Lecture Analysis 3, 12 October 2020

$$(f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

5) $f * 1 \neq f$

$f(t) = t$

$$f*1 = \int_0^t \tau \, d\tau = \frac{\tau^2}{2}\Big|_0^t = \frac{t^2}{2} \neq t$$

6) $f*f \not\geq 0$ $\quad (f\cdot f = f^2 \geq 0)$

$f(t) = \sin(t)$

$$\sin(t) * \sin(t) = \int_0^t \sin(\tau)\sin(t-\tau)\, d\tau$$

$$= \int_0^t \frac{\cos(\tau-(t-\tau)) - \cos(\cancel{\tau}+t-\cancel{\tau})}{2}\, d\tau$$

$$= \frac{1}{2}\int_0^t (\cos(2\tau - t) - \cos t)\, d\tau$$

$$= \frac{1}{2}\, \frac{\sin(2\tau - t)}{2}\Big|_0^t - \frac{1}{2}\cos t \int_0^t d\tau$$

$$= \frac{1}{2}\frac{1}{2}\Big[\sin(t) - \underbrace{\sin(-t)}_{=-\sin(t)}\Big] - \frac{1}{2}\cos t \; t\Big|_0^t$$

$$= \frac{1}{2}\, \frac{\cancel{2}\sin(t)}{\cancel{2}} - \frac{1}{2}\cos(t)\, t = \underbrace{\frac{\sin t}{2} - \frac{t\cos t}{2}}_{} \not\geq 0$$

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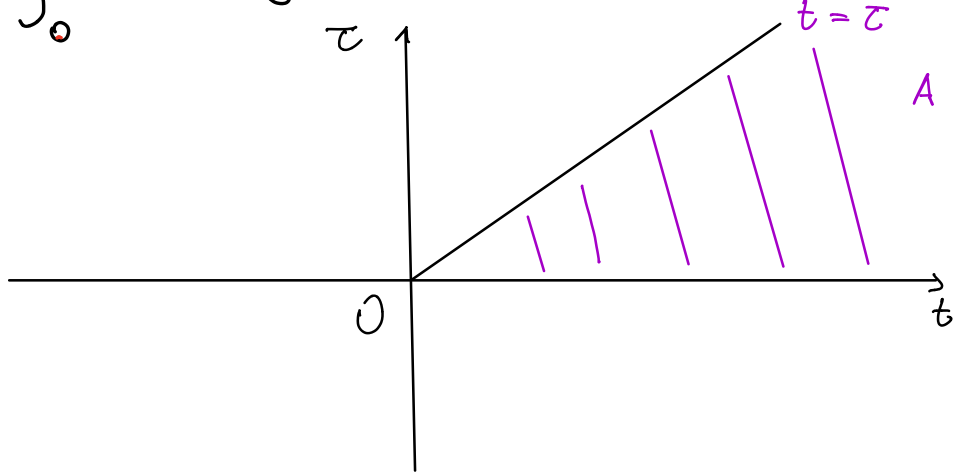
7) $\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\,\mathcal{L}\{g\}$

$\boxed{\text{VERIFICATION}}$

$$\mathcal{L}\{f*g\}(s) = \int_0^\infty (f*g)(t)\, e^{-st}\, dt =$$

$$= \int_0^{+\infty}\int_0^t f(\tau)\, g(t-\tau)\, e^{-st}\, d\tau\, dt$$



$$A = \{(t,\tau) \in \mathbb{R}^2 :\ 0 \le t < +\infty,\quad 0 \le \tau \le t\}$$
$$= \{(t,\tau) \in \mathbb{R}^2 :\ 0 \le \tau < +\infty,\quad \tau \le t < +\infty\}$$

$$= \int_0^{+\infty}\int_0^t f(\tau)\, g(t-\tau)\, e^{-st}\, d\tau\, dt$$
$$= \int_0^{+\infty}\Bigg(\underbrace{\int_\tau^{+\infty} f(\tau)\, g(t-\tau)\, e^{-st}\, dt}_{1} \Bigg) d\tau = (*)$$

