J. Eur. Math. Soc. 10, 1061–1104

F. Da Lio · N. Forcadel · R. Monneau

Convergence of a non-local eikonal equation to anisotropic mean curvature motion. Application to dislocation dynamics

Received April 20, 2006 and in revised form November 20, 2006

Abstract. We prove the convergence at a large scale of a non-local first order equation to an anisotropic mean curvature motion. The equation is an eikonal-type equation with a velocity depending in a non-local way on the solution itself, which arises in the theory of dislocation dynamics. We show that if an anisotropic mean curvature motion is approximated by equations of this type then it is always of variational type, whereas the converse is true only in dimension two.

Keywords. Dislocation dynamics, asymptotic behaviour, non-local equations, eikonal equation, mean curvature motion, viscosity solutions

1. Introduction

1.1. Physical motivation

In this paper, we study the asymptotic behaviour of an equation modelling dislocation dynamics. More precisely, we show that, at a large scale, dislocation dynamics is given by a mean curvature motion (we refer to Subsection 1.3 for the exact setting of the result). Dislocations are line defects in crystals whose typical length in metallic alloys is of the order of 10^{-6} m and thickness of the order of 10^{-9} m. The concept of dislocations in crystals was put forward in the XXth century, as the main microscopic explanation of the macroscopic plastic behaviour of metallic crystals (see the physical monograph of Hirth and Lothe [31]). Since the beginning of the 90's, the research field of dislocations is enjoying a new development, in particular thanks to the power of computers which allows simulations with a large number of dislocations.

Recently Rodney, Le Bouar and Finel introduced in [37] a new model called the *phase field model of dislocation*. In this model, the dislocation line in the crystal moves in its slip plane with a normal velocity which is proportional to the Peach–Koeller force acting on this line. In the case where there is no exterior stress, this force is simply the self-force

F. Da Lio: Dipartimento di Matematica Pura ed Applicata, Università degli Studi di Padova, Via Belzoni, 7, 35131 Padova, Italy; e-mail: dalio@math.unipd.it

N. Forcadel, R. Monneau: CERMICS, École Nationale des Ponts et Chaussées, 6 et 8 avenue Blaise Pascal, Cité Descartes, Champs-sur-Marne, 77455 Marne-la-Vallée Cedex 2, France;
e-mail: forcadel@cermics.enpc.fr, monneau@cermics.enpc.fr

Mathematics Subject Classification (2000): 35F25, 35D05, 35Q99, 35B40, 35G25, 49L25

created by the elastic field generated by the dislocation line itself. In [5], [4], Alvarez, Hoch, Le Bouar and Monneau proposed to rewrite this model as a non-local Hamilton–Jacobi equation. Using viscosity solutions (we refer to the monographs of Barles [7] and Bardi and Capuzzo-Dolcetta [6] and to the paper of Crandall, Ishii and Lions [21] for a good introduction to this theory), Alvarez et al. [5], [4] proved a short time existence and uniqueness result. Then Alvarez, Cardaliaguet and Monneau [1] and Barles and Ley [10] proved a long time result under certain assumptions. We also refer to Forcadel [27] for a uniqueness and existence result for dislocation dynamics with a mean curvature term. This equation was also numerically studied by Alvarez, Carlini, Monneau and Rouy [2], [3].

Mathematically, a dislocation line is represented by the boundary of a bounded domain $\Omega \subset \mathbb{R}^2$ which moves with normal speed given by

$$V_n = \overline{c}_0 \star \rho$$

where the kernel $\overline{c}_0 = \overline{c}_0(x)$ depends only on the space variables, $\star$ denotes convolution in space and ρ is the characteristic function of the set Ω, i.e.

$$\rho(x) = \begin{cases} 1 & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

In this paper, we consider a simplification of the model proposed by Alvarez et al. [5], [4]. We assume that the negative part of the kernel $\overline{c}_0$ is concentrated at one point, i.e., $\overline{c}_0 = c_0 - (\int_{\mathbb{R}^2} c_0)\delta_0$ where c_0 is now a positive kernel. Because of the formal half contribution of the Dirac mass to $\overline{c}_0 \star \rho$ on the dislocation line $\partial\Omega$, we can rewrite (formally on the dislocation line)

$$V_n = c_0 \star \rho - \frac{1}{2}\int_{\mathbb{R}^2} c_0.$$

For this model, we will be able to prove, in the framework of the Slepčev level set formulation (see [39]), a long time existence and uniqueness result for the solution of this equation (see Section 2).

Physically, the kernel c_0 is assumed to behave like $1/|x|^3$ at infinity. We refer to Section 4.1 for more details and note that the decay of our kernel fails to satisfy the natural integrability condition (1.10). For this reason, we can rescale the characteristic function ρ, defining

$$\rho^\varepsilon(x,t) = \rho\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2|\ln\varepsilon|}\right).$$

This is almost the parabolic scaling. Here the presence of the logarithm is a well-known factor in physics (see for instance Barnet Gavazza [14], Brown [18] or Hirth and Lothe [31]). We then show that at a large scale (i.e. $\varepsilon \to 0$), the normal speed of the dislocation line associated to ρ^ε is given by anisotropic mean curvature of the line. More precisely, we show that the solution of the non-local Hamilton–Jacobi equation modelling dislocation dynamics converges, at a large scale, to the solution of a mean curvature motion. We also study the link between the energy of dislocations and the energy associated to the mean curvature motion, and we prove a formal convergence of the energies. We show that the mean curvature motion we can approach with this type of non-local eikonal equations is always of variational type. Finally, we show that in the two-dimensional case, essentially all mean curvature motions of variational type can be approximated, which is not true in higher dimensions.

This result is very natural for dislocation dynamics. Indeed, in many references in physics, the authors describe dislocation dynamics by line tension terms deriving from an energy associated to the dislocation line. See for instance Brown [18] and Barnet and Gavazza [14] for physical references, and Garroni and Müller [29], [28] for a variational approach. As far as we know, our result is the first rigorous proof for the convergence of dislocation dynamics to mean curvature motion.

Similar results have already been proved for general kernels in relation with the Merriman–Bence–Osher algorithm for computing mean curvature motion [36]. We refer to Barles and Georgelin [9], Evans [25], Ishii [33] and Ishii, Pires and Souganidis [34] for results of this kind. We also refer to Souganidis [40] for an example where the kernels are fractional laplacian. Nevertheless, our kernel does not satisfy the assumptions of these papers. We refer to Subsection 4.1 for a comparison with other related works. Moreover, we show in Section 7 that the limit mean curvature motion obtained by convolution is of variational type.

1.2. Mathematical setting of the problem

Given a function g defined on the unit sphere $\mathbf{S}^{n-1}$ of $\mathbb{R}^n$ with

$$g \in C^0(\mathbf{S}^{n-1}), \quad g(-\theta) = g(\theta) \geq 0, \quad \forall \theta \in \mathbf{S}^{n-1}, \tag{1.1}$$

we consider kernels $c_0 \in L^\infty(\mathbb{R}^n)$ satisfying

$$\begin{cases} c_0(x) = \dfrac{1}{|x|^{n+1}} g\left(\dfrac{x}{|x|}\right) & \text{if } |x| \geq 1, \\ c_0(-x) = c_0(x) \geq 0, & \forall x \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

We want to see what happens for large dislocation, i.e., at a large scale. Up to a change of variable, this is equivalent to concentrating the kernel. Since c_0 behaves like $1/|x|^{n+1}$ at infinity (see (1.2)), the "natural scaling" is then the following one for $0 < \varepsilon < 1$:

$$c_0^\varepsilon(x) = \frac{1}{\varepsilon^{n+1}|\ln \varepsilon|} c_0\left(\frac{x}{\varepsilon}\right). \tag{1.3}$$

The presence of the logarithm comes out naturally in the proofs (see Subsection 4.1) but it is also expected from a physical point of view.

We will use the level set formulation in the sense that the dislocation line (here in any dimension $n \geq 1$) is represented by any level set of a continuous function u^ε, solving the following equation (in the sense of Definition 2.1):

$$\begin{cases} u_t^\varepsilon(x,t) = \left((c_0^\varepsilon \star 1_{\{u^\varepsilon(\cdot,t) > u^\varepsilon(x,t)\}})(x) - \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^n} c_0^\varepsilon \right) |Du^\varepsilon(x,t)| & \text{in } \mathbb{R}^n \times (0,T), \\ u^\varepsilon(\cdot,0) = u_0(\cdot) & \text{in } \mathbb{R}^n, \end{cases} \tag{1.4}$$

where Du^ε indicates the gradient of u^ε with respect to the space variables, the convolution is done in space only and 1_A is the characteristic function of the set A. Here, we consider the simultaneous evolutions of all the level sets of the function u^ε. This approach has been introduced by Slepčev [39] (see also Da Lio, Kim and Slepčev [23]).

We will prove that the unique viscosity solution of (1.4) converges to the unique solution of a mean curvature type equation.

1.3. Main results

We denote by $C^{1,1/2}_{x,t}(\mathbb{R}^n \times [0,T])$ the set of continuous functions satisfying a Lipschitz condition in x and a Hölder condition in t of exponent $1/2$, and by $\mathrm{Lip}(\mathbb{R}^n)$ the set of Lipschitz continuous functions.

Theorem 1.1 (Existence, uniqueness and regularity for the ε-problem). *Let $n \geq 1$. Assume that $u_0 \in \mathrm{Lip}(\mathbb{R}^n)$ and $c_0 \in W^{1,1}(\mathbb{R}^n)$. Then for all $\varepsilon \in (0,1)$, there exists a unique viscosity solution u^ε of* (1.4) *in the sense of Definition* 2.1. *Moreover, u^ε is $C^{1,1/2}_{x,t}(\mathbb{R}^n \times [0,T])$ uniformly in ε for $\varepsilon \in (0,1/2)$. Namely, we have the following estimates for $\varepsilon \in (0,1/2)$:*

$$|Du^\varepsilon(\cdot,t)|_{L^\infty(\mathbb{R}^n)} \leq |Du_0|_{L^\infty(\mathbb{R}^n)}, \qquad \forall t \geq 0,$$

and

$$|u^\varepsilon(x,t+h) - u^\varepsilon(x,t)| \leq C|Du_0|_{L^\infty(\mathbb{R}^n)}\sqrt{h}, \qquad \forall x \in \mathbb{R}^n,\ t \geq 0,\ h \in [0,1],$$

where the constant C depends only on n and $\sup_{\mathbb{R}^n} c_0$.

We are interested in the limit problem satisfied by the limit u^0 of u^ε as ε goes to zero. For this purpose, we consider the following problem:

$$\begin{cases} u^0_t(x,t) + F(D^2u^0, Du^0) = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u^0(\cdot,0) = u_0(\cdot) & \text{in } \mathbb{R}^n, \end{cases} \tag{1.5}$$

with

$$F(M,p) = -\,\mathrm{trace}\left(M \cdot A\left(\frac{p}{|p|}\right)\right) \tag{1.6}$$

with

$$A\left(\frac{p}{|p|}\right) = \int_{\theta \in \mathbf{S}^{n-2} = \mathbf{S}^{n-1} \cap \{\langle x, p/|p|\rangle = 0\}} \left(\frac{1}{2} g(\theta)\theta \otimes \theta\right) d\theta. \tag{1.7}$$

Hereafter $M \cdot A$ and $\langle \cdot,\cdot \rangle$ denote respectively the product of the two matrices and the usual scalar product.

Remark 1.2. In particular, F is geometric (see Barles, Soner and Souganidis [12]), because $M \mapsto F(M,p)$ is linear and

$$F(M,p) = F\left(\left(\mathrm{Id} - \frac{p}{|p|} \otimes \frac{p}{|p|}\right) \cdot M, \frac{p}{|p|}\right).$$

Remark 1.3. In the particular case where $g \equiv 1$, we get

$$A = \frac{|\mathbf{S}^{n-2}|}{2(n-1)} \operatorname{Id}_{\{\langle x,p\rangle=0\}}$$

where $|\mathbf{S}^{n-2}|$ is the Lebesgue measure of $\mathbf{S}^{n-2}$, and then

$$F(M,p) = \frac{-|\mathbf{S}^{n-2}|}{2(n-1)} \operatorname{trace}\left(\left(\operatorname{Id} - \frac{p}{|p|} \otimes \frac{p}{|p|}\right) \cdot M\right).$$

We recover the classical mean curvature motion up to the factor $|\mathbf{S}^{n-2}|/2(n-1)$.

We prove the following result:

Theorem 1.4 (Convergence of dislocation dynamics to mean curvature motion). *Let $n \geq 1$. Given $u_0 \in \operatorname{Lip}(\mathbb{R}^n)$ and $c_0 \in W^{1,1}(\mathbb{R}^n)$, we consider the solution u^ε of problem* (1.4) *with the kernel c_0^ε defined in* (1.1)–(1.3)*. Then the solution u^ε converges locally uniformly on compact sets in $\mathbb{R}^n \times [0,\infty)$ to the unique viscosity solution u^0 of* (1.5)–(1.7).

Remark 1.5. This result also suggests a natural scheme to compute the mean curvature motion numerically. This is the subject of a paper in preparation [22].

From expression (1.6)–(1.7) it is not clear if the anisotropic mean curvature motion (1.5) is of a variational type or not. Theorem 1.7 below will show that this mean curvature motion is indeed of variational type. Before stating it, we need the following definition:

Definition 1.6. *Let $g \in C^0(\mathbb{R}^n \setminus \{0\})$ satisfy $g(\lambda p) = g(p)/|\lambda|^{n+1}$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $p \in \mathbb{R}^n \setminus \{0\}$. We then associate to g a tempered distribution L_g defined by*

$$\langle L_g, \varphi \rangle = \int_{\mathbb{R}^n} dx\, g(x)(\varphi(x) - \varphi(0) - x \cdot D\varphi(0)\, 1_{B_1(0)}(x))$$

for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of test functions, and $B_1(0)$ denotes the unit ball centred at zero.

