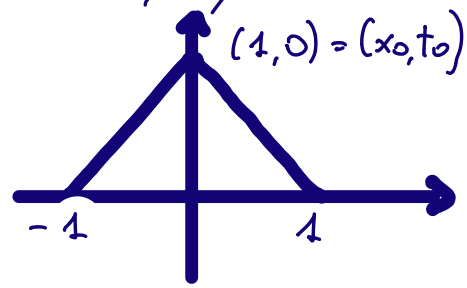
Typical questions:

1) $\forall x \in \mathbb{R}$: $\lim\limits_{t\to+\infty} u(x,t) = \frac{1}{2}\int_{-1}^{1} g(s)\,ds$

$$= \frac{1}{2}\, 2 = 1$$

2) Determine the max value of $u(x,t)$

The MAX value of $u(x,t)$ is attained at the point $(x_0,t_0) \in \mathrm{II}$ satisfying:



THE DOMAIN OF DEPENDENCE OF (x_0,t_0) COINCIDES WITH $[-1,1]$

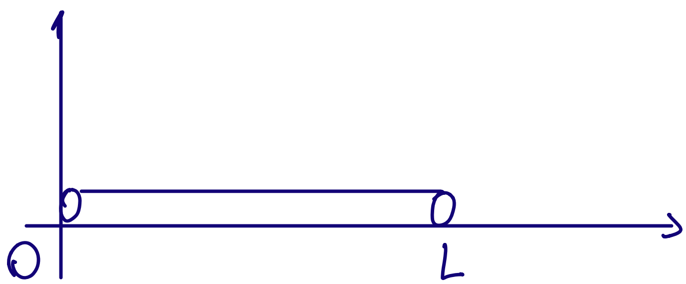
$$\begin{cases} x_0 - t_0 = -1 \\ x_0 + t_0 = 1 \end{cases}$$

$$x_0 = 0 \quad \& \quad t_0 = 1$$

$$u(0,1) = \frac{1}{2}\left[f(1) + f(-1)\right] + \frac{1}{2}\int_{-1}^{1} 1 \, ds$$

$$= 1 + \frac{1}{\cancel{2}} \cdot \cancel{2} = 2$$

Heat EQUATION VIA FOURIER SERIES.



$$\begin{cases} u_t - c^2 u_{xx} = 0 & 0 < x < L, \ t > 0 \\ u(0,t) = u(L,t) = 0 & t \geq 0 \\ u(x,0) = f(x) & 0 \leq x \leq L \end{cases}$$

$$c^2 = \frac{k}{\rho\sigma}$$

Sol: Method of separation of VARIABLES

1^{st} step $\quad u(x,t) = F(x)\,G(t)$

$u_t = F(x)\,G'(t)$

$u_{xx} = F''(x)\,G(t)$

$F(x)\,G'(t) - c^2 F''(x)\,G(t) = 0$

$\to c^2 G(t)\,F(x)$

$$\frac{F''(x)}{F(x)} = \frac{G'(t)}{c^2 G(t)} = K \in \mathbb{R} \ \forall x \in (0,L),\ \forall t > 0$$

Step 2

$\Rightarrow \quad F''(x) = K F(x) \quad , \quad G'(t) = K c^2 G(t)$

$(BC) \Rightarrow \quad F(0) = F(L) = 0$

$$(P_F) \quad \begin{cases} F''(x) - K F(x) = 0 \\ F(0) = F(L) = 0 \end{cases}$$

$(P_G) : G'(t) - K c^2 G(t) = 0$

(P_F) : the only case where there are nontrivial SOLUTIONS IS WHEN $K < 0$:

$F(x) = A \cos(\sqrt{-K}\,x) + B \sin(\sqrt{-K}\,x)$

$F(0) = 0 \quad \Rightarrow \quad A = 0$

$F(L) = 0 \quad \Rightarrow \quad \sqrt{-K}\,L = n\pi \quad \Rightarrow \quad \sqrt{-K} = \frac{n\pi}{L}$

$\Rightarrow \quad K = -\left(\frac{n\pi}{L}\right)^2 \quad K \geq 1$

For EVERY $n \geq 1$ we find

$F_n(x) = \sin\left(\frac{n\pi}{L}x\right)$

(P_G) $K_n = -\left(\frac{n\pi}{L}\right)^2$

$$G_n'(t) + c^2\left(\frac{n\pi}{L}\right)^2 G_n(t) = 0$$

$$\Rightarrow \quad G_n(t) = B_n e^{-c^2\left(\frac{n\pi}{L}\right)^2 t}$$

<u>Recall</u> $\quad y'(t) + a\, y(t) = 0$

$$\Leftrightarrow \quad y(t) = C_0 e^{-at}$$

$$a = c^2\left(\frac{n\pi}{L}\right)^2$$

$\forall n \geq 1$

$$u_n(x,t) = B_n e^{-c^2\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

<u>Step 3</u>

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 t}$$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

Hence B_n are the Fourier coeff. of the ODD $2L$-periodic extension of f, that is:

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

EXAMPLE

Copper bar of length 80 cm

$L = 80$ cm, $\quad u(0,t) = u(L,t) = 0$

$u(x,0) = f(x)$.

Q: Find $u(x,t)$ and compute how long it will take for the MAXIMUM TEMPERATURE to drop to 50 °C in two cases:

1) $f(x) = 100 \sin\left(\frac{\pi x}{80}\right)$

2) $f(x) = 100 \sin\left(\frac{3\pi}{80} x\right)$

<u>Sol</u> $c^2 = 1$ $\quad u_t - u_{xx} = 0$

1) $f(x) = 100 \sin\left(\frac{\pi x}{80}\right)$

$B_1 = 100$, $B_n = 0 \quad \forall n > 1$

$\Rightarrow u(x,t) = 100 \sin\left(\frac{\pi x}{80}\right) e^{-\left(\frac{\pi}{80}\right)^2 t}$

$u_{MAX}(x,t) = 100\, e^{-\left(\frac{\pi}{80}\right)^2 t} = 50$

$\Leftrightarrow e^{-\left(\frac{\pi}{80}\right)^2 t} = \frac{1}{2}$

$\Leftrightarrow -\left(\frac{\pi}{80}\right)^2 t = \log_e \frac{1}{2} = -\log(2)$

$\Leftrightarrow t = \left(\frac{80}{\pi}\right)^2 \log(2)$

2) $f(x) = 100 \sin\left(\frac{3\pi}{80} x\right)$

$B_3 = 100 \quad B_n = 0 \quad \forall n \neq 3$

$u(x,t) = 100 \sin\left(\frac{3\pi}{80} x\right) e^{-\left(\frac{3\pi}{80}\right)^2 t}$

$$u_{MAX}(x,t) = 100\, e^{-\left(\frac{3\pi}{80}\right)^2 t} = 50$$

$$\Leftrightarrow \quad t = \left(\frac{80}{3\pi}\right)^2 \log(2) \quad \blacksquare$$