1 $$\int_\tau^{+\infty} f(\tau)\, g(t-\tau)\, e^{-st}\, dt \underset{y = t - \tau}{=}$$
$$= \int_0^{+\infty} f(\tau)\, g(y)\, e^{-s(y+\tau)}\, dy$$

$$(*) = \int_0^{+\infty} \left(\int_0^{+\infty} f(\tau)\, g(y)\, e^{-s(y+\tau)} dy \right) d\tau$$

$$f(\tau) e^{-s\tau} \cdot g(y) e^{-sy}$$

$$= \left(\int_0^{+\infty} f(\tau) e^{-s\tau} d\tau \right) \left(\int_0^{+\infty} g(y) e^{-sy} dy \right)$$

$$= \mathcal{L}\{f\}(s) \cdot \mathcal{L}\{g\}(s)$$

$\square$

EXAMPLE

$\mathcal{L}^{-1}(F(s)) \qquad F(s) = \dfrac{1}{s^2(s+1)}$

Sol : $F(s) = \dfrac{1}{s^2} \cdot \dfrac{1}{s+1}$

$$= \mathcal{L}\{t\} \cdot \mathcal{L}\{e^{-t}\}$$

$$\Rightarrow \mathcal{L}\{f * g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}$$

$$f * g = \mathcal{L}^{-1}[\mathcal{L}\{f\} \cdot \mathcal{L}\{g\}]$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = t * e^{-t}$$

$$= \int_0^t \tau e^{-(t-\tau)} d\tau = e^{-t} + t - 1$$

ANOTHER APPLICATION:

INTEGRAL EQUATIONS

$$f(t) = g(t) + \int_0^t f(\tau) h(t-\tau) d\tau$$

$$= g(t) + (f * h)(t)$$

$F, G, H \rightarrow$ LT of f, g, h

$$F(s) = G(s) + \mathcal{L}\{f * h\}(s)$$

$$= G(s) + \underbrace{F(s) H(s)}$$

$$F(s)(1 - H(s)) = G(s)$$

$$F(s) = \frac{G(s)}{1 - H(s)}$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left\{ \frac{G(s)}{1 - H(s)} \right\}$$

EXAMPLE

$$f(t) = e^{-t} + \int_0^t f(\tau) \sinh(t-\tau) d\tau$$

By applying the above formula:

$$f(t) = \mathcal{L}^{-1}\left\{ \frac{G(s)}{1-H(s)} \right\}$$

$$h(t) = \sinh(t) = \frac{e^{t}-e^{-t}}{2}, \quad g(t) = e^{-t}$$

$$G(s) = \frac{1}{s+1}, \qquad H(s) = \mathcal{L}\{\sinh(t)\}$$

$$= \frac{1}{s^2-1}$$

$$f(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s+1} \cdot \frac{1}{1-\frac{1}{s^2-1}} \right\} \underset{\text{ex}}{=} \mathcal{L}^{-1}\left\{ \frac{s-1}{s^2-2} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{s}{s^2-2} \right\} - \frac{1}{\sqrt{2}} \mathcal{L}^{-1}\left\{ \frac{\sqrt{2}}{s^2-2} \right\}$$

$$= \cosh(\sqrt{2}\, t) - \frac{1}{\sqrt{2}} \sinh(\sqrt{2}\, t)$$

$$\cosh(t) = \frac{e^{t}+e^{-t}}{2}$$

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LAST PROPERTIES (Sect 2.5 & 2.8 Iozzi's Notes)

- $$\mathcal{L}\left\{ \int_0^t f(x)\,dx \right\} = \frac{1}{s} \mathcal{L}\{f\}(s)$$

  EXERCISE: Compute $\mathcal{L}\left\{ \int_0^t \cos(ax)\,dx \right\}$

- $$\left(\mathcal{L}\{f\}(s)\right)' = -\mathcal{L}\{tf\}$$

- $\lim_{t \to 0^+} \frac{f(t)}{t}$ exists

  $$\Rightarrow \quad \mathcal{L}\left\{ \frac{f(t)}{t} \right\} = \int_s^{+\infty} \mathcal{L}\{f\}(\tilde{s})\,d\tilde{s}$$

EXERCISE

$$F(s) = \ln\left(1+\frac{w^2}{s^2}\right) \qquad Q: \mathcal{L}^{-1}\{F\}\ ?$$

Solution

1) $\left(\mathcal{L}\{f(t)\}\right)' = -\mathcal{L}\{t\,f(t)\}$

$$F(s) = \mathcal{L}\{f(t)\}$$

$$F'(s) = \left(\ln\left(1+\frac{w^2}{s^2}\right)\right)' = \left(\ln(s^2+w^2)\right)' - \left(\ln s^2\right)' =$$

$$\left[\ln\left(1+\frac{w^2}{s^2}\right) = \ln\left(\frac{s^2+w^2}{s^2}\right) = \ln(s^2+w^2) - \ln(s^2)\right]$$

$$\left(\ln(s^2+w^2)\right)' - \left(\ln s^2\right)' = \frac{1}{s^2+w^2}\cdot 2s - \frac{2}{s}$$

$$F'(s) = \frac{2s}{s^2+w^2} - \frac{2}{s} = 2\mathcal{L}(\cos wt) - 2\mathcal{L}\{1\}$$

$$= \mathcal{L}\{2\cos(wt) - 2\}$$

$$= -\mathcal{L}\{t\,f(t)\}$$

$$\Longrightarrow \quad -t\,f(t) = 2\cos(wt) - 2$$

$$\Longrightarrow \quad f(t) = \frac{2 - 2\cos wt}{t} \qquad \blacksquare$$

2) $$\int_s^{+\infty} \mathcal{L}\{f\}(\tilde{s})\,d\tilde{s} = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