We define the Fourier transform of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ as

$$\mathcal{F}(\varphi)(\xi) = \int_{\mathbb{R}^n} dx\, \varphi(x) e^{-i\xi\cdot x}.$$

We have the following theorem:

Theorem 1.7 (Variational origin of the anisotropic mean curvature motion). *Let $n \geq 2$. Let $g \in C^0(\mathbb{R}^n \setminus \{0\})$ satisfy $g(\lambda p) = g(p)/|\lambda|^{n+1}$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $p \in \mathbb{R}^n \setminus \{0\}$. Then*

$$\int_{\mathbf{S}^{n-1}\cap\{\langle x, p/|p|\rangle=0\}} \frac{1}{2} g(\theta)\theta \otimes \theta\, d\theta = D^2 G\left(\frac{p}{|p|}\right) \quad \textit{with} \quad G := -\frac{1}{2\pi}\mathcal{F}(L_g) \qquad (1.8)$$

where $\mathcal{F}(L_g)$ is the Fourier transform of L_g. Moreover $G(\lambda p) = |\lambda| G(p)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $p \in \mathbb{R}^n$ and, with A defined in (1.7)*, if $u^0 \in C^2(\mathbb{R}^n)$ with $|Du^0| \neq 0$, then*

$$\frac{1}{|Du^0|} \operatorname{trace}\left(A\left(\frac{Du^0}{|Du^0|} \right) \cdot D^2 u^0 \right) = \operatorname{div}\left(\nabla G\left(\frac{Du^0}{|Du^0|} \right) \right), \tag{1.9}$$

which means that the mean curvature motion derives from the energy $\int G(Du^0)$.

Moreover, if $g \geq 0$, then G is convex.

The converse is true in the two-dimensional case: if $G \in C^0(\mathbb{R}^2) \cap C^2(\mathbb{R}^2 \setminus \{0\})$ is convex and satisfies $G(\lambda p) = |\lambda| G(p)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $p \in \mathbb{R}^2$, then there exists a non-negative function g such that $L_g = -(2\pi)^{N-1} \mathcal{F}(G)$.

A different non-local equation for a mean field model describing a spin flip dynamics has been studied by De Masi, Orlandi, Presutti and Triolo [24], Katsoulakis and Souganidis [35] and Barles, Souganidis [13]. In [15], Bellettini, Buttà and Presutti have proved that the limit dynamics is related to the Hessian of an energy.

Proposition 1.8 (Counter-example). *The converse of Theorem* 1.7 *is false in dimension $n \geq 3$, i.e., there exists g which changes its sign such that $A(p) = D^2 G(p) \geq 0$.*

Remark 1.9. If g is a positive measure, we can formally approximate crystalline curvature by our non-local eikonal equation.

Remark 1.10. Physically, only $\mathcal{F}(L_g)$ is known. We see that formula (1.8) allows one to compute g easily in dimension $n = 2$ and then to check that $g \geq 0$ or not. See Hirth and Lothe [31, Chapter 13-8] for an example where g is not non-negative, and Head [30] for examples in cubic elasticity.

In the simplest case of applications for dislocation dynamics, the crystal is described by isotropic elasticity (see [31]). When the Burgers vector is along the x_1 direction, we have

$$G(p) = \frac{p_2^2 + \frac{1}{1-\nu} p_1^2}{|p|} \quad \text{with} \quad \nu \in (-1, 1/2)$$

where ν is the Poisson ratio of the material, and

$$g(\theta) = \frac{(2\gamma - 1)(\theta_1)^2 + (2-\gamma)(\theta_2)^2}{|\theta|^5} \geq 0 \quad \text{with} \quad \gamma = \frac{1}{1-\nu} \in (1/2, 2).$$

It is well-known that we can approach generalized mean curvature motion with the Merriman–Bence–Osher [36] construction with a general kernel K_0 satisfying $K_0(-x) = K_0(x)$ and for every $p \in \mathbf{S}^{n-1}$,

$$\int_{p^\perp} K_0(x)|x|^2 < \infty \tag{1.10}$$

where $p^\perp = \{x : \langle x, p \rangle = 0\}$ and with the "parabolic scaling" $K_0^\varepsilon = \varepsilon^{-n-1} K_0(x/\varepsilon)$. We refer, for instance to Barles and Georgelin [9], Evans [25], Ishii [33] and Ishii, Pires and Souganidis [34] (we also refer to Subsection 4.1 for a formal proof).

More precisely, under the additional assumption that $\int_{p^\perp} K_0(x) = 1$ for any $p \in \mathbf{S}^{n-1}$, the limit motion found in [34, Section 3] with the threshold $\theta = 1/2$ is (1.5)–(1.6), with (1.7) replaced by

$$A\left(\frac{p}{|p|}\right) = \int_{p^\perp} \left(\frac{1}{2} K_0(x) \cdot x \otimes x\right) dx. \tag{1.11}$$

Up to our knowledge, it has not been known in this general setting if the limit mean curvature motion associated to (1.11) is of variational type (cf. (1.9)). It turns out that this is a simple consequence of our Theorem 1.7:

Theorem 1.11 (Variational property of the limit motion). *Every mean curvature motion of the form of* (1.5)–(1.6) *with A defined in* (1.11) *is of variational type.*

The problem we consider is formally associated to the following energy:

$$\mathcal{E}^\varepsilon(u^\varepsilon) = \int_\lambda \overline{\mathcal{E}^\varepsilon}(\lambda)\, d\lambda \tag{1.12}$$

where

$$\overline{\mathcal{E}^\varepsilon}(\lambda) = \int_{\mathbb{R}^n} -\frac{1}{2}(\overline{c}_0^\varepsilon \star \rho_\lambda^\varepsilon)\rho_\lambda^\varepsilon$$

with

$$\rho_\lambda^\varepsilon = 1_{\{u^\varepsilon > \lambda\}}, \qquad \overline{c}_0^\varepsilon = c_0^\varepsilon - \left(\int_{\mathbb{R}^n} c_0^\varepsilon\right)\delta_0.$$

We will show formally in Section 8 that this energy is non-increasing in time and that there is a convex function G such that $\mathcal{E}^\varepsilon(u^\varepsilon) \to \int G(Du^0)$, which is the energy associated to the mean curvature motion of the limit solution u^0.

1.4. Organisation of the paper

Let us now explain how this paper is organised: Section 2 is devoted to the study of the ε-problem. In Section 3, we recall some known results on the limit problem. Then we give, in Section 4, a result on the convergence of the velocity for a test function. The regularity result of Theorem 1.1 is proved in Section 5 (see Corollary 5.3) together with estimates at initial time. The convergence result (Theorem 1.4) is proved in Section 6. The variational property of the limit motion (Theorems 1.7 and 1.11) and the counter-example (Proposition 1.8) are proved in Section 7. In Section 8, we study very formally the link between energy and mean curvature motion. Finally, in an appendix, we give some technical lemmata on Fourier transform.

2. Existence and uniqueness for the ε-problem

We will denote by $B_{\mathrm{loc}}USC(\mathbb{R}^n \times [0,T])$ and $B_{\mathrm{loc}}LSC(\mathbb{R}^n \times [0,T])$ respectively the set of locally bounded upper semicontinuous and lower semicontinuous functions in $\mathbb{R}^n \times [0,T]$.

Definition 2.1. *A function* $u^\varepsilon \in B_{\mathrm{loc}}USC(\mathbb{R}^n \times [0,T])$ *is a* viscosity subsolution of (1.4) *if it satisfies:*

(i) $u^\varepsilon(x,0) \le u_0(x)$ *in* $\mathbb{R}^n$,
(ii) *for every* $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ *and every test function* $\Phi \in C^\infty(\mathbb{R}^n \times [0,T))$ *such that* $u^\varepsilon - \Phi$ *has a maximum at* (x_0,t_0),

$$\Phi_t(x_0,t_0) \le \left((c_0^\varepsilon \star 1_{\{u^\varepsilon(\cdot,t_0) \ge u^\varepsilon(x_0,t_0)\}})(x_0) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon \right) |D\Phi(x_0,t_0)|. \tag{2.13}$$

A function $u^\varepsilon \in B_{\mathrm{loc}}LSC(\mathbb{R}^n \times [0,T])$ *is a* viscosity supersolution of (1.4) *if it satisfies:*

(i) $u^\varepsilon(x,0) \ge u_0(x)$ *in* $\mathbb{R}^n$,
(ii) *for every* $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ *and every test function* $\Phi \in C^\infty(\mathbb{R}^n \times [0,T))$ *such that* $u^\varepsilon - \Phi$ *has a minimum at* (x_0,t_0),

$$\Phi_t(x_0,t_0) \ge \left((c_0^\varepsilon \star 1_{\{u^\varepsilon(\cdot,t_0) > u^\varepsilon(x_0,t_0)\}})(x_0) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon \right) |D\Phi(x_0,t_0)|. \tag{2.14}$$

A continuous function u^ε *is a* viscosity solution of (1.4) *if it is a subsolution and supersolution of* (1.4).

This definition comes from the definition of viscosity solution for non-local equations given by Slepčev [39] (see also Da Lio, Kim and Slepčev [23]) and it permits one to extend to non-local equations all properties enjoyed by viscosity solutions of local equations.

Note the difference in the choice of the set in the indicator function in the definition of a subsolution and a supersolution. This is crucial to extending all the properties of viscosity solutions to non-local, geometric parabolic equations (see Slepčev [39]), in particular for the stability of the solution, i.e., the lim sup of subsolutions is a subsolution (and so the existence by Perron's method).

Next we prove a comparison result for locally bounded semicontinuous viscosity subsolutions and supersolutions to the equation (1.4).

Theorem 2.2 (Comparison principle for the ε-problem). *Assume* $c_0 \in W^{1,1}(\mathbb{R}^n)$. *Let* $u \in B_{\mathrm{loc}}USC(\mathbb{R}^n \times [0,T])$ *and* $v \in B_{\mathrm{loc}}LSC(\mathbb{R}^n \times [0,T])$ *be respectively a viscosity subsolution and supersolution of* (1.4). *If* $u(x,0) \le v(x,0)$ *for all* $x \in \mathbb{R}^n$ *then* $u(x,t) \le v(x,t)$ *for all* $(x,t) \in \mathbb{R}^n \times [0,T]$.

To prove this result, we need the analogue of Ishii's lemma for non-local equations. We first recall the definition of the limit subdifferential and superdifferential:

$$\bar{\mathcal{P}}^+ u(x,t) = \left\{ \begin{array}{l} (p,a) \in \mathbb{R}^n \times \mathbb{R} : \exists (x_n,t_n,p_n,a_n) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \\ \text{such that } (p_n,a_n) \in \mathcal{P}^+ u(x_n,t_n) \\ \text{and } (x_n,t_n,u(x_n,t_n),p_n,a_n) \to (x,t,u(x,t),p,a) \end{array} \right\}$$

where $\mathcal{P}^+$ is the classical superdifferential. The set $\bar{\mathcal{P}}^- u(x,t)$ is defined in a similar way. It is well known that we have an equivalent definition of a viscosity solution by using subdifferentials and superdifferentials (cf. Crandall, Ishii and Lions [21]). We claim that the definition remains equivalent if we replace the classical subdifferentials and superdifferentials by the limit ones. Indeed, let $u \in B_{\mathrm{loc}}USC(\mathbb{R}^n \times [0,T])$ be a viscosity subsolution of (1.4). We will show that

$$(p,a) \in \bar{\mathcal{P}}^+ u(x,t) \;\Rightarrow\; a \le \left(c_0^\varepsilon \star 1_{\{u(\cdot,t) \ge u(x,t)\}}(x) - \frac{1}{2}\int c_0^\varepsilon \right)|p|. \tag{2.15}$$

Let $(x_n,t_n,p_n,a_n) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ be such that $(p_n,a_n) \in \mathcal{P}^+ u(x_n,t_n)$ and $(x_n,t_n,u(x_n,t_n),p_n,a_n) \to (x,t,u(x,t),p,a)$. We then have, by definition,

$$\begin{aligned} a_n &\le \left(c_0^\varepsilon \star 1_{\{u(\cdot,t_n) \ge u(x_n,t_n)\}}(x_n) - \frac{1}{2}\int c_0^\varepsilon \right)|p_n| \\ &\le \left(c_0^\varepsilon \star 1_{\{u(\cdot,t_n) \ge u(x_n,t_n)\} \cup \{u(\cdot,t) \ge u(x,t)\}}(x_n) - \frac{1}{2}\int c_0^\varepsilon \right)|p_n|. \end{aligned}$$

We just have to show that

$$c_0^\varepsilon \star 1_{\{u(\cdot,t_n) \ge u(x_n,t_n)\} \cup \{u(\cdot,t) \ge u(x,t)\}}(x_n) \to c_0^\varepsilon \star 1_{\{u(\cdot,t) \ge u(x,t)\}}(x).$$

To do this, we use the following decomposition:

$$\begin{aligned} &c_0^\varepsilon \star 1_{\{u(\cdot,t_n) \ge u(x_n,t_n)\} \cup \{u(\cdot,t) \ge u(x,t)\}}(x_n) - c_0^\varepsilon \star 1_{\{u(\cdot,t) \ge u(x,t)\}}(x) \\ &\quad = c_0^\varepsilon \star 1_{\{u(\cdot,t_n) \ge u(x_n,t_n)\} \cup \{u(\cdot,t) \ge u(x,t)\}}(x_n) - c_0^\varepsilon \star 1_{\{u(\cdot,t_n) \ge u(x_n,t_n)\} \cup \{u(\cdot,t) \ge u(x,t)\}}(x) \\ &\qquad + c_0^\varepsilon \star 1_{\{u(\cdot,t_n) \ge u(x_n,t_n)\} \cup \{u(\cdot,t) \ge u(x,t)\} \setminus \{u(\cdot,t) \ge u(x,t)\}}(x). \end{aligned}$$

The first part clearly goes to zero as n goes to infinity. For the second part, we need the following lemma:

Lemma 2.3. *Let f_n be a sequence of measurable functions on $\mathbb{R}^n$ and*

$$f(x) \ge \sup\left\{ \limsup_{n\to\infty} f_n(y) : y \to x \right\}.$$

Let a_n be a sequence converging to zero. Then

$$\mathcal{L}(\{f_n \ge a_n\} \setminus \{f \ge 0\}) \to 0 \quad \text{as } n \to \infty,$$

where, for any measurable set A, $\mathcal{L}(A)$ denotes the Lebesgue measure of A.

