Lecture ANALYSIS 3-7.12.2020

Last week

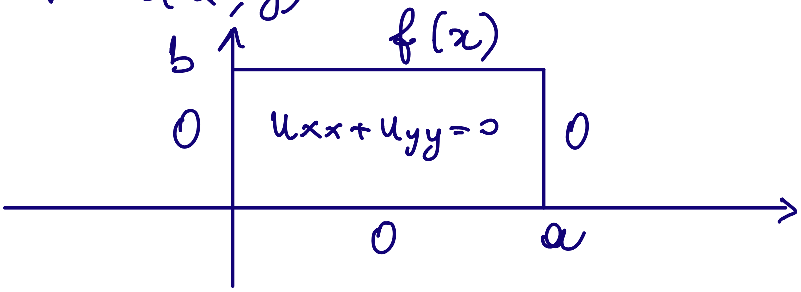
- Characteristic lines associated to 1-D wave equation
- Solution Heat Equation via Fourier series

Today

- Laplace equation in a rectangular domain
- Heat equation on an infinite bar

Laplace Equation on a rectangular domain

$$\begin{cases} u_{xx} + u_{yy} = 0 & 0 < x < a, \quad 0 < y < b \\ u(x,0) = 0 \\ u(x,b) = f(x) \\ u(0,y) = 0 \\ u(a,y) = 0 \end{cases}$$



METHOD OF SEPARATION OF VARIABLES

- $u(x,y) = F(x)\,G(y)$

$$u_{xx} = F''(x)\,G(y)$$

$$u_{yy} = F(x)\,G''(y)$$

$$F''(x)\,G(y) + F(x)\,G''(y) = 0 \qquad \forall x \in (0,a) \quad \forall y \in (0,b)$$

We divide by $F(x)G(y)$:

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = -K \qquad K \in \mathbb{R}$$

$$F''(x) + K\,F(x) = 0 \qquad\qquad G''(y) - K\,G(y) = 0$$

(BC)

1 $u(0,y) = u(a,y) \quad \forall y \in (0,b)$

$F(0)G(y) = F(a)G(y)$

$$\Rightarrow \begin{cases} F(0) = 0 \\ F(a) = 0 \end{cases}$$

2 $u(x,0) = 0 \qquad \forall x \in (0,a)$

$\Rightarrow \quad G(0) = 0$

$$\begin{cases} F''(x) + K\,F(x) = 0 \\ F(0) = 0 \\ F(a) = 0 \end{cases}$$

If $K > 0 \Rightarrow \forall n \geq 1 \qquad K_n = \left(\frac{n\pi}{a}\right)^2$

$F_n(x) = \sin\left(\frac{n\pi}{a}x\right)$

$$G''(y) - \left(\frac{n\pi}{a}\right)^2 G(y) = 0 \qquad \forall y \in (0,b)$$

$$G_n(y) = A_n^* e^{\frac{n\pi}{a}y} + B_n^* e^{-\frac{n\pi}{a}y}$$

$$G_n(0) = 0 \Rightarrow A_n^* + B_n^* = 0 \Rightarrow A_n^* = -B_n^*$$

$$G_n(y) = \underbrace{2A_n^*}_{A_n}\left[\frac{e^{\frac{n\pi}{a}y} - e^{-\frac{n\pi}{a}y}}{2}\right] = A_n \sinh\left(\frac{n\pi y}{a}\right)$$

$\forall\, n \geq 1$

$$u_n(x,t) = A_n \sin\left(\frac{n\pi}{a} x\right) \cdot \sinh\left(\frac{n\pi}{a} y\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a} x\right) \sinh\left(\frac{n\pi}{a} y\right)$$

$$f(x) = u(x,b) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a} x\right) \sinh\left(\frac{n\pi}{a} b\right)$$

$$\Rightarrow A_n \sinh\left(\frac{n\pi}{a} b\right) = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi}{a} x\right) dx$$