We apply such a rule to the function $G(s) = F'(s) = \left(\frac{2s}{s^2+w^2} - \frac{2}{s}\right)$

$$\int_s^{+\infty} G(\tilde{s})\,d\tilde{s} = \mathcal{L}\left\{\frac{g(t)}{t}\right\}$$

$$g(t) = \mathcal{L}^{-1}\{G\}(t) = \mathcal{L}^{-1}\left\{\frac{2s}{s^2+\omega^2} - \frac{2}{s}\right\}$$

$$= 2\cos(\omega t) - 2$$

RECALL: FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

$$F(b) - F(a) = \int_a^b F'(\tilde{s})\,d\tilde{s} = \int_a^b G(\tilde{s})\,d\tilde{s}$$

"$b = +\infty$" "$a = s$"

$$\underset{\overset{\shortparallel}{0}}{F(+\infty)} - F(s) = \int_s^{+\infty} G(\tilde{s})\,d\tilde{s} = \mathcal{L}\left\{\frac{2\cos\omega t - 2}{t}\right\}$$

$$\underset{\overset{\shortparallel}{\mathcal{L}\{f(t)\}}}{+F(s)} = \mathcal{L}\left\{\frac{-2\cos\omega t + 2}{t}\right\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{2 - 2\cos(\omega t)}{t}\right\}$$

$$\Rightarrow f(t) = \frac{2 - 2\cos(\omega t)}{t} \quad \blacksquare$$

CHAPTER 2: Fourier Analysis

Jean Baptiste Joseph Fourier (1768-1830)

had crazy idea (1807):

Any periodic function can be rewritten as a weighted sum of sines and cosines of different frequencies.

Don't believe it?

- Neither did Lagrange, Laplace, Poisson and other big wigs
- Not translated into English until 1878!

But it's true!

- called Fourier Series

2.1 Background Material

DEFINITION 2.1

$f : \mathbb{R} \to \mathbb{R}$ is a PERIODIC FUNCTION if $f(x)$ is defined for all $x \in \mathbb{R}$, except possibly at some POINTS AND if there is $P > 0$ CALLED THE PERIOD OF $f(x)$ SUCH THAT

$$f(x+P) = f(x)$$ for every x in the domain of definition of f.

→ The SMALLEST POSITIVE NUMBER P FOR WHICH f is PERIODIC IS CALLED THE FUNDAMENTAL PERIOD of f.

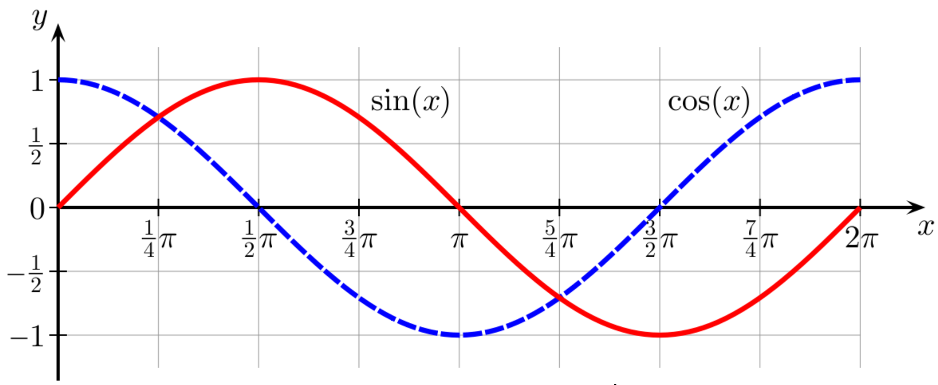
→ If $f(x)$ has PERIOD P THEN IT ALSO HAS PERIOD mP $\forall m \in \mathbb{N} \setminus \{0\}$