For the proof, we refer to Slepčev [39].

Applying this lemma with $f_n = u(\cdot,t_n) - u(x,t)$, $a_n = u(x_n,t_n) - u(x,t)$ and $f = u(\cdot,t) - u(x,t)$ yields (2.15). The proof for a supersolution is analogous.

Using (2.15), we can rewrite Ishii's lemma (see [21, Lemma 8.3]) for non-local equations:

Lemma 2.4 (Ishii's lemma for non-local equations). *Let U and V be open sets in $\mathbb{R}^n$, and for $T > 0$, let $u \in B_{\text{loc}}USC(U \times (0,T))$ and $v \in B_{\text{loc}}LSC(V \times (0,T))$ be respectively a subsolution and supersolution of* (1.4)*. Let $\phi : U \times V \times (0,T) \to (0,\infty)$ be of class C^∞. Assume that $(x,y,t) \mapsto u(x,t) - v(y,t) - \phi(x,y,t)$ reaches a local maximum at $(\bar{x},\bar{y},\bar{t}) \in U \times V \times (0,T)$. Set $\tau = \partial_t\phi(\bar{x},\bar{y},\bar{t})$, $p_1 = D_x\phi(\bar{x},\bar{y},\bar{t})$, and $p_2 = -D_y\phi(\bar{x},\bar{y},\bar{t})$ Then there exist $\tau_1, \tau_2 \in \mathbb{R}$ such that*

$$\tau = \tau_1 - \tau_2, \quad (p_1,\tau_1) \in \bar{\mathcal{P}}^+ u(\bar{x},\bar{t}), \quad (p_2,\tau_2) \in \bar{\mathcal{P}}^- v(\bar{y},\bar{t}),$$

and so

$$\tau_1 \le \left(c_0^\varepsilon \star 1_{\{u(\cdot,\bar{t}) \ge u(\bar{x},\bar{t})\}}(\bar{x}) - \frac{1}{2}\int c_0^\varepsilon \right)|p|$$

and

$$\tau_2 \ge \left(c_0^\varepsilon \star 1_{\{v(\cdot,\bar{t}) > v(\bar{y},\bar{t})\}}(\bar{y}) - \frac{1}{2}\int c_0^\varepsilon \right)|q|.$$

Proof of Theorem 2.2. The proof is inspired by Barles, Cardaliaguet, Ley and Monneau [8]. Let $u \in B_{\text{loc}}USC(\mathbb{R}^n \times [0,T])$ and $v \in B_{\text{loc}}LSC(\mathbb{R}^n \times [0,T])$ be respectively a viscosity subsolution and supersolution of (1.4). Since the equation is geometric we may assume that u and v are bounded (see Slepčev [39, property (P1)]). Suppose for contradiction that $M = \sup_{\mathbb{R}^n \times [0,T]}(u(x,t) - v(x,t)) > 0$. Then for $\eta \in (0,1)$ small enough we have $M_\eta = \sup_{t\in[0,T]} \limsup_{|x-y|\to 0}(u(x,t) - v(y,t) - \eta t) > 0$ as well.

For all $\gamma > 0$ and $\alpha > 0$ with $\alpha \ll \gamma$, we define the auxiliary function $\Phi_{\gamma,\alpha} : \mathbb{R}^n \times \mathbb{R}^n \times [0,T] \to \mathbb{R}$ by

$$\Phi_{\gamma,\alpha}(x,y,t) = u(x,t) - v(y,t) - \eta t - |x-y|^2/\gamma^2 - \alpha(|x|^2 + |y|^2). \tag{2.16}$$

We observe that $\limsup_{|x|,|y|\to\infty} \Phi_{\gamma,\alpha}(x,y,t) = -\infty$, thus $\Phi_{\gamma,\alpha}(x,y,t)$ reaches its maximum at a point $(x_{\gamma,\alpha}, y_{\gamma,\alpha}, t_{\gamma,\alpha}) \in \mathbb{R}^n \times \mathbb{R}^n \times [0,T]$. Standard arguments show that

$$\alpha(|x_{\gamma,\alpha}|^2 + |y_{\gamma,\alpha}|^2),\ |x_{\gamma,\alpha} - y_{\gamma,\alpha}|^2/\gamma^2 \le C_0, \tag{2.17}$$

with $C_0 > 0$ depending on $\|u\|_\infty, \|v\|_\infty$. In particular,

$$\lim_{\gamma\to 0} \limsup_{\alpha\to 0} |x_{\gamma,\alpha} - y_{\gamma,\alpha}| = 0.$$

Then

$$\begin{aligned} &\limsup_{\gamma\to 0} \limsup_{\alpha\to 0} \Phi_{\gamma,\alpha}(x_{\gamma,\alpha}, y_{\gamma,\alpha}, t_{\gamma,\alpha}) \\ &\qquad \le \limsup_{\gamma\to 0} \limsup_{\alpha\to 0}(u(x_{\gamma,\alpha}, t_{\gamma,\alpha}) - v(y_{\gamma,\alpha}, t_{\gamma,\alpha}) - \eta t_{\gamma,\alpha}) \le M_\eta. \end{aligned} \tag{2.18}$$

We also have

$$\liminf_{\gamma\to 0} \liminf_{\alpha\to 0} \Phi_{\gamma,\alpha}(x_{\gamma,\alpha}, y_{\gamma,\alpha}, t_{\gamma,\alpha}) \ge M_\eta. \tag{2.19}$$

Indeed, by definition, for all $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times [0, T]$ we have

$$\begin{aligned} u(x,t) - v(y,t) - \eta t - |x-y|^2/\gamma^2 - \alpha(|x|^2 + |y|^2) \\ \le \Phi_{\gamma,\alpha}(x_{\gamma,\alpha}, y_{\gamma,\alpha}, t_{\gamma,\alpha}) \le u(x_{\gamma,\alpha}, t_{\gamma,\alpha}) - v(y_{\gamma,\alpha}, t_{\gamma,\alpha}) - \eta t_{\gamma,\alpha}. \end{aligned}$$

We first take $\liminf_{\alpha\to 0}$ to get

$$\begin{aligned} u(x,t) - v(y,t) - \eta t - |x-y|^2/\gamma^2 &\le \liminf_{\alpha\to 0} \Phi_{\gamma,\alpha}(x_{\gamma,\alpha}, y_{\gamma,\alpha}, t_{\gamma,\alpha}) \\ &\le \liminf_{\alpha\to 0}(u(x_{\gamma,\alpha}, t_{\gamma,\alpha}) - v(y_{\gamma,\alpha}, t_{\gamma,\alpha}) - \eta t_{\gamma,\alpha}). \end{aligned} \tag{2.20}$$

Then we take $\limsup_{|x-y|\to 0}$ to get

$$\sup_{t\in[0,T]} \limsup_{|x-y|\to 0} (u(x,t) - v(y,t) - \eta t) \le \liminf_{\alpha\to 0} \Phi_{\gamma,\alpha}(x_{\gamma,\alpha}, y_{\gamma,\alpha}, t_{\gamma,\alpha}),$$

and finally take $\liminf_{\gamma\to 0}$ to get (2.19).

By combining (2.19) and (2.18) we get

$$\begin{aligned} M_\eta &\le \liminf_{\gamma\to 0} \liminf_{\alpha\to 0} \Phi_{\gamma,\alpha}(x_{\gamma,\alpha}, y_{\gamma,\alpha}, t_{\gamma,\alpha}) \\ &\le \limsup_{\gamma\to 0} \limsup_{\alpha\to 0} \Phi_{\gamma,\alpha}(x_{\gamma,\alpha}, y_{\gamma,\alpha}, t_{\gamma,\alpha}) \le M_\eta. \end{aligned}$$

Therefore

$$\lim_{\gamma\to 0} \liminf_{\alpha\to 0} \Phi_{\gamma,\alpha}(x_{\gamma,\alpha}, y_{\gamma,\alpha}, t_{\gamma,\alpha}) = \lim_{\gamma\to 0} \limsup_{\alpha\to 0} \Phi_{\gamma,\alpha}(x_{\gamma,\alpha}, y_{\gamma,\alpha}, t_{\gamma,\alpha}) = M_\eta.$$

In an analogous way, we can deduce (using (2.18) and (2.20)) that

$$\begin{aligned} M_\eta &= \lim_{\gamma\to 0} \liminf_{\alpha\to 0}(u(x_{\gamma,\alpha}, t_{\gamma,\alpha}) - v(y_{\gamma,\alpha}, t_{\gamma,\alpha}) - \eta t_{\gamma,\alpha}) \\ &= \lim_{\gamma\to 0} \limsup_{\alpha\to 0}(u(x_{\gamma,\alpha}, t_{\gamma,\alpha}) - v(y_{\gamma,\alpha}, t_{\gamma,\alpha}) - \eta t_{\gamma,\alpha}). \end{aligned}$$

We then get

$$\lim_{\gamma\to 0} \limsup_{\alpha\to 0}(|x_{\gamma,\alpha} - y_{\gamma,\alpha}|^2/\gamma^2 + \alpha(|x_{\gamma,\alpha}|^2 + |y_{\gamma,\alpha}|^2)) = 0. \tag{2.21}$$

Fix $\gamma_0 > 0$ such that for all $\gamma \le \gamma_0$ and for all α small enough we have

$$M_{\gamma,\alpha} = \Phi_{\gamma,\alpha}(x_{\gamma,\alpha}, y_{\gamma,\alpha}, t_{\gamma,\alpha}) > M_\eta/2$$

and

$$\begin{aligned} \limsup_{\alpha\to 0}(\|Dc_0^\varepsilon\|_1(2|x_{\gamma,\alpha} - y_{\gamma,\alpha}|^2/\gamma^2 + \alpha|y_{\gamma,\alpha}|^2|x_{\gamma,\alpha} - y_{\gamma,\alpha}| + \alpha|x_{\gamma,\alpha} - y_{\gamma,\alpha}|) \\ + \tfrac{3}{2}\|c_0\|_1\alpha(2 + |x_{\gamma,\alpha}|^2 + |y_{\gamma,\alpha}|^2)) \le \eta/3. \end{aligned} \tag{2.22}$$

We claim that there is $\gamma \le \gamma_0$ such that $t_{\gamma,\alpha} > 0$ for all α small enough. Indeed, if for each $\gamma \le \gamma_0$, there is $\alpha \in (0, \gamma)$ such that $t_{\gamma,\alpha} = 0$, then

$$\begin{aligned} M_\eta/2 < M_{\gamma,\alpha} &\le u(x_{\gamma,\alpha}, 0) - v(y_{\gamma,\alpha}, 0) \le u_0(x_{\gamma,\alpha}) - u_0(y_{\gamma,\alpha}) \le \|Du_0\|\,|x_{\gamma,\alpha} - y_{\gamma,\alpha}| \\ &\le C\|Du_0\|\gamma, \end{aligned}$$

where we have used (2.17). Thus we get a contradiction if γ is small enough, and the claim is proved. Hence, by Lemma 2.4 (if $t_{\gamma,\alpha} = T$, we use the fact that u (resp. v) is a subsolution (resp. supersolution) in $(0, T]$, see Lemma 2.8 of Barles [7]), there are $(a, p) \in \bar{D}^+u(x_{\gamma,\alpha}, t_{\gamma,\alpha})$ and $(b, q) \in \bar{D}^-v(y_{\gamma,\alpha}, t_{\gamma,\alpha})$ such that

$$a - b = \eta;$$
$$p = 2(x_{\gamma,\alpha} - y_{\gamma,\alpha})/\gamma^2 + 2\alpha x_{\gamma,\alpha};$$
$$q = 2(x_{\gamma,\alpha} - y_{\gamma,\alpha})/\gamma^2 - 2\alpha y_{\gamma,\alpha};$$
$$a - \left((c_0^\varepsilon \star 1_{\{u(\cdot,t_{\gamma,\alpha}) \ge u(x_{\gamma,\alpha},t_{\gamma,\alpha})\}})(x_{\gamma,\alpha}) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon\right)|p| \le 0; \tag{2.23}$$
$$b - \left((c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha}) > v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}})(y_{\gamma,\alpha}) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon\right)|q| \ge 0. \tag{2.24}$$

By subtracting (2.24) from (2.23) we get

$$\begin{aligned} \eta + \left((c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha}) > v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}})(y_{\gamma,\alpha}) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon\right)|q| \\ - \left((c_0^\varepsilon \star 1_{\{u(\cdot,t_{\gamma,\alpha}) \ge u(x_{\gamma,\alpha},t_{\gamma,\alpha})\}})(x_{\gamma,\alpha}) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon\right)|p| \le 0. \end{aligned} \tag{2.25}$$

From the fact that $\Phi_{\gamma,\alpha}(x_{\gamma,\alpha}, y_{\gamma,\alpha}, t_{\gamma,\alpha}) \ge \Phi_{\gamma,\alpha}(x, x, t_{\gamma,\alpha})$ it follows that

$$\begin{aligned} v(x, t_{\gamma,\alpha}) - v(y_{\gamma,\alpha}, t_{\gamma,\alpha}) &\ge u(x, t_{\gamma,\alpha}) - u(x_{\gamma,\alpha}, t_{\gamma,\alpha}) - 2\alpha|x|^2 \\ &\quad + |x_{\gamma,\alpha} - y_{\gamma,\alpha}|^2/\gamma^2 + \alpha(|x_{\gamma,\alpha}|^2 + |y_{\gamma,\alpha}|^2). \end{aligned}$$

In particular, from the above inequality we deduce that

$$\{u(\cdot, t_{\gamma,\alpha}) \ge u(x_{\gamma,\alpha}, t_{\gamma,\alpha})\} \cap \{v(\cdot, t_{\gamma,\alpha}) \le v(y_{\gamma,\alpha}, t_{\gamma,\alpha})\} \subset \{|x|^2 \ge R^2_{\alpha,\gamma}\},$$

where

$$R^2_{\alpha,\gamma} = \frac{1}{2\alpha}(|x_{\gamma,\alpha} - y_{\gamma,\alpha}|^2/\gamma^2 + \alpha(|x_{\gamma,\alpha}|^2 + |y_{\gamma,\alpha}|^2)).$$

Thus

$$\{u(\cdot, t_{\gamma,\alpha}) \ge u(x_{\gamma,\alpha}, t_{\gamma,\alpha})\} \subset \{v(\cdot, t_{\gamma,\alpha}) > v(y_{\gamma,\alpha}, t_{\gamma,\alpha})\} \cup \{|x|^2 \ge R^2_{\alpha,\gamma}\}. \tag{2.26}$$

For given $\gamma \le \gamma_0$ the following two cases may occur.