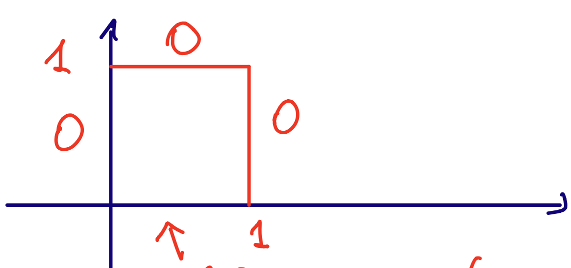
$\Rightarrow$

$$A_n = \frac{1}{\sinh\left(\frac{n\pi}{a} b\right)} \cdot \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi}{a} x\right) dx$$

EXERCISE

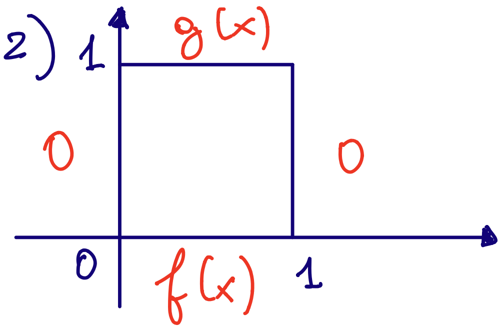
1) Find the solution of

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } [0,1] \times [0,1] \\ u(x,0) = f(x) = \sin(2\pi x) & \\ u(x,1) = 0 & x \in [0,1] \\ u(0,y) = 0 & \\ u(1,y) = 0 & y \in [0,1] \end{cases}$$



$f(x) = \sin(2\pi x)$

2)



$$\begin{cases} u_{xx} + u_{yy} = 0 \\ \begin{cases} u(0,y) = 0 \\ u(1,y) = 0 \end{cases} & y \in [0,1] \\ \begin{cases} u(x,0) = f(x) \\ u(x,1) = g(x) \end{cases} & x \in [0,1] \end{cases}$$

Heat Equation on an infinite bar

$$\begin{cases} u_t - c^2 u_{xx} = 0 & x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \end{cases}$$

We require that the solution u is bounded:

$$-M \le u(x,t) \le M \qquad \forall (x,t) \in \mathbb{R} \times (0, +\infty)$$
$$M > 0$$

Solution

$$u(x,t) = F(x)\, G(t)$$

$$u_t = F(x)\, G'(t), \quad u_{xx} = F''(x)\, G(t)$$

$$\frac{F''(x)}{F(x)} = \frac{G'(t)}{c^2 G(t)} = k \in \mathbb{R}$$

$$\begin{cases} F''(x) - k\, F(x) = 0 & \text{in } \mathbb{R} \\ G'(t) - c^2 k\, G(t) = 0 & \text{in } t > 0 \end{cases}$$

If $k > 0$ we get:

$F(x) = A e^{\sqrt{k}\,x} + B e^{-\sqrt{k}\,x}$ $\quad A, B \in \mathbb{R}$

$G(t) = D e^{kc^2 t}$ $\quad D \in \mathbb{R}$

NOTE : $\lim_{t \to 0} D e^{kc^2 t} = +\infty$

$\Rightarrow$ NO PHYSICAL MEANING.

$\Rightarrow$ $k \leq 0$, we denote $k = -p^2$ $\quad p \geq 0$

In this case we can write the product solution as follows:

$$u_p(x,t) = F_p(t)\, G_p(t)$$
$$= \left[A(p)\cos(px) + B(p)\sin(px)\right] e^{-c^2 p^2 t}$$

Because of the superposition principle:

$$u(x,t) = \int_0^{+\infty} u_p(x,t)\, dp$$

$$= \int_0^{+\infty} \left[A(p)\cos(px) + B(p)\sin(px)\right] e^{-c^2p^2t}\, dp$$

$$u(x,0) = \underbrace{\int_0^{+\infty} A(p)\cos(px) + B(p)\sin(px)\, dp = f(x)}_{1}$$