For instance if $m = 2$:

$$f(x+2P) = f(x) \qquad \forall x$$

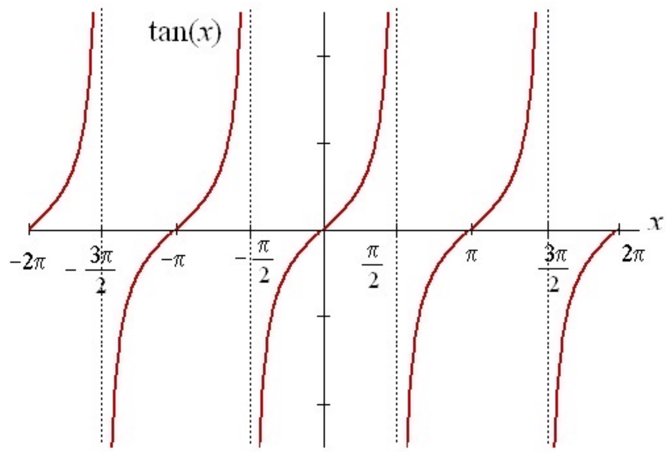
$$f(x+2P) = f(x+P+P) = f(x+P) = f(x).$$

1



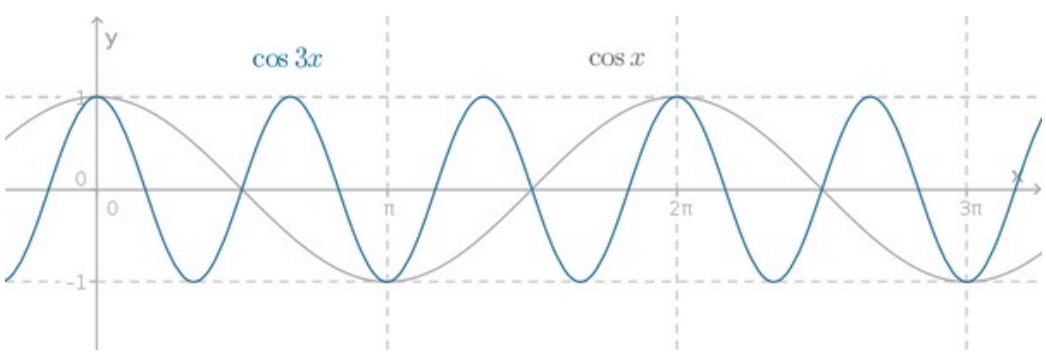
$\cos(x)$, $\sin(x)$ has period $P = 2\pi$

2



$\mathbb{R} \setminus \{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$

$\tan(x)$ has period $P = \pi$

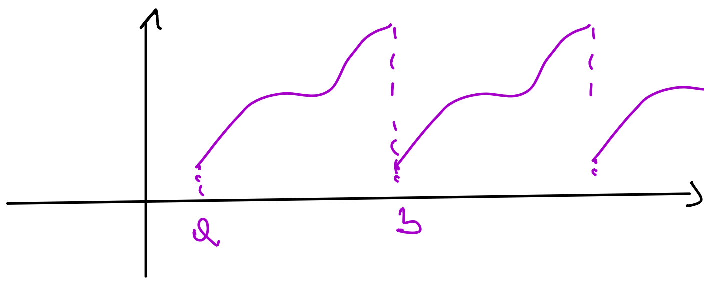


3 $\cos(3x)$, $\cos(2x)$, $\cos(\alpha x)$ $\alpha \in \mathbb{R}^+$
$\sin(3x)$, $\sin(\alpha x)$... are PERIODIC
For every $\alpha \in \mathbb{R}^+$ $\cos(\alpha x)$ has

PERIOD $\quad P = \left(\frac{2\pi}{\alpha}\right)$

For instance $\cos(3x)$ has period $p = \frac{2\pi}{3}$

4



$P = b - a$

5 f, g PERIODIC WITH PERIOD P
$\Rightarrow f+g$ IS PERIODIC OF PERIOD P
(EX : the fundamental period of $f+g$ IS NOT IN GENERAL P)

6 f, g CONTINUOUS, PERIODIC OF PERIOD P_1 AND P_2
$\Rightarrow f+g$ IS PERIODIC $\Longleftrightarrow \frac{P_1}{P_2} \in \mathbb{Q}$

$\Rightarrow \sin(3x) + \sin(\pi x)$ IS NOT PERIODIC

$P_1 = \frac{2\pi}{3} \qquad P_2 = \frac{2\cancel{\pi}}{\cancel{\pi}} \qquad \frac{P_1}{P_2} = \frac{2\pi}{3} \cdot \frac{1}{2}$

EXERCISE $f(x+P) = f(x)$
$\Rightarrow f(\alpha x)$ has period $\frac{P}{\alpha}$
$\forall \alpha > 0$.