Case 1. For all α small and for some $\tilde{C}_\gamma > 0$ we have

$$|x_{\gamma,\alpha} - y_{\gamma,\alpha}|^2/\gamma^2 \ge \tilde{C}^2_\gamma.$$

In this case we have

$$\{|x - x_{\alpha,\gamma}| \geq R_{\alpha,\gamma}\} \subset \{|x| \geq \tilde{R}_{\alpha,\gamma}\}, \tag{2.27}$$

where $\tilde{R}_{\alpha,\gamma} = -|x_{\alpha,\gamma}| + R_{\alpha,\gamma}$ satisfies the estimate in the following lemma whose proof is postponed:

Lemma 2.5. *We have*

$$\tilde{R}_{\alpha,\gamma} = R_{\alpha,\gamma} - |x_{\alpha,\gamma}| \geq \frac{\tilde{C}_\gamma^2}{8\sqrt{C_0}\sqrt{\alpha}}.$$

Now choose $\delta > 0$ such that $\delta C_\gamma \leq \eta/3$, $C_\gamma > 0$ being an upper bound of $|p|$, $|q|$ depending on γ and independent of α small enough. Since $c_0^\varepsilon \in W^{1,1}(\mathbb{R}^n)$, for α small we have

$$\int_{B^c(0,\tilde{R}_{\alpha,\gamma})} c_0^\varepsilon(x)\,dx \leq \delta$$

and

$$|(c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha})>v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}})(x_{\gamma,\alpha}) - (c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha})>v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}})(y_{\gamma,\alpha})| \\ \leq \|Dc_0^\varepsilon\|_1 |x_{\gamma,\alpha} - y_{\gamma,\alpha}|.$$

By using the inclusions (2.26) and (2.27), from (2.25) we get

$$\begin{aligned}
0 &\geq \eta + |q| c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha})>v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}}(y_{\gamma,\alpha}) - |p| c_0^\varepsilon \star 1_{\{u(\cdot,t_{\gamma,\alpha})\geq u(x_{\gamma,\alpha},t_{\gamma,\alpha})\}}(x_{\gamma,\alpha}) \\
&\quad - \frac{1}{2}\int c_0^\varepsilon(x)\,dx\,(|q| - |p|) \\
&\geq \eta + |q| c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha})>v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}}(y_{\gamma,\alpha}) - |p| c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha})>v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}}(x_{\gamma,\alpha}) \\
&\quad - |p| c_0^\varepsilon \star 1_{B^c(0,R_{\alpha,\gamma})}(x_{\gamma,\alpha}) - \frac{1}{2}\|c_0^\varepsilon\|_1 |p - q| \\
&\geq \eta - \|Dc_0^\varepsilon\|_1 |x_{\gamma,\alpha} - y_{\gamma,\alpha}|(2|x_{\gamma,\alpha} - y_{\gamma,\alpha}|/\gamma^2 + \alpha + \alpha|y_{\gamma,\alpha}|^2) \\
&\quad - \frac{3}{2}\|c_0\|_1 \{2\alpha + \alpha(|x_{\gamma,\alpha}|^2 + |y_{\gamma,\alpha}|^2)\} - |p| \int_{B^c(0,\tilde{R}_{\alpha,\gamma})} c_0^\varepsilon(x)\,dx \\
&\geq \eta - \|Dc_0^\varepsilon\|_1 |x_{\gamma,\alpha} - y_{\gamma,\alpha}|(2|x_{\gamma,\alpha} - y_{\gamma,\alpha}|/\gamma^2 + \alpha + \alpha|y_{\gamma,\alpha}|^2) \\
&\quad - \frac{3}{2}\|c_0\|_1 \{2\alpha + \alpha(|x_{\gamma,\alpha}|^2 + |y_{\gamma,\alpha}|^2)\} - \delta C_\gamma.
\end{aligned} \tag{2.28}$$

By taking $\limsup_{\alpha\to 0}$ in (2.28) and using (2.22) we get a contradiction.

Case 2. There is a subsequence $\alpha_n > 0$, which we still denote by α, such that

$$|x_{\gamma,\alpha} - y_{\gamma,\alpha}|^2/\gamma^2 \to 0 \quad \text{as } \alpha \to 0.$$

In this case we have $\lim_{\alpha\to 0}|p| = 0$ and $\lim_{\alpha\to 0}|q| = 0$. On the other hand, from (2.25),

$$0 \geq \eta - \frac{1}{2}\|c_0^\varepsilon\|_{L^1}(|p| + |q|). \tag{2.29}$$

By letting $\alpha \to 0$ in (2.29), we get a contradiction.

Proof of Lemma 2.5. By assumptions, we have $|x_{\gamma,\alpha} - y_{\gamma,\alpha}|^2/\gamma^2 \geq \tilde{C}_\gamma^2$. Hence

$$\begin{aligned}
R_{\gamma,\alpha}^2 - |x_{\gamma,\alpha}|^2 &\geq \frac{\tilde{C}_\gamma^2}{2\alpha} - \frac{1}{2}(|x_{\gamma,\alpha}|^2 - |y_{\gamma,\alpha}|^2) \\
&\geq \frac{\tilde{C}_\gamma^2}{2\alpha} - \frac{1}{2}|x_{\gamma,\alpha} - y_{\gamma,\alpha}|(|x_{\gamma,\alpha}| + |y_{\gamma,\alpha}|) \\
&\geq \frac{\tilde{C}_\gamma^2}{2\alpha} - \frac{\gamma C_0}{\sqrt{\alpha}} \\
&\geq \frac{\tilde{C}_\gamma^2}{4\alpha} \quad \text{if } \alpha \text{ is small enough}
\end{aligned}$$

where we have used (2.17) for the third line. Moreover, using (2.17), we deduce

$$R_{\gamma,\alpha} \leq \sqrt{C_0/\alpha}$$

so

$$\tilde{R}_{\gamma,\alpha} = R_{\gamma,\alpha} - |x_{\gamma,\alpha}| = \frac{R_{\gamma,\alpha}^2 - |x_{\gamma,\alpha}|^2}{R_{\gamma,\alpha} + |x_{\gamma,\alpha}|} \geq \frac{\tilde{C}_\gamma^2}{4\alpha} \frac{1}{2\sqrt{C_0/\alpha}} \geq \frac{\tilde{C}_\gamma^2}{8\sqrt{C_0}\sqrt{\alpha}}.$$

This ends the proof of the lemma.

Theorem 2.6 (Existence and uniqueness for the ε-problem). *Let $u_0 \in \mathrm{Lip}(\mathbb{R}^n)$ be such that*

$$|Du_0| < B_0 \quad \textit{in } \mathbb{R}^n. \tag{2.30}$$

Then there is a unique solution of (1.4).

Proof. The uniqueness comes from the comparison principle, and the existence is a straightforward consequence of Perron's method (see Da Lio, Kim and Slepčev [23, Theorem 1.2]). Indeed, it suffices to remark that $u^{\pm}(x,t) = u_0(x) \pm \|c_0^\varepsilon\|_1 B_0 t$ are respectively a supersolution and subsolution of (1.4).

Proposition 2.7 (Lipschitz estimates in space). *The unique solution of* (1.4) *is Lipschitz continuous:*

$$|Du^\varepsilon(\cdot,t)|_{L^\infty(\mathbb{R}^n)} \leq |Du^\varepsilon(\cdot,0)|_{L^\infty(\mathbb{R}^n)}. \tag{2.31}$$

Proof. The estimate (2.31) follows from the fact that the equation is invariant under space translation. Indeed, if we set

$$v(x,t) = u^\varepsilon(x+h,t) + |Du_0|_{L^\infty(\mathbb{R}^n)}|h|,$$

then it is easy to check that v is still a supersolution to the problem (1.4). Moreover, $v(x,0) \geq u(x,0)$, so, by the comparison principle, $v(x,t) \geq u(x,t)$ for all $t \in [0,\infty)$, i.e. $u(x,t) - u(x+h,t) \leq |Du_0|_{L^\infty(\mathbb{R}^n)}|h|$. Using similarly a subsolution, we deduce the result.

3. The limit problem

Definition 3.1. *A function* $u^0 \in B_{\rm loc}USC(\mathbb{R}^n \times [0,T])$ *is a* viscosity subsolution of (1.5)–(1.7) *if it satisfies:*

(i) $u^0(x,0) \le u_0(x)$ *in* $\mathbb{R}^n$,
(ii) *for every* $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$ *and every test function* $\Phi \in C^\infty(\mathbb{R}^n \times [0,\infty))$ *such that* $u^0 - \Phi$ *has a maximum at* (x_0,t_0),

$$\frac{\partial \Phi}{\partial t}(x_0,t_0) + F_*(D\Phi, D^2\Phi) \le 0. \tag{3.32}$$

A function $u^0 \in B_{\rm loc}LSC(\mathbb{R}^n \times [0,T])$ *is a* viscosity supersolution of (1.5)–(1.7) *if it satisfies:*

(i) $u^0(x,0) \ge u_0(x)$ *in* $\mathbb{R}^n$,
(ii) *for every* $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$ *and every test function* $\Phi \in C^\infty(\mathbb{R}^n \times [0,\infty))$ *such that* $u^0 - \Phi$ *has a minimum at* (x_0,t_0),

$$\frac{\partial \Phi}{\partial t}(x_0,t_0) + F^*(D\Phi, D^2\Phi) \ge 0. \tag{3.33}$$

A continuous function u^0 *is a* viscosity solution of (1.5)–(1.7) *if it is a subsolution and supersolution of* (1.5)–(1.7).

This definition comes from the general definition of viscosity solution for discontinuous Hamiltonians first given by Ishii [32] (see also Crandall, Ishii and Lions [21]). We need an equivalent definition which eliminates, at least partially, the difficulty related to the fact that $D\Phi$ may be equal to zero.

Theorem 3.2 (Equivalent definition for mean curvature type motions). *In Definition* 3.1 *we can replace condition* (3.32) *by*

$$\frac{\partial \Phi}{\partial t}(x_0,t_0) + F(D\Phi, D^2\Phi) \le 0 \quad \text{if } D\Phi(x_0,t_0) \ne 0$$

or

$$\frac{\partial \Phi}{\partial t}(x_0,t_0) \le 0 \quad \text{if } D\Phi(x_0,t_0) = 0 \text{ and } D^2\Phi(x_0,t_0) = 0,$$

and condition (3.33) *by*

$$\frac{\partial \Phi}{\partial t}(x_0,t_0) + F(D\Phi, D^2\Phi) \ge 0 \quad \text{if } D\Phi(x_0,t_0) \ne 0$$

or

$$\frac{\partial \Phi}{\partial t}(x_0,t_0) \le 0 \quad \text{if } D\Phi(x_0,t_0) = 0 \text{ and } D^2\Phi(x_0,t_0) = 0,$$

and the definition remains equivalent.

The equivalence between these two definitions was first proved by Barles and Georgelin [9] for the isotropic mean curvature motion and their proof adapts here without any difficulty.

It is well known that this problem admits a unique viscosity solution. See for instance Bellettini and Novaga [16], [17], Chen, Giga and Goto [20] and Evans and Spruck [26]. Moreover, we have the following comparison principle:

Theorem 3.3 (Comparison principle for the limit problem). *If $u \in B_{\mathrm{loc}}USC(\mathbb{R}^n \times [0,T])$ is a subsolution of* (1.5) *and $v \in B_{\mathrm{loc}}LSC(\mathbb{R}^n \times [0,T])$ is a supersolution of* (1.5) *satisfying $u(x,0) \le v(x,0)$ for all $x \in \mathbb{R}^n$, then $u(x,t) \le v(x,t)$ for all $(x,t) \in \mathbb{R}^n \times (0,T)$.*

In this theorem, we do not need any assumption on the behaviour of the solution at infinity, since the equation is geometric.