1 is the Fourier integral of f :

$$A(p) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(v)\cos(pv)\, dv$$

$$B(p) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(v)\sin(pv)\,dv$$

We can write u as follows:

$$u(x,t) = \frac{1}{\pi}\int_0^{+\infty}\left(\int_{-\infty}^{+\infty} f(v)\left[\cos(pv)\cos(px) + \sin(pv)\sin(px)\right]dv\right)e^{-c^2p^2t}\,dp$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\alpha = pv, \quad \beta = px$$

$$\Rightarrow u(x,t) = \frac{1}{\pi}\int_0^{+\infty}\left(\int_{-\infty}^{+\infty}\left(f(v)\cos(p(x-v))\right)dv\right)e^{-c^2p^2t}\,dp$$

$$= \frac{1}{\pi}\int_{-\infty}^{+\infty} f(v)\left(\int_0^{+\infty}\cos(p(x-v))\,e^{-c^2p^2t}\,dp\right)dv$$

We are going to use the following formula:

$$\int_0^{+\infty}\cos(2bs)\,e^{-s^2}\,ds = \frac{\sqrt{\pi}}{2}e^{-b^2}$$

$$\int_0^{+\infty}\cos(p(x-v))\,e^{-c^2p^2t}\,dp = \circledast$$

$$s^2 = c^2tp^2 \qquad p^2 = \frac{s^2}{c^2t} \Rightarrow p = \frac{s}{c\sqrt{t}}$$

$$dp = \frac{ds}{c\sqrt{t}}$$

$$\circledast = \int_0^{+\infty} \cos\left(\frac{(x-v)}{c\sqrt{t}} \cdot s\right) e^{-s^2} \frac{ds}{c\sqrt{t}}$$

We set $2b = \dfrac{x-v}{c\sqrt{t}} \Rightarrow b = \dfrac{x-v}{2c\sqrt{t}}$

From the previous formula we get that

$$\circledast = \frac{\sqrt{\pi}}{2c\sqrt{t}} \quad e^{-\left(\frac{x-v}{2c\sqrt{t}}\right)^2}$$

$$u(x,t) = \frac{1}{2\sqrt{\pi}\, c\, t} \int_{-\infty}^{+\infty} f(v)\, e^{-\frac{(x-v)^2}{4c^2 t}}\, dv.$$

PROOF OF FORMULA $\circledast$

$$\int_0^{+\infty} \cos(2bs)\, e^{-s^2} ds = \frac{\sqrt{\pi}}{2} e^{-b^2}$$

$$I(b) = \int_0^{+\infty} \cos(2bs)\, e^{-s^2} ds$$

$$I'(b) = \int_0^{+\infty} -2s \sin(2bs)\, e^{-s^2} ds$$

$$= \int_0^{+\infty} \sin(2bs) \frac{d}{ds}\left(e^{-s^2}\right) ds$$

$$= \underbrace{e^{-s^2} \sin(2bs) \Big|_0^{+\infty}}_{=0} - \underbrace{\int_0^{+\infty} e^{-s^2} \cos(2bs)\, 2b\, ds}_{2b\, I(b)}$$

$$I'(b) = -2b\, I(b)$$

$$\Rightarrow I(b) = I(0)\, e^{-b^2} \qquad I(0) = \underbrace{\int_0^{+\infty} e^{-s^2}\, ds}_{\text{GAUSS INTEGRAL}} = \frac{\sqrt{\pi}}{2}$$

