A LEBESGUE MEASURABLE SET THAT IS NOT BOREL

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1. Outline

(1) Ternary Expansions
(2) The Cantor Set
(3) The Cantor Ternary Function (a.k.a. The Devil's Staircase Function)
(4) Properties of the Cantor Ternary Function
   - Continuous
   - Monotone
   - Maps $\mathcal{C}$ onto $[0, 1]$
   - Constant on each interval in complement of Cantor set $\mathcal{C}$
(5) Brief review of Vitali set
(6) Problem #28, pp. 71 - 72, [Roy]

2. Ternary Expansions

We're quite comfortable using decimal expansions for real numbers, i.e., writing

$$x = \sum_{n=-\infty}^{N} d_n 10^n = d_N \cdot 10^N + \cdots + d_1 \cdot 10 + d_0 + \frac{d_{-1}}{10} + \cdots$$

with $d_n \in \{0, 1, \ldots, 9\}$. But the choice of 10 as our base is quite arbitrary (mathematically, not evolutionarily). In this construction we will be using ternary expansions, that is, writing

$$x = \sum_{n=-\infty}^{N} t_n 3^n = t_N \cdot 3^N + \cdots + t_1 \cdot 3 + t_0 + \frac{t_{-1}}{3} + \cdots$$

with $a_n \in \{0, 1, 2\}$. For instance,

$$\begin{aligned} 197.2 &= 1 \cdot 10^2 + 9 \cdot 10 + 7 + \frac{2}{10} \\ &= 2 \cdot 3^4 + 1 \cdot 3^3 + 0 \cdot 3^2 + 2 \cdot 3 + 2 + \frac{1}{3} + \frac{2}{3^2} + \frac{1}{3^3} + \cdots \\ &= 21022.121\ldots_3 \ . \end{aligned}$$

In defining the Cantor ternary function, we will be using ternary expansions for $x \in [0, 1]$, which can be expressed as $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$. (How can you express 1 in this way?)

3. The Cantor Set

Recall that the Cantor set $\mathcal{C}$ can be constructed by starting with the interval $[0,1]$ and iteratively removing the middle third of the remaining intervals. (Draw picture.) At each stage we are removing intervals of the form $\left(\frac{3k-2}{3^m}, \frac{3k-1}{3^m}\right)$ with $k \in \{1, \ldots, 3^{m-1}\}$.

It can be shown that the Cantor set is also the set of all numbers in $[0,1]$ that have ternary expansions with no 1s. (Discuss how at n^{th} stage numbers in left, middle and right thirds have 0, 1, and 2 as the n^{th} digit of their ternary expansions, respectively. Use picture.)

4. The Cantor Ternary Function

We define a function $f : [0,1] \to [0,1]$ as follows. Given $x \in [0,1]$ with $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$, let N be the smallest n such that $a_n = 1$. If no such n exists, let $N = \infty$. Define

$$b_n = \begin{cases} a_n/2 & \text{if } n < N \\ 1 & \text{if } n = N\,. \end{cases}$$

Define f by

$$f(x) = f\left(\sum_{n=1}^{\infty} \frac{a_n}{3^N}\right) = \sum_{n=1}^{N} \frac{b_n}{2^n}\,.$$

Note that we should check that f is well-defined since numbers of the form $\frac{a}{3^b}$ have two ternary expansions. Observe that $f(x) = \sum_{n=1}^{N} \frac{b_n}{2^n}$ is a binary expansion of a number in $[0,1]$. (Show *Mathematica* plot.)

Lemma. f is continuous.

Proof. Fix $\epsilon > 0$ and $c \in [0,1]$. Idea: make δ small enough so that the ternary expansions of x and c agree sufficiently far. Choose N such that $2^N > 1/\epsilon$. Let $\delta = \frac{1}{3^{N+1}}$. Given x with $|x - c| < \delta$, then x and c have ternary expansions (x_n) and (c_n) such that $x_n = c_n$ for all $n \le N$. Let (y_n) and (d_n) be the binary expansions of $f(x)$ and $f(c)$, i.e.,

$$f(x) = \sum_{n=1}^{\infty} \frac{y_n}{2^n} \qquad\qquad f(c) = \sum_{n=1}^{\infty} \frac{d_n}{2^n}\,.$$

Then $y_n = d_n$ for all $n \le N$. Then

$$\begin{aligned} |f(x) - f(c)| &= \left|\sum_{n=1}^{\infty} \frac{y_n}{2^n} - \sum_{n=1}^{\infty} \frac{d_n}{2^n}\right| = \left|\sum_{n=1}^{\infty} \frac{y_n - d_n}{2^n}\right| = \left|\sum_{n=1}^{N} \frac{y_n - d_n}{2^n} + \sum_{n=N+1}^{\infty} \frac{y_n - d_n}{2^n}\right| \\ &= \left|\sum_{n=N+1}^{\infty} \frac{y_n - d_n}{2^n}\right| = \frac{1}{2^N}\left|\sum_{n=1}^{\infty} \frac{y_{N+1+n} - d_{N+1+n}}{2^n}\right| \le \frac{1}{2^N} < \epsilon\,. \end{aligned}$$

$\square$

Lemma. f is monotone.