4. Convergence of the velocity for a test function

4.1. Link with other works

In this subsection, we show in a heuristic way the links and the differences between our result and previous strongly related works such as Barles and Georgelin [9], Chambolle and Novaga [19], Evans [25], Ishii [33] and Ishii, Pires and Souganidis [34]. In particular, we explain the term $1/|\ln \varepsilon|$ in our scaling. We make the computation formally for a general kernel K_0 with the parabolic scaling, i.e.

$$K_0^\varepsilon(x) = \frac{1}{\varepsilon^{n+1}} K_0\left(\frac{x}{\varepsilon}\right).$$

We assume that K_0 is symmetric, i.e. $K_0(-x) = K_0(x)$, and admits a moment of order two for every section, i.e. for every $p \in \mathbf{S}^{n-1}$,

$$\int_{p^\perp} K_0(x)|x|^2 < \infty. \tag{4.34}$$

We want to show formally that for every regular function φ, the velocity

$$c^\varepsilon = K_0^\varepsilon \star 1_{\{\varphi \ge 0\}}(0) - \frac{1}{2}\int K_0^\varepsilon$$

converges to anisotropic mean curvature. To simplify the computation, we finally assume that the zero level set of φ is the graph of a function h, i.e. more precisely that $\varphi(x', x_n) =$

$h(x') - x_n$ where $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$ and $D_{x'}h(0) = 0$. We have

$$\begin{aligned} c^\varepsilon &= \int_{\{x_n \le h(x')\}} K_0^\varepsilon - \int_{\{x_n \le 0\}} K_0^\varepsilon = \frac{1}{\varepsilon} \int_{\{0 \le x_n \le h(\varepsilon x')/\varepsilon\}} K_0(x)\, dx \\ &\simeq \int_{x' \in \mathbb{R}^{n-1}} \left(\frac{1}{\varepsilon} \int_0^{\frac{\varepsilon}{2} D^2 h(0)(x',x')} K_0(x', x_n)\, dx_n \right) dx' \\ &\simeq \int_{x' \in \mathbb{R}^{n-1}} \frac{1}{2} K_0(x', 0) D^2 h(0)(x', x')\, dx' \\ &= \operatorname{trace}(A(p)(\mathrm{Id} - p \otimes p)(D^2\varphi)) \qquad \text{with } |D\varphi(0)| = 1, \end{aligned}$$

where $p = D\varphi/|D\varphi|$ and $A(p) = \int_{x \in p^\perp \simeq \mathbb{R}^{n-1}} \frac{1}{2} K_0(x) x \otimes x dx$. So, formally, if (4.34) holds, then the velocity c^ε converges to anisotropic mean curvature. Barles and Georgelin [9] and Evans [25] used this result to prove the convergence of the Merriman–Bence–Osher scheme [36]. For the proof, they used the kernel

$$K_0(x) = \frac{1}{(4\pi)^{n/2}} e^{-x^2/4}$$

which satisfied the assumptions. This result was then generalised for a threshold dynamics by Ishii [33] and Ishii, Pires and Souganidis [34] to more general kernels assuming also the symmetry of the kernel and (4.34). A by-product of our work is that for general kernels, our limit mean curvature motion is of variational type (see Theorem 1.11).

The main difference in our case is that c_0 behaves like $1/|x|^{n+1}$ and so (4.34) does not hold. This explains the presence of the small term $1/|\ln \varepsilon|$ in our scaling which will compensate the bad decay of our kernel. Indeed to make a renormalization of the integral $\int_{x' \in \mathbb{R}^{n-1}} \frac{1}{2} K_0(x', 0) D^2 h(0)(x', x')\, dx'$ finite, we have to multiply by a term going to zero faster. We denote by $J(\varepsilon)$ this term (i.e. we use the scaling $c_0^\varepsilon(x) = (J(\varepsilon)/\varepsilon^{n+1}) c_0(x/\varepsilon)$). Using the same computation as above, we obtain

$$\begin{aligned} c^\varepsilon &= J(\varepsilon) \frac{1}{\varepsilon} \int_{\{0 \le x_n \le h(\varepsilon x')/\varepsilon\} \cap \{|x'| \le \delta/\varepsilon\}} c_0(x)\, dx + J(\varepsilon) \mathcal{I}_1 \\ &\simeq J(\varepsilon) \int_{\{|x'| \le \delta/\varepsilon\}} \frac{1}{2} c_0(x', 0) D^2 h(0)(x', x')\, dx' + J(\varepsilon) \mathcal{I}_1 \end{aligned}$$

where

$$\mathcal{I}_1 = \frac{1}{\varepsilon} \int_{\{0 \le x_n \le h(\varepsilon x')/\varepsilon\} \cap \{|x'| \ge \delta/\varepsilon\}} c_0(x)\, dx \le \frac{1}{\varepsilon} \int_{(B_{\delta/\varepsilon}(0))^c} c_0(x)\, dx.$$

Using the particular form of c_0 for $|x| \ge 1$, we deduce that

$$\mathcal{I}_1 \le \frac{1}{\varepsilon} \int_{\delta/\varepsilon}^{\infty} dr\, \frac{1}{r^2} \int_{\theta \in \mathbf{S}^{n-1}} d\theta\, g(\theta)$$

and so $\mathcal{I}_1$ is bounded as $\varepsilon \to 0$. This implies that the last term $J(\varepsilon)\mathcal{I}_1$ goes to zero as $\varepsilon \to 0$. We then decompose the first integral:

$$
\begin{aligned}
J(\varepsilon) \int_{\{|x'|\le \delta/\varepsilon\}} \frac{1}{2} c_0(x',0) D^2h(0)(x',x')\,dx' \\
&= J(\varepsilon) \int_{|x'|\le 1} \frac{1}{2} c_0(x',0) D^2h(0)(x',x')\,dx' \\
&\quad + J(\varepsilon) \int_{|x'|\in(1,\delta/\varepsilon)} \frac{1}{2} c_0(x',0) D^2h(0)(x',x')\,dx'.
\end{aligned}
$$

Since c_0 is bounded, the first term goes to zero as ε goes to zero. Thus, the only interesting term is the second one. Using again the particular form of c_0 for $|x| \ge 1$, we deduce that

$$
\begin{aligned}
J(\varepsilon) \int_{|x'|\in(1,\delta/\varepsilon)} \frac{1}{2} c_0(x',0) D^2h(0)(x',x')\,dx' \\
&= J(\varepsilon) \int_{\theta\in\mathbf{S}^{n-2}} d\theta\, \frac{1}{2} D^2h(0)(\theta,\theta) g(\theta) \int_1^{\delta/\varepsilon} \frac{1}{r}\,dr \\
&= J(\varepsilon) \left(\ln \frac{\delta}{\varepsilon}\right) \int_{\theta\in\mathbf{S}^{n-2}} \frac{1}{2} g(\theta) D^2h(0)(\theta,\theta)\,d\theta \\
&= J(\varepsilon) \left(\ln \frac{\delta}{\varepsilon}\right) \operatorname{trace}(A(p)D^2\varphi).
\end{aligned}
$$

So the correct scaling is to take $J(\varepsilon) = |\ln \varepsilon|^{-1}$ and we finally obtain

$$c^\varepsilon \to \operatorname{trace}(A(p)D^2\varphi) \quad \text{when } |D\varphi(0)| = 1.$$

4.2. *Proof of convergence*

In this section, we prove rigorously the convergence result for test functions. Define (for $M = D^2\varphi$, $p = D\varphi$)

$$G(M,p) = \frac{-1}{|p|} F(M,p).$$

For an $n \times n$ matrix M we introduce the norm

$$|M| = \sup_{\xi\in B_1(0)} |M\cdot\xi|. \tag{4.35}$$

We define the modulus of continuity of the function g by

$$\omega_g(r) = \sup_{|\theta'-\theta|\le r,\, \theta,\theta'\in\mathbf{S}^{n-1}} |g(\theta') - g(\theta)|.$$

Then we have the following fundamental estimate for test functions independent of time:

Proposition 4.1 (Error estimate on the velocity for a test function). *Assume that* $\varphi \in C^2(\mathbb{R}^n)$ *and that* $D\varphi(x_0) \neq 0$. *For*

$$c_0^\varepsilon(\cdot) = \frac{1}{\varepsilon^{n+1}|\ln \varepsilon|} c_0\left(\frac{\cdot}{\varepsilon}\right),$$

define

$$c^\varepsilon = (c_0^\varepsilon \star 1_{\{\varphi(\cdot)>\varphi(x_0)\}})(x_0) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon.$$

Set $b = |D\varphi(x_0)|$, *and for any* $a \geq |D^2\varphi|_{L^\infty(B_1(x_0))}$ *and* $0 < r < 1$, *define the relative modulus of continuity of* $D^2\varphi$ *at* x_0 *by*

$$\omega(r) = \begin{cases} \sup\limits_{x \in B_r(x_0)} \dfrac{|D^2\varphi(x) - D^2\varphi(x_0)|}{a} & \text{if } a \neq 0, \\ 0 & \text{if } a = 0. \end{cases}$$

Fix $\delta_1 \leq 1$ *such that*

$$\omega(\delta_1) \leq 1.$$

Set $\delta_0 = \min(1, b/3a, \delta_1)$. *Then there exists a constant* $C = C(n, \sup_{\mathbb{R}^n} c_0) > 0$ *such that for* $0 < \varepsilon < \delta$ *and* $0 < \delta \leq \delta_0/2$, *we have*

$$|c^\varepsilon - G(D^2\varphi(x_0), D\varphi(x_0))| \leq Ce(\varepsilon, \delta, \delta_0)$$

with

$$e(\varepsilon, \delta, \delta_0) = \frac{1}{|\ln \varepsilon|}\left(\frac{1}{\delta} + \frac{1}{\delta_0}|\ln \delta|\right) + \frac{1}{\delta_0}\left(\omega_g\left(\frac{\delta}{\delta_0}\right) + \omega(2\delta) + \frac{\delta}{\delta_0}\right).$$

Before proving Proposition 4.1, let us give a corollary.

Corollary 4.2 (Convergence of the velocity for a test function). *Assume that* $\varphi \in C^2(\mathbb{R}^n \times (0, \infty))$ *and* $D\varphi(x_0, t_0) \neq 0$. *If* $(x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$, *then*

$$c^\varepsilon := (c_0^\varepsilon \star 1_{\{\varphi(\cdot, t_\varepsilon)>\varphi(x_\varepsilon, t_\varepsilon)\}})(x_\varepsilon, t_\varepsilon) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon \to G(D^2\varphi(x_0, t_0), D\varphi(x_0, t_0)).$$

Proof. This is a straightforward consequence of the fact that we can choose the relative modulus of continuity ω uniformly in a neighbourhood of (x_0, t_0) and then estimate $c^\varepsilon - G(D^2\varphi(x_\varepsilon, t_\varepsilon), D\varphi(x_\varepsilon, t_\varepsilon))$ using Proposition 4.1. We conclude by choosing $\delta = \delta(\varepsilon) = 1/\sqrt{|\ln \varepsilon|}$.

Proof of Proposition 4.1. Up to a change of coordinates, we can assume that $x_0 = 0$, $\varphi(x_0) = 0$, $D\varphi(x_0) = be_n$ with $b > 0$. We write $x' = (x_1, \ldots, x_{n-1})$ for a point of $\mathbb{R}^{n-1}$ and $x = (x', x_n) \in \mathbb{R}^n$. Then using the implicit function theorem, we can assume that there exists a neighbourhood

$$Q_\delta = B_\delta^{n-1} \times (-\delta, \delta) \subset \mathbb{R}^n$$

of the origin such that the level set $\{\varphi = 0\}$ can be written as

$$\{\varphi = 0\} \cap Q_\delta = \{(x', x_n) \in Q_\delta : x_n = h(x')\}$$

for a suitable function $h \in C^2(B_\delta^{n-1}; (-\delta, \delta))$.

Then we have the following result which will be proved later:

Lemma 4.3. *Let δ_0 be as defined in Proposition 4.1. For $0 < \delta \le \delta_0/2$, we have*

$$\forall x' \in B_\delta^{n-1},\ (x', h(x')) \in Q_\delta \text{ and } \left| \frac{h(x') - \frac{1}{2} D^2 h(0) \cdot (x', x')}{|x'|^2} \right| \le \frac{a}{b} \left(\omega(2\delta) + 8 \frac{\delta}{\delta_0} \right).$$

Moreover,

$$\frac{\partial^2 h}{\partial x_i \partial x_j}(0) = -\frac{1}{|D\varphi(0)|} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(0), \qquad i, j = 1, \dots, n-1,$$

and

$$|h(x')| \le 6 \frac{a}{b} |x'|^2 \quad \text{for } x' \in B_\delta^{n-1}.$$

We have

$$\begin{aligned} c^\varepsilon &= (c_0^\varepsilon \star 1_{\{\varphi(\cdot)>0\}})(0) - \frac{1}{2} \int_{\mathbb{R}^n} c_0^\varepsilon = (c_0^\varepsilon \star 1_{\{\varphi(\cdot)>0\}})(0) - (c_0^\varepsilon \star 1_{\{x_n>0\}})(0) \\ &= -(c_0^\varepsilon \star 1_{\{\varphi(\cdot)\le 0\} \cap \{x_n>0\}})(0) + (c_0^\varepsilon \star 1_{\{\varphi(\cdot)>0\} \cap \{x_n<0\}})(0) = -\{(I)_\varepsilon + (II)_\varepsilon\} \end{aligned}$$

where

$$\begin{aligned} (I)_\varepsilon &= (c_0^\varepsilon \star 1_{Q_\delta \cap \{\varphi(\cdot)\le 0\} \cap \{x_n>0\}})(0) - (c_0^\varepsilon \star 1_{Q_\delta \cap \{\varphi(\cdot)>0\} \cap \{x_n<0\}})(0) \\ (II)_\varepsilon &= (c_0^\varepsilon \star 1_{(\mathbb{R}^n \setminus Q_\delta) \cap \{\varphi(\cdot)\le 0\} \cap \{x_n>0\}})(0) - (c_0^\varepsilon \star 1_{(\mathbb{R}^n \setminus Q_\delta) \cap \{\varphi(\cdot)>0\} \cap \{x_n<0\}})(0). \end{aligned}$$