Recall

$$\begin{cases} y'(t) = a(t)\, y(t) \\ y(0) = a_0 \end{cases}$$

$$y(t) = a_0\, e^{A(t)} \quad \text{where} \quad A(t) = \int a(s)\, ds$$

$$a(b) = -2b \qquad \int -2b\, db = -\cancel{2}\, \frac{b^2}{\cancel{2}} = -b^2$$

$$\Rightarrow \quad I(b) = \frac{\sqrt{\pi}}{2}\, e^{-b^2} \qquad \square$$

LOOK AT BONUS EXERCISE SERIE 8

$$\mathcal{F}(e^{-x^2})(\xi) = \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{+\infty} e^{-x^2} e^{-ix\xi}\, dx}_{\overset{!!}{I(\xi)}}$$

ALTERNATIVE METHOD BY FOURIER TRANSFORM.

$$\begin{cases} u_t - c^2 u_{xx} = 0 & x \in \mathbb{R}, \ t > 0 \\ u(x,0) = f(x) \end{cases}$$

Take$_x$ the FT of both sides the equation w.r.t

$$\begin{cases} \mathcal{F}(u_t - c^2 u_{xx})(\xi) = \hat{0} = 0 \\ \mathcal{F}(u(x,0))(\xi) = \mathcal{F}(f(x))(\xi) \end{cases}$$

$$\mathcal{F}(u_t) - c^2 \mathcal{F}(u_{xx}) = 0$$

$$\mathcal{F}(u(x,t))(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(x,t)\, e^{-ix\cdot\xi}\, dx$$

$$\Rightarrow \mathcal{F}(u_t)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u_t(x,t)\, e^{-ix\cdot\xi}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{d}{dt} \left(\int_{-\infty}^{+\infty} u(x,t)\, e^{-ix\cdot\xi}\, dx \right)$$

$$= \frac{d}{dt} \left[\mathcal{F}(u)(\xi,t) \right]$$

$$\mathcal{F}(u_{xx}) = (i\xi)^2 \mathcal{F}(u) = -\xi^2 \mathcal{F}(u)$$

We get:

$$\frac{d}{dt} \mathcal{F}(u)(\xi,t) + c^2 \xi^2 \mathcal{F}(u)(\xi,t) = 0$$

$$\hat{u}(\xi,t) = \mathcal{F}(u)(\xi,t)$$

$$\frac{d}{dt} \hat{u}(\xi,t) + c^2 \xi^2 \hat{u}(\xi,t) = 0$$

$$\Rightarrow \hat{u}(\xi,t) = \hat{u}(\xi,0)\, e^{-c^2 t \xi^2} = \hat{f}(\xi)\, e^{-c^2 t \xi^2}$$

Remark The role of $a(t) = -c^2\xi^2$

$$\int a(t)\,dt = \int -c^2\xi^2\,dt = -c^2\xi^2 t$$

Apply inversion formula

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\xi)\, e^{-c^2\xi\, t}\, e^{ix\xi}\, d\xi$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(v)\, e^{-i\xi v}\, dv \right) e^{-c^2\xi^2 t}\, e^{ix\xi}\, d\xi$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(v) \underbrace{\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-c^2\xi^2 t}\, e^{-i\xi(v-x)}\, d\xi \right)}_{= \mathcal{F}\left(e^{-c^2 t\xi^2}\right)(v-x)\ \boxed{*}} dv = \circledast$$

$$= \boxed{*} = \frac{1}{\sqrt{2c^2 t}}\, e^{-\frac{(v-x)^2}{4c^2 t}}$$

RECALL : $\mathcal{F}\left(e^{-ax^2}\right) = \dfrac{1}{\sqrt{2a}}\, e^{-\frac{\xi^2}{4a}}$

$$\circledast = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{+\infty} f(v)\, e^{-\frac{(v-x)^2}{4c^2 t}}\, dv \right) \frac{1}{\sqrt{2}\, c\sqrt{t}}$$

$$= \frac{1}{2\sqrt{\pi}\, c\sqrt{t}} \int_{-\infty}^{+\infty} f(v)\, e^{-\frac{(v-x)^2}{4c^2 t}}\, dv. \quad \blacksquare$$