Proof. Idea: If $x < y$, then their ternary expansions (x_n) and (y_n) must differ at some point N and at that point $x_N < y_N$. $\square$

Lemma. f is constant on each interval in $[0,1] \setminus \mathcal{C}$.

Proof. Suppose $x, y \in \left(\frac{3k-2}{3^M}, \frac{3k-1}{3^M}\right)$ with ternary expansions (x_n) and (y_n). Without loss of generality, assume that M is the smallest positive integer such that $x_M = 1 = y_M$. Then $x_n = y_n$ for all $n < M$, so

$$f(x) = \sum_{n=1}^{M-1} \frac{(1/2)x_n}{2^n} + \frac{1}{2^M} = \sum_{n=1}^{M-1} \frac{(1/2)y_n}{2^n} + \frac{1}{2^M} = f(y)\,.$$

$\square$

Lemma. f maps $\mathcal{C}$ onto $[0,1]$.

Proof. Suppose $y \in [0,1]$ has binary expansion (y_n). For each n, let $x_n = 2y_n$. Then $x_n = 0$ or 2 for all n, so $x := \sum_{n=1}^{\infty} \frac{x_n}{3^n} \in \mathcal{C}$. Since

$$f(x) = \sum_{n=1}^{\infty} \frac{(1/2)x_n}{2^n} = \sum_{n=1}^{\infty} \frac{y_n}{2^n} = y$$

then f maps $\mathcal{C}$ onto $[0,1]$. $\square$

5. Facts About Nonmeasurable Sets

Recall that we constructed the Vitali set $\mathcal{V}$ by choosing representatives for the equivalence classes of the equivalence relation given by $x \sim y$ if and only if $x - y \in \mathbb{Q}$ (i.e., coset representatives for the quotient group $\mathbb{R}/\mathbb{Q}$). We showed that these representatives could be chosen to all lie in $[0,1]$, but also noted that they could be chosen to lie in any interval $[0, 1/10^n]$ by choosing a suitable decimal approximation. We proved that $\mathcal{V}$ was nonmeasurable by letting (q_n) be an enumeration of the rational numbers in $[0,1]$ and defining $\mathcal{V}_n = \mathcal{V} + q_n = \{v + q_n : v \in \mathcal{V}\}$. We will use this construction once again in the following propositions.

Proposition. If E is measurable and $E \subseteq \mathcal{V}$, then $\lambda(E) = 0$.

Proof. As in the construction of $\mathcal{V}$, let (q_i) be an enumeration of the rational numbers in $[-1,1]$. Leting $E_i = E + q_i$ for each i, then (E_i) is a disjoint sequence and $\lambda(E_i) = \lambda(E)$ for all i. (This follows by the same reasoning used in the construction of $\mathcal{V}$.) Since $E \subseteq \mathcal{V} \subseteq [0,1]$, then $\bigcup_{i \in \mathbb{Z}_{>0}} E_i \subseteq [-1,2]$. Then

$$3 \geq \lambda[-1,2] \geq \lambda\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \lambda(E_i) = \sum_{i=1}^{\infty} \lambda(E) = \lambda(E) \sum_{i=1}^{\infty} 1 = \begin{cases} 0 & \text{if } \lambda(E) = 0 \\ \infty & \text{if } \lambda(E) > 0\,. \end{cases}$$

Since $\sum_{i=1}^{\infty} \lambda(E) \leq 3$, then $\lambda(E) = 0$. $\square$

Proposition. If $A \subseteq \mathbb{R}$ with $\lambda^*(A) > 0$, then there exists $E \subseteq A$ with E nonmeasurable.