For $\delta > \varepsilon$ we have

$$|(II)_\varepsilon| \le \int_{\mathbb{R}^n \setminus Q_\delta} c_0^\varepsilon = \frac{1}{\varepsilon |\ln \varepsilon|} \int_{\mathbb{R}^n \setminus Q_{\delta/\varepsilon}} c_0 \le \frac{C}{\delta |\ln \varepsilon|}.$$

Let us now compute the term $(I)_\varepsilon$. For $\delta \le \delta_0/2$ we have

$$(I)_\varepsilon = \int_{B_\delta^{n-1}} dx' \int_0^{h(x')} dx_n \frac{1}{\varepsilon^{n+1} |\ln \varepsilon|} c_0 \left(\frac{x}{\varepsilon} \right).$$

Define (with $x = (x', x_n) = |x|\theta$)

$$(I)'_\varepsilon = \int_{B_\delta^{n-1} \setminus B_\varepsilon^{n-1}} dx' \int_0^{h(x')} dx_n \frac{1}{|\ln \varepsilon|} \frac{g(\theta)}{(|x'|^2 + |x_n|^2)^{(n+1)/2}}.$$

Then

$$
\begin{aligned}
|(I)_\varepsilon - (I)'_\varepsilon| &\le \int_{B_\varepsilon^{n-1}} dx' \int_0^{|h(x')|} dx_n \, \frac{1}{\varepsilon^{n+1}|\ln\varepsilon|} c_0\left(\frac{x}{\varepsilon}\right) \\
&\le \int_{B_\varepsilon^{n-1}} dx' \int_0^{6\frac{a}{b}|x'|^2} dx_n \, \frac{1}{\varepsilon^{n+1}|\ln\varepsilon|} \sup_{\mathbb{R}^n} c_0 \le \frac{6a}{b|\ln\varepsilon|} \, \frac{\sup_{\mathbb{R}^n} c_0}{n+1}
\end{aligned}
$$

where we have used the fact that $|h(x')| \le 6\frac{a}{b}|x'|^2$ for $|x'| \le \delta \le \delta_0/2$. Now,

$$
\begin{aligned}
(I)'_\varepsilon &= \int_{B_\delta^{n-1}\setminus B_\varepsilon^{n-1}} dx' \int_0^{h(x')/|x'|^2} d\zeta \, \frac{|x'|^2}{|\ln\varepsilon|} \, \frac{g\left(\frac{(x',|x'|^2\zeta)}{\sqrt{|x'|^2+(|x'|^2\zeta)^2}}\right)}{(|x'|^2+(|x'|^2\zeta)^2)^{(n+1)/2}} \\
&= \int_{B_\delta^{n-1}\setminus B_\varepsilon^{n-1}} \frac{1}{|\ln\varepsilon|} \, \frac{dx'}{|x'|^{n-1}} \left(\int_0^{h(x')/|x'|^2} d\zeta \, \frac{g\left(\frac{(x'/|x'|,|x'|\zeta)}{\sqrt{1+|x'|^2\zeta^2}}\right)}{(1+|x'|^2\zeta^2)^{(n+1)/2}} \right).
\end{aligned}
$$

Define

$$
\begin{aligned}
(I)''_\varepsilon &= \int_{B_\delta^{n-1}\setminus B_\varepsilon^{n-1}} \frac{1}{|\ln\varepsilon|} \, \frac{dx'}{|x'|^{n-1}} \left(\int_0^{\frac{1}{2}D^2h(0)\cdot(x'/|x'|,x'/|x'|)} d\zeta \; g\left(\frac{x'}{|x'|}\right) \right) \\
&= \frac{\ln(\delta/\varepsilon)}{|\ln\varepsilon|} \left(\int_{\theta\in\mathbf{S}^{n-2}\subset\{x_n=0\}} d\theta \, \frac{1}{2} g(\theta)\cdot D^2h(0)\cdot(\theta,\theta) \right).
\end{aligned}
$$

Moreover, we define

$$
(I)''_0 = \int_{\theta\in\mathbf{S}^{n-2}\subset\{x_n=0\}} d\theta \left(\frac{1}{2} g(\theta)\cdot D^2h(0)\cdot(\theta,\theta) \right).
$$

From Lemma 4.3 we have

$$
\begin{aligned}
-(I)''_0 &= \int_{\theta\in\mathbf{S}^{n-2}\subset\{x_n=0\}} d\theta \left(\frac{1}{2} g(\theta)\cdot \frac{1}{|D\varphi(0)|} D^2\varphi(0)\cdot(\theta,\theta) \right) \\
&= \frac{1}{|D\varphi(0)|} \operatorname{trace}\left(D^2\varphi(0)\cdot A\left(\frac{D\varphi(0)}{|D\varphi(0)|}\right)\right) = G(D^2\varphi(0), D\varphi(0))
\end{aligned}
$$

where A is defined in (1.7). Then

$$
\begin{aligned}
|(I)''_\varepsilon - (I)''_0| &\le \frac{|\ln\delta|}{|\ln\varepsilon|} \left(\int_{\theta\in\mathbf{S}^{n-2}} d\theta \right) \frac{1}{2} |D^2h(0)| \sup_{\mathbf{S}^{n-1}} g \\
&\le \frac{|\ln\delta|}{|\ln\varepsilon|} (n-1)|B_1^{n-1}| \frac{a}{2b} \sup_{\mathbf{S}^{n-1}} g. \qquad (4.36)
\end{aligned}
$$

We now want to estimate the difference between $(I)'_\varepsilon$ and $(I)''_\varepsilon$. To this end, we first set $v = (x'/|x'|, |x'|\zeta)$ and $\theta = (x'/|x'|, 0)$. Then using only the fact that $|\theta| = 1$ and the identity $\langle v-\theta,\theta\rangle = 0$ for the scalar product, we get $0 \le |v|-1 \le |v-\theta|$, and $|v/|v|-\theta| \le 2|v-\theta|$. Hence

$$\begin{aligned}\left|\frac{g(v/|v|)}{|v|^{n+1}} - g(\theta)\right| &\le |g(v/|v|) - g(\theta)| + g(\theta)(|v|^{n+1}-1)\\ &\le \omega_g(|v/|v|-\theta|) + (n+1)|v|^n(|v|-1)\sup_{\mathbf{S}^{n-1}} g\\ &\le \omega_g(2|v-\theta|) + (n+1)(1+|v-\theta|)^n|v-\theta|\sup_{\mathbf{S}^{n-1}} g\\ &\le \omega_g(2|v-\theta|) + (n+1)2^n|v-\theta|\sup_{\mathbf{S}^{n-1}} g\end{aligned}$$

where for the last line, we have used the fact that $|v-\theta| \le 1$ when $|x'| \le \delta$, $|\zeta| \le \frac{1}{2}|D^2h(0)\cdot(\theta,\theta)| \le a/2b$, and $\delta \le \delta_0/2$.

Using $|v-\theta| \le \delta a/2b$, we bound the last term by the quantity

$$e_1 = \omega_g\left(\frac{\delta a}{b}\right) + (n+1)2^{n-1}\frac{\delta a}{b}\sup_{\mathbf{S}^{n-1}} g.$$

Using Lemma 4.3 with

$$e_2 = \frac{a}{b}\left(\omega(2\delta) + 8\frac{\delta}{\delta_0}\right)$$

we then estimate

$$\begin{aligned}|(I)''_\varepsilon - (I)'_\varepsilon| &\le \int_{B^{n-1}_\delta\setminus B^{n-1}_\varepsilon} \frac{1}{|\ln\varepsilon|}\frac{dx'}{|x'|^{n-1}}\left\{e_2\cdot\sup_{\mathbf{S}^{n-1}} g + \frac{a}{2b}\cdot e_1\right\}\\ &\le \frac{\ln(\delta/\varepsilon)}{|\ln\varepsilon|}\left(\int_{\theta\in\mathbf{S}^{n-2}} d\theta\right)\left\{e_2\cdot\sup_{\mathbf{S}^{n-1}} g + \frac{a}{2b}\cdot e_1\right\}\\ &\le (n-1)|B_1^{n-1}|\left\{e_2\cdot\sup_{\mathbf{S}^{n-1}} g + \frac{a}{2b}\cdot e_1\right\}.\end{aligned}$$

Finally, we get (using $3a/b \le 1/\delta_0$ and $\delta \le \delta_0/2 \le 1/2$),

$$\begin{aligned}|c^\varepsilon + (I)''_0| &\le |(II)_\varepsilon| + |(I)_\varepsilon - (I)'_\varepsilon| + |(I)'_\varepsilon - (I)''_\varepsilon| + |(I)''_\varepsilon - (I)''_0|\\ &\le \frac{C}{|\ln\varepsilon|}\left(\frac{1}{\delta} + \frac{a}{b}|\ln\delta|\right) + C\left(\frac{a}{b}\omega_g\left(\frac{\delta a}{b}\right) + \frac{a}{b}\omega(2\delta) + \frac{a}{b}\frac{\delta}{\delta_0}\right)\\ &\le \frac{C}{|\ln\varepsilon|}\left(\frac{1}{\delta} + \frac{1}{\delta_0}|\ln\delta|\right) + C\frac{1}{\delta_0}\left(\omega_g\left(\frac{\delta}{\delta_0}\right) + \omega(2\delta) + \frac{\delta}{\delta_0}\right)\end{aligned}$$

where the constant C only depends on the dimension n and c_0. More precisely, we have $C = C(n, \int_{\mathbb{R}^n\setminus B_1} c_0, \sup_{\mathbb{R}^n} c_0, \sup_{\mathbf{S}^{n-1}} g) = C(n, \sup_{\mathbb{R}^n} c_0)$. This ends the proof of Proposition 4.1.

Proof of Lemma 4.3. Using the notations $\varphi_i = \partial\varphi/\partial x_i$ and $\varphi_{ij} = \partial^2\varphi/\partial x_i \partial x_j$, and taking the derivatives of the relation $\varphi(x', h(x')) = 0$, we get

$$\begin{cases} h_i = -\dfrac{\varphi_i}{\varphi_n}, \quad i = 1, \dots, n-1, \\ h_{ij} = -\dfrac{1}{\varphi_n}(\varphi_{ij} + \varphi_{in}h_j + \varphi_{jn}h_i + \varphi_{nn}h_i h_j), \quad i, j = 1, \dots, n-1. \end{cases}$$

Now, by definition of a, we have $|D^2\varphi(x)| \le a$ for $x \in B_1$. Therefore, for $0 < \delta \le 1$,

$$|D\varphi(x) - D\varphi(0)| \le a\delta \quad \text{for } x \in B_\delta.$$

Let $\delta_0'' \in (0, \infty)$ be such that $a\delta_0'' = \frac{1}{2}|D\varphi(0)|$, and set $\delta_0' = \min(1, \delta_0'')$. Then for $b = |D\varphi(0)| = \varphi_n(0)$ and $0 < \delta \le \delta_0'$ we get

$$a\delta_0 \le a\delta_0' \le b/2 \le \varphi_n(x) \le |D\varphi(x)| \quad \text{for } x \in B_\delta.$$

Using the elementary estimate

$$\begin{aligned} \forall x \in B_\delta, \quad & \left| \frac{f(x)}{g(x)} - \frac{f(0)}{g(0)} \right| \\ & \le \frac{1}{g(0) \inf_{B_\delta} g} (|f(x) - f(0)| g(0) + |f(0)|\,|g(x) - g(0)|) \end{aligned} \tag{4.37}$$

and the fact that $\varphi_i(0) = 0$ for $i = 1, \dots, n-1$, we get

$$Dh(0) = 0 \quad \text{and} \quad |Dh(x')| \le \delta/\delta_0 \quad \text{for } (x', h(x')) \in B_\delta.$$

Still using (4.37), we get, for $(x', h(x')) \in B_\delta$ and $0 < \delta \le \delta_0$,

$$\begin{aligned} |D^2h(x') - D^2h(0)| &\le \frac{2}{b^2}((a\omega(\delta)) \cdot b + a \cdot (a\delta)) + \frac{2}{b}\left(2a\frac{\delta}{\delta_0} + a\left(\frac{\delta}{\delta_0}\right)^2\right) \\ &\le \frac{2a}{b}\left(\omega(\delta) + 4\frac{\delta}{\delta_0}\right) \end{aligned}$$

where we have used the fact that $a/b \le 1/2\delta_0$.

Using the Taylor formula with $h(0) = 0 = Dh(0)$, we get

$$\left| h(x') - \frac{1}{2}D^2h(0) \cdot (x', x') \right| \le \int_0^1 dt \int_0^t ds\, |D^2h(sx') - D^2h(0)| \cdot |x'|^2$$

and so for $(x', h(x')) \in B_\delta$,

$$\left| \frac{h(x') - \frac{1}{2}D^2h(0) \cdot (x', x')}{|x'|^2} \right| \le \frac{a}{b}\left(\omega(\delta) + 4\frac{\delta}{\delta_0}\right) =: J(\delta).$$

Now assume that $0 < 2\delta \le \delta_0$. Then $Q_\delta = B_\delta^{n-1} \times (-\delta, \delta) \subset B_{2\delta}$, and for $x' \in B_\delta^{n-1}$ we have (using $|D^2 h(0)| \le a/b$)

$$|h(x')| \le \delta^2 \left(\frac{1}{2} \frac{a}{b} + J(2\delta) \right) < \delta$$

while $\omega(2\delta) \le 1$ and $6a\delta/b \le 1$. Therefore for $0 < 2\delta \le \delta_0$, we get $(x', h(x')) \in Q_\delta \subset B_{2\delta}$ if $x' \in B_\delta^{n-1}$, and hence

$$\left| \frac{h(x') - \frac{1}{2} D^2 h(0) \cdot (x', x')}{|x'|^2} \right| \le \frac{a}{b} \left(\omega(2\delta) + 8 \frac{\delta}{\delta_0} \right).$$

We then deduce

$$|h(x')| \le |x'|^2 \frac{a}{b} \left(\omega(2\delta) + 8 \frac{\delta}{\delta_0} + \frac{1}{2} \right) \le 6 \frac{a}{b} |x'|^2,$$

which ends the proof of Lemma 4.3.