Proof. Without loss of generality, take $A \subseteq [0, 1)$. (Since $\lambda^*(A) > 0$, then it must the case that $\lambda^*(A \cap [n, n+1)) > 0$ for some n. Let $B := (A \cap [n, n+1)) - n$. Since Lebesgue measure is translation-invariant, then $\lambda^*(B) > 0$ and $B \subseteq [0, 1)$.) For each i, let $E_i = A \cap \mathcal{V}_i$ where $\mathcal{V}_i$ is, as before, the translate of $\mathcal{V}$ by q_i. For contradiction, suppose that E_i is measurable for all i. Since $E_i \subseteq \mathcal{V}_i = \mathcal{V} + q_i$, then $E_i - q_i \subseteq \mathcal{V}$. Since E_i is measurable, then $E_i - q_i$ is measurable, so $\lambda(E_i - q_i) = 0$ by the previous proposition. Thus $\lambda(E_i) = \lambda(E_i - q_i) = 0$.

Since

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} (A \cap \mathcal{V}_i) = A \cap \left(\bigcup_{i=1}^{\infty} V_i \right) \supseteq A \cap [0, 1) = A$$

then

$$0 < \lambda^*(A) \leq \lambda^* \left(\bigcup_{i=1}^{\infty} E_i \right) \leq \sum_{i=1}^{\infty} \lambda^*(E_i) = 0\,,$$

which is a contradiction. Thus E_i is nonmeasurable for some i, and $E_i \subseteq A$. □

6. Constructing A Measurable Non-Borel Set

We follow the construction indicated in Exercise 3.28, pp. 71-72 of [Roy]. Let f be the Cantor ternary function as defined above, and let $g(x) = f(x) + x$.

Lemma. $g : [0, 1] \to [0, 2]$ is a homeomorphism, i.e., g is a continuous bijection with a continuous inverse.

Proof.

- One-to-one: g is increasing
- Continuous: Since f is continuous, then g is a sum of continuous functions, hence continuous.
- Onto: Since $g(0) = 0$ and $g(1) = 2$ and g is continuous, then g attains every value between 0 and 2 by the Intermediate Value Theorem.
- Continuous Inverse: Let $h = g^{-1}$. Suppose $U \subseteq [0, 1]$ is open. Then $[0, 1] \setminus U$ is closed and bounded, hence compact. Since g is continuous, then $g([0, 1] \setminus U)$ is compact. Now

$$g([0, 1] \setminus U) = h^{-1}([0, 1] \setminus U) = [0, 2] \setminus h^{-1}(U)\,,$$

so $[0, 2] \setminus h^{-1}(U)$ is compact, hence closed and bounded. Then $h^{-1}(U) \subseteq [0, 2]$ is open, hence h is continuous.

Thereforefore g is a homeomorphism.

□

Lemma. $g(\mathcal{C})$ has measure 1.

Proof. Recall that f is constant on any interval in $[0, 1] \setminus \mathcal{C}$. Thus for any interval $(a, b) \subseteq [0, 1] \setminus \mathcal{C}$,

$$\lambda(g(a), g(b)) = g(b) - g(a) = f(b) + b - f(a) - a = b - a\,.$$

Let $\{I_{n,k}\}_{k=1}^{2^{n-1}}$ denote the collection of intervals removed at stage n in the construction of $\mathcal{C}$. Then

$$\lambda([0,2]\setminus\mathcal{C}) = \lambda(g([0,1]\setminus\mathcal{C})) = \lambda\left(g\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^{n-1}} I_{n,k}\right)\right) = \lambda\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^{n-1}} g(I_{n,k})\right)$$
$$= \sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}} \lambda(g(I_{n,k})) = \sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}} \lambda(I_{n,k}) = 1$$

since the total measure of intervals removed is 1. Since $[0,2] = g(\mathcal{C}) \uplus ([0,2]\setminus g(\mathcal{C}))$, then

$$2 = \lambda[0,2] = \lambda(g(\mathcal{C})) + \lambda([0,2]\setminus g(\mathcal{C})) = \lambda(g(\mathcal{C})) + 1\,,$$

hence $\lambda(g(\mathcal{C})) = 1$. □

Since $\lambda(g(\mathcal{C})) > 0$, then there exists a nonmeasurable $E \subseteq g(\mathcal{C})$. Let $A = g^{-1}(E)$. Since $A \subseteq \mathcal{C}$, then $\lambda^*(A) \leq \lambda^*(\mathcal{C}) = 0$. Thus A has outer measure zero, hence is measurable, but $g(A) = E$ is nonmeasurable.

Since $g^{-1} = h$ is continuous, hence measurable. We claim that A is not a Borel set. For contradiction, suppose A is Borel. Since h is measurable, then $h^{-1}(A) = g(A) = E$ is measurable, which is a contradiction. Therefore A is not a Borel set.

References

[Roy] H.L. Royden, *Real Analysis*. Prentice Hall, New Jersey, 3rd Edition, 1988.