Corollary 4.4 (Error estimate for a particular test function). *For $B, \eta > 0$, consider the function*

$$\varphi(x) = B\sqrt{\eta^2 + |x|^2}.$$

Then there exists a constant $C' = C'(n, \sup_{\mathbb{R}^n} c_0) > 0$ such that for

$$c^\varepsilon(x) = (c_0^\varepsilon \star 1_{\{\varphi(\cdot) > \varphi(x)\}})(x) - \frac{1}{2} \int_{\mathbb{R}^n} c_0^\varepsilon$$

we have pointwise, for $|x_0| \ge 6\sqrt{2}\,\varepsilon$ and $3 \ge \eta \ge 6\sqrt{2}\,\varepsilon$,

$$|c^\varepsilon(x_0)|D\varphi(x_0)| + F(D^2\varphi(x_0), D\varphi(x_0))| \le C'B/\eta.$$

Proof. Let us first remark that we do not change the result if we divide φ by B (because F is geometric), so we can assume that $B = 1$.

For all x, we have

$$D\varphi(x) = \frac{x}{\sqrt{\eta^2 + |x|^2}}, \qquad D^2\varphi(x) = \frac{1}{\sqrt{\eta^2 + |x|^2}} (\mathrm{Id} - p(x) \otimes p(x))$$

where $p(x) = D\varphi(x)$ and $|p(x)| \le 1$ for all x. Next, for all x and x_0,

$$\begin{aligned} D^2\varphi(x) - D^2\varphi(x_0) = &\left(\frac{1}{\sqrt{\eta^2 + |x|^2}} - \frac{1}{\sqrt{\eta^2 + |x_0|^2}} \right) (\mathrm{Id} - p(x) \otimes p(x)) \\ &- \frac{1}{\sqrt{\eta^2 + |x_0|^2}} (p(x) \otimes (p(x) - p(x_0)) + (p(x) - p(x_0)) \otimes p(x_0)). \end{aligned}$$

Moreover,

$$\begin{aligned}\left|\frac{1}{\sqrt{\eta^2+|x|^2}}-\frac{1}{\sqrt{\eta^2+|x_0|^2}}\right| &\le \frac{|\sqrt{\eta^2+|x|^2}-\sqrt{\eta^2+|x_0|^2}|}{\eta^2}\\ &\le \frac{|\,|x|^2-|x_0|^2|}{\eta^2(\sqrt{\eta^2+|x|^2}+\sqrt{\eta^2+|x_0|^2})}\\ &\le \frac{|\,|x|-|x_0|\,|(|x|+|x_0|)}{\eta^2(|x|+|x_0|)}\le\frac{|x-x_0|}{\eta^2}\end{aligned}$$

and using the bound $|D^2\varphi|\le 1/\eta$ we get

$$|p(x)-p(x_0)|=|D\varphi(x)-D\varphi(x_0)|\le|x-x_0|/\eta.$$

We set $a=1/\eta\ge|D^2\varphi|$. Then, with the notation of Proposition 4.1,

$$|D^2\varphi(x)-D^2\varphi(x_0)|/a\le 3|x-x_0|/\eta,\qquad \omega(r)\le 3r/\eta.$$

We now apply Proposition 4.1 with $a=1/\eta$, $b=|D\varphi(x_0)|>0$, $\delta_1=\eta/3$, $2\delta=\delta_0=\min(b/3a,\delta_1)=b/3a$ (because $b\le 1$). We deduce that there exists a constant $C'=C'(n,\sup_{\mathbb{R}^n}c_0)>0$ such that for $\delta>\varepsilon>0$,

$$|c^\varepsilon(x_0)|D\varphi(x_0)|+F(D^2\varphi(x_0),D\varphi(x_0))|\le C'\left(\frac{1}{\eta}+\frac{1}{\eta|\ln\varepsilon|}\right)\le\frac{C'}{\eta}.$$

Moreover, the condition $\delta>\varepsilon$ is equivalent to $b>6\varepsilon/\eta$. We then deduce conditions on $|x_0|$ and η:

1. If $|x_0|\le\eta$, then $b\ge|x_0|/\sqrt{2}\,\eta$ and it suffices to take $|x_0|>6\sqrt{2}\,\varepsilon$.
2. If $|x_0|\ge\eta$, then $b\ge 1/\sqrt{2}$ and it suffices to take $\eta>6\sqrt{2}\,\varepsilon$.

5. A priori estimate at initial time

Proposition 5.1 (Modulus of continuity in time). *There exists a constant* $C''=C''(n,\sup_{\mathbb{R}^n}c_0)>0$ *such that for all* $x_0\in\mathbb{R}^n$, $t>0$, $\eta>6\sqrt{2}\,\varepsilon$ *and* $\varepsilon\in(0,1/2)$,

$$|u^\varepsilon(x_0,t)-u_0(x_0)|\le|Du_0|_{L^\infty(\mathbb{R}^n)}\{\eta+tC''/\eta\}.$$

Remark 5.2. Since $|Du^\varepsilon(\cdot,t)|_{L^\infty(\mathbb{R}^n)}\le|Du_0|_{L^\infty(\mathbb{R}^n)}$ (see Proposition 2.7), we also have, for $\varepsilon\in(0,1/2)$ and $\eta>6\sqrt{2}\,\varepsilon$,

$$|u^\varepsilon(x_0,t+s)-u^\varepsilon(x_0,s)|\le|Du_0|_{L^\infty(\mathbb{R}^n)}\{\eta+tC''/\eta\}.$$

Proof of Proposition 5.1. We consider the function

$$\varphi(x,t) = B_0\sqrt{\eta^2+|x|^2} + u_0(x_0) - B_0|x_0| + Lt$$

with $B_0 = |Du_0|_{L^\infty(\mathbb{R}^n)}$ and L that will be specified later. It suffices to show that for $L = C''B_0/\eta$ and C'' large enough, φ is a supersolution of (1.4). Indeed, by the comparison principle (Theorem 2.2), we will then have

$$u^\varepsilon(x_0,t) \le \varphi(x_0,t) \le B_0(\eta + tC''/\eta) + u_0(x_0).$$

Let $(x,t) \in \mathbb{R}^n \times (0,\infty)$. Since $\varphi \in C^\infty(\mathbb{R}^n \times (0,\infty))$, to prove that φ is a supersolution of (1.4) at (x,t), it suffices to show that it satisfies the equation pointwise, i.e.

$$\varphi_t(x,t) \ge c^\varepsilon |D\varphi(x,t)|.$$

The proof is now decomposed into two cases:

1. $|x| \le 6\sqrt{2}\,\varepsilon$. In this case,

$$c^\varepsilon|D\varphi(x,t)| \le \frac{\|c_0\|_{L^1}}{\varepsilon|\ln\varepsilon|}\,\frac{B_0|x|}{\eta} \le \frac{6\sqrt{2}\,\|c_0\|_{L^1}B_0}{|\ln\varepsilon|\eta}.$$

So it suffices to take

$$L \ge \frac{6\sqrt{2}\,\|c_0\|_{L^1}}{|\ln\frac{1}{2}|}\,\frac{B_0}{\eta}.$$

2. $|x| \ge 6\sqrt{2}\,\varepsilon$. In this case we will show that φ is a supersolution of

$$\varphi_t + F(D^2\varphi, D\varphi) \ge L - L_0 \quad \text{for } L_0 = \frac{B_0}{\eta}\sup_{q\in\mathbf{S}^{n-1}} \operatorname{trace}\left(A\left(\frac{q}{|q|}\right)\right) \tag{5.38}$$

and then we will use Corollary 4.4. We set $M = D^2\varphi$. We can choose a basis such that

$$A\left(\frac{p}{|p|}\right) = \begin{pmatrix} A_{n-1}\left(\frac{p}{|p|}\right) & 0 \\ 0 & 0 \end{pmatrix},$$

where the last vector of the basis is $p/|p|$ with $p = D\varphi$. We set

$$M = B_0\begin{pmatrix} M_{n-1} & M_n \\ {}^tM_n & M_{nn} \end{pmatrix},$$

where $M_{n-1} = \frac{1}{\sqrt{\eta^2+|x|^2}}\,\mathrm{Id}$, M_n is a vector and $M_{nn} = \eta^2/(\eta^2+|x|^2)^{3/2}$. Then

$$\operatorname{trace}\left(MA\left(\frac{p}{|p|}\right)\right) = \frac{B_0}{\sqrt{\eta^2+|x|^2}}\operatorname{trace}(A_{n-1}) \le \frac{B_0}{\eta}\operatorname{trace}\left(A\left(\frac{p}{|p|}\right)\right).$$

Hence

$$\varphi_t(x,t) + F(D^2\varphi, D\varphi) = L - \operatorname{trace}\left(MA\left(\frac{p}{|p|}\right)\right)$$
$$\geq L - \frac{B_0}{\eta}\sup_{\mathbf{S}^{n-1}}\operatorname{trace}\left(A\left(\frac{p}{|p|}\right)\right) = L - L_0.$$

We now prove that φ is a supersolution of (1.4), i.e. $\varphi_t(x,t) \geq c^\varepsilon|D\varphi(x,t)|$, where $c^\varepsilon = (c_0^\varepsilon \star 1_{\{\varphi(\cdot,t)>\varphi(x,t)\}})(x,t) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon$. We have pointwise

$$\begin{aligned}\varphi_t &\geq -F(D^2\varphi, D\varphi) + L - L_0 \geq c^\varepsilon|D\varphi| + L - L_0 - F(D^2\varphi, D\varphi) - c^\varepsilon|D\varphi| \\ &\geq c^\varepsilon|D\varphi| + L - L_0 - C'B_0/\eta,\end{aligned}$$

where we have used Corollary 4.4. It is sufficient to take $L \geq B_0C''/\eta$ with

$$C'' = \sup_{q\in\mathbf{S}^{n-1}}\operatorname{trace}\left(A\left(\frac{q}{|q|}\right)\right) + C' + \frac{6\sqrt{2}\,|c_0|_{L^1}}{\ln\frac{1}{2}}. \tag{5.39}$$

Moreover, trace(A) is bounded by $|g|_{L^\infty}$, which is controlled by $|c_0|_{L^\infty}$ (since $c_0(x) = g(x)$ if $|x| = 1$). So, by Corollary 4.4, $C'' = C''(n, \sup_{\mathbb{R}^n} c_0)$.

Using similarly a subsolution, we deduce the result. This ends the proof of the proposition.

Corollary 5.3. *There exists a constant $C > 0$ such that for any $\varepsilon \leq 1/2$, the solution u^ε of* (1.4) *satisfies*

$$|u^\varepsilon(x,t+h) - u^\varepsilon(x,t)| \leq C|Du_0|_{L^\infty(\mathbb{R}^n)}\sqrt{h}, \qquad \forall x \in \mathbb{R}^n,\ t \geq 0,\ h \in [0, h_0],$$

with $h_0 = h_0(n, \sup_{\mathbb{R}^n} C_0) > 0$.

Proof. We can optimise the estimate of Remark 5.2 to obtain, for $\eta = \sqrt{tC''} \leq 3$,

$$|u^\varepsilon(x_0, t+s) - u^\varepsilon(x_0, s)| \leq 2|Du_0|_{L^\infty(\mathbb{R}^n)}\sqrt{C''}\sqrt{t} \qquad \text{if } \sqrt{t} > \frac{6\sqrt{2}\,\varepsilon}{\sqrt{C''}}.$$

Moreover, for all ε, we have

$$|u^\varepsilon(x_0, t+s) - u^\varepsilon(x_0, s)| \leq \frac{t}{\varepsilon|\ln\varepsilon|}|c_0|_{L^1}|Du_0|_{L^\infty(\mathbb{R}^n)}.$$

But, for $\sqrt{t} \leq 6\sqrt{2}\varepsilon/\sqrt{C''}$ and $\varepsilon \leq 1/2$, the following holds (using (5.39)):

$$\frac{t}{\varepsilon|\ln\varepsilon|}|c_0|_{L^1}|Du_0|_{L^\infty(\mathbb{R}^n)} \leq |Du_0|_{L^\infty(\mathbb{R}^n)}\sqrt{t}\,\frac{6\sqrt{2}}{\sqrt{C''}}\,\frac{|c_0|_{L^1}}{|\ln\frac{1}{2}|} \leq |Du_0|_{L^\infty(\mathbb{R}^n)}\sqrt{t}\sqrt{C''},$$

so, for all $t \leq 9/C''$ and s, we have

$$|u^\varepsilon(x_0, t+s) - u^\varepsilon(x_0, s)| \leq 2|Du_0|_{L^\infty(\mathbb{R}^n)}\sqrt{C''}\sqrt{t}.$$

Iterating the estimate on time intervals of length T satisfying $\sqrt{TC''} \leq 3$, we get the result. This ends the proof of the corollary.

6. Proof of the convergence theorem

Proof of Theorem 1.4. We use the half-relaxed limits introduced by Barles and Perthame [11], and defined by

$$\overline{u}(x,t) = \limsup_{\varepsilon\to 0,\, y\to x,\, s\to t} u^\varepsilon(y,s), \qquad \underline{u}(x,t) = \liminf_{\varepsilon\to 0,\, y\to x,\, s\to t} u^\varepsilon(y,s).$$

We will show that $\overline{u}$ (resp. $\underline{u}$) is a viscosity subsolution (resp. supersolution) of (1.5)–(1.7).

We argue by contradiction. Assume that there exists $\phi \in C^2$ such that $\overline{u} - \phi$ reaches a global strict maximum at (x_0, t_0) and

$$\phi_t(x_0,t_0) + F_*(D^2\phi, D\phi) = \theta > 0. \tag{6.40}$$

Two cases may occur:

1. $|D\phi(x_0,t_0)| \neq 0$. Then there exist $(x_\varepsilon, t_\varepsilon) \to (x_0,t_0)$ such that $u^\varepsilon - \phi$ reaches a maximum at $(x_\varepsilon, t_\varepsilon)$. As u^ε has linear growth, we can assume (by adding a term like $|x-x_0|^4 + |t-t_0|^2$ to ϕ if necessary) that this maximum is global. Since u^ε is a solution of (1.4), we have

$$\phi_t(x_\varepsilon,t_\varepsilon) \le \left((c_0^\varepsilon \star 1_{\{u^\varepsilon(\cdot,t_\varepsilon) \ge u^\varepsilon(x_\varepsilon,t_\varepsilon)\}})(x_\varepsilon) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon \right) |D\phi(x_\varepsilon,t_\varepsilon)|.$$

   Moreover, for all $x \neq x_\varepsilon$, we have $u^\varepsilon(x,t_\varepsilon) - \phi(x,t_\varepsilon) < u^\varepsilon(x_\varepsilon,t_\varepsilon) - \phi(x_\varepsilon,t_\varepsilon)$. So $\{u^\varepsilon(\cdot,t_\varepsilon) \ge u^\varepsilon(x_\varepsilon,t_\varepsilon)\} \subset \{\phi(\cdot,t_\varepsilon) > \phi(x_\varepsilon,t_\varepsilon)\} \cup \{x_\varepsilon\}$. Thus,

$$\phi_t(x_\varepsilon,t_\varepsilon) \le \left((c_0^\varepsilon \star 1_{\{\phi(\cdot,t_\varepsilon) > \phi(x_\varepsilon,t_\varepsilon)\}})(x_\varepsilon) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon \right) |D\phi(x_\varepsilon,t_\varepsilon)|.$$

   We can use Corollary 4.2 and pass to the limit as $\varepsilon \to 0$ to obtain

$$\begin{aligned}\phi_t(x_0,t_0) &\le G(D^2\phi(x_0,t_0), D\phi(x_0,t_0))|D\phi(x_0,t_0)| \\ &= -F(D^2\phi(x_0,t_0), D\phi(x_0,t_0)),\end{aligned}$$

   which contradicts (6.40) (since $F(M,p) = F_*(M,p)$ for $p \neq 0$).
2. $|D\phi(x_0,t_0)| = 0$ and $|D^2\phi(x_0,t_0)| = 0$. As in the first case, there exist $(x_\varepsilon,t_\varepsilon) \to (x_0,t_0)$ such that $u^\varepsilon - \phi$ reaches a global maximum at $(x_\varepsilon,t_\varepsilon)$ (up to adding a term like $|x-x_0|^4 + |t-t_0|^2$ to ϕ if necessary). We set

$$c^\varepsilon[\phi](x_\varepsilon,t_\varepsilon) = \left((c_0^\varepsilon \star 1_{\{\phi(\cdot,t_\varepsilon) > \phi(x_\varepsilon,t_\varepsilon)\}})(x_\varepsilon) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon \right).$$

   By assumptions, for all $\eta > 0$, there exists $r > 0$ such that

$$|D^2\phi(x,t)| \le \eta \quad \text{if } (x,t) \in Q_{2r}(x_0,t_0)$$

   where $Q_r(x_0,t_0) = B_r(x_0) \times (t_0 - r, t_0 + r)$. There are two subcases:

A: $|D\phi(x_\varepsilon, t_\varepsilon)| > 12\varepsilon\eta r$. We set

$$\mathcal{I}(x,t) = c^\varepsilon[\phi](x,t)|D\phi| + F_*(D^2\phi, D\phi), \qquad \phi^r(x,t) = \frac{1}{r^2}\phi(x_0 + rx, t_0 + rt).$$

Straightforward computations give, with $\bar{x}_\varepsilon = x_\varepsilon/r$ and $\bar{t}_\varepsilon = t_\varepsilon/r$,

$$\begin{aligned}\mathcal{I}(x_\varepsilon, t_\varepsilon) &= F_*(D^2\phi^r, D\phi^r) + \frac{|\ln\frac{\varepsilon}{r}|}{|\ln\varepsilon|}|D\phi^r|c^{\varepsilon/r}[\phi^r](\bar{x}_\varepsilon, \bar{t}_\varepsilon)\\ &= F_*(D^2\phi^r, D\phi^r) + \left(1 - \frac{|\ln r|}{|\ln\varepsilon|}\right)|D\phi^r|c^{\varepsilon/r}[\phi^r](\bar{x}_\varepsilon, \bar{t}_\varepsilon)\\ &= \left(1 - \frac{|\ln r|}{|\ln\varepsilon|}\right)\mathcal{I}_1 + \mathcal{I}_2\end{aligned}$$

where

$$\mathcal{I}_1 = F_*(D^2\phi^r, D\phi^r) + |D\phi^r|c^{\varepsilon/r}[\phi^r](\bar{x}_\varepsilon, \bar{t}_\varepsilon), \qquad \mathcal{I}_2 = \frac{|\ln r|}{|\ln\varepsilon|}F_*(D^2\phi^r, D\phi^r).$$

We can then apply Proposition 4.1 to $\mathcal{I}_1$ with

$$a = 2\eta \geq |D^2\phi^r|, \qquad b = |D\phi^r(\bar{x}_\varepsilon, \bar{t}_\varepsilon)| \to 0, \qquad 2\delta = \delta_0 = b/6\eta$$

and get (with an abuse of notation for a generic constant C)

$$|\mathcal{I}_1| \leq Cb\left\{\frac{1}{\delta_0} + \frac{1}{\delta_0}\frac{|\ln\delta|}{|\ln\varepsilon|} + \frac{1}{\delta_0|\ln\varepsilon|}\right\} \leq C\left\{\eta + \eta + \frac{\eta}{|\ln\varepsilon|}\right\} \leq C\eta$$

for ε small enough to get b small enough. We then deduce that for ε small enough we have $|\mathcal{I}(x_\varepsilon, t_\varepsilon)| \leq C\eta$ and so

$$\begin{aligned}&\phi_t(x_\varepsilon, t_\varepsilon) + F_*(D^2\phi, D\phi)\\ &\qquad = \phi_t(x_\varepsilon, t_\varepsilon) - c^\varepsilon[\phi](x_\varepsilon, t_\varepsilon) + F_*(D^2\phi, D\phi) + c^\varepsilon[\phi](x_\varepsilon, t_\varepsilon)\\ &\qquad \leq |\mathcal{I}(x_\varepsilon, t_\varepsilon)| \leq C\eta.\end{aligned}$$

B: $|D\phi(x_\varepsilon, t_\varepsilon)| \leq 12\varepsilon\eta r$. Then

$$c^\varepsilon[\phi](x_\varepsilon, t_\varepsilon)|D\phi| \leq \frac{|c_0|_{L^1}}{\varepsilon|\ln\varepsilon|}|D\phi| \leq \frac{12\eta r}{|\ln\varepsilon|}|c_0|_{L^1}$$

and using $F_*(D^2\phi, D\phi) = 0$ in (x_0, t_0), we also deduce that for ε small enough,

$$\phi_t(x_\varepsilon, t_\varepsilon) + F_*(D^2\phi, D\phi) \leq C\eta.$$

Letting $\varepsilon \to 0$, we get

$$\phi_t(x_0, t_0) + F_*(D^2\phi, D\phi) \leq C\eta$$

and so $\theta \leq C\eta$, which is a contradiction for η small enough.

Finally, we have shown that $\overline{u}$ is a subsolution. The proof that $\underline{u}$ is a supersolution is exactly the same.

Moreover, by Corollary 5.3, we have

$$|u^\varepsilon(\cdot,t)-u_0(\cdot)| \le C|t|^{1/2}, \qquad \text{for } 0 \le t \le 1$$

where C is a constant which depends only on n, $\sup_{\mathbb{R}^n} c_0$ and $|Du_0|_{L^\infty}$. So $\overline{u}(\cdot,0) = \underline{u}(\cdot,0) = u_0(\cdot)$. Since $\overline{u}$ is a subsolution and $\underline{u}$ is a supersolution, we deduce by the comparison principle (Theorem 3.3) that $\overline{u}(x,t) \le \underline{u}(x,t)$ for all (x,t) and so $\overline{u} = \underline{u} = u^0$, i.e. u^ε converges locally uniformly on compact subsets of $\mathbb{R}^n \times [0,\infty)$ to u^0 which is the unique solution of (1.5)–(1.7). This ends the proof of the theorem.

7. Proof of Theorem 1.7

We now prove Theorem 1.7. We need the following proposition:

Proposition 7.1 (The matrix A is a Hessian). *Let $n \ge 2$. Let $g \in C^0(\mathbb{R}^n \setminus \{0\})$ with $g(\lambda p) = g(p)/|\lambda|^{n+1}$. Set*

$$A\left(\frac{p}{|p|}\right) = \int_{\theta\in \mathbf{S}^{n-2} = \mathbf{S}^{n-1}\cap\{\langle x, p/|p|\rangle = 0\}} \left(\frac{1}{2} g(\theta)\theta\otimes\theta\right) d\theta$$

with $A(\lambda p) = A(p)/|\lambda|$ for $\lambda \neq 0$. Then the function $G := -(2\pi)^{-1}\mathcal{F}(L_g)$ (where L_g and the Fourier transform are given in Definition 1.6) is such that $G(\lambda p) = |\lambda| G(p)$ and

$$A(p) = D^2 G(p).$$

For the proof, we will need the following lemma:

Lemma 7.2 (The curl of the matrix A). *Under the assumptions of Proposition 7.1, the curl of A defined by $\operatorname{curl}(A) = (\partial_k A_{ij} - \partial_i A_{kj})_{i,j,k}$ is zero, and there exists a distribution Φ such that $A(p) = D^2\Phi(p)$. Moreover, $\Phi \in C^0(\mathbb{R}^n)\cap C^2(\mathbb{R}^n\setminus\{0\})$, and Φ is unique if we assume $\Phi(-p) = \Phi(p)$ and $\Phi(0) = 0$. We then have $\Phi(\lambda p) = |\lambda|\Phi(p)$ for all $\lambda \in \mathbb{R}\setminus\{0\}$ and $p \in \mathbb{R}^n$.*

Proof. In this proof, we denote by $e \cdot f$ the scalar product of e and f. First, we compute $\partial_k A_{ij}(p)$ for $p \neq 0$ and $g \in C^1(\mathbb{R}^n \setminus \{0\})$ and with ∂_k indicating the derivation in the direction e_k. Because we will compute the curl of row vectors of the matrix, it is sufficient to choose an orthonormal basis $(e_1,\ldots,e_n)$ such that e_1 is parallel to p. Then two cases may occur:

1. e_k is parallel to p ($e_k \parallel p$). Then

$$\partial_k A_{ij}(p) = -\frac{p\cdot e_k}{|p|^2} A_{ij}(p).$$

2. e_k is perpendicular to p ($e_k \perp p$). In this case (see Figure 1), we have to consider variations at the first order of the integral defining $A(p)$ for $\theta \in p^\perp \cap \mathbf{S}^{n-1}$ to $\theta \in (p+\varepsilon e_k)^\perp \cap \mathbf{S}^{n-1}$ for ε arbitrarily small. Consider a unit vector $\theta \in p^\perp \cap \mathbf{S}^{n-1}$ that

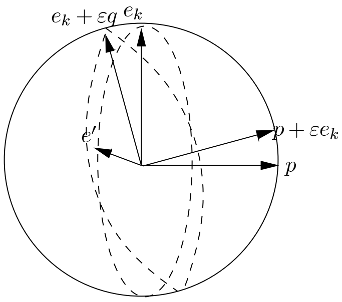

Fig. 1. Computation at the first order of $\partial_k A_{ij}(p)$, case $|p| = 1$.

we write

$$\theta = (\cos\alpha)e' + (\sin\alpha)e_k$$

with $\sin\alpha = \theta \cdot e_k$ and $e' \perp p$, $e' \perp e_k$. At the first order, this vector becomes (by infinitesimal rotation)

$$(\cos\alpha)e' + (\sin\alpha)(e_k + \varepsilon q) \in (p + \varepsilon e_k)^{\perp} \cap \mathbf{S}^{n-1},$$

where we define

$$q = \frac{-p}{|p|}, \qquad \bar{g}(\theta) = g(\theta)\theta \otimes \theta.$$

Then

$$\partial_k A_{ij}(p) = \frac{1}{|p|^2} \int_{\mathbf{S}^{n-1} \cap p^{\perp}} d\theta \, \frac{1}{2} q \cdot \nabla \bar{g}(\theta)(\theta \cdot e_k)(e_i, e_j).$$

Moreover,

$$\begin{aligned} & q \cdot \nabla \bar{g}(\theta)(\theta \cdot e_k)(e_i, e_j) \\ & \quad = (q \cdot \nabla g(\theta))(\theta \cdot e_k)(\theta \cdot e_i)(\theta \cdot e_j) + g(\theta)(\theta \cdot e_k)(q \cdot e_i(\theta \cdot e_j) + (\theta \cdot e_i) q \cdot e_j) \\ & \quad = q \cdot \nabla g(\theta)(\theta \cdot e_k)(\theta \cdot e_i)(\theta \cdot e_j) + (q \cdot e_i)\bar{g}(\theta)(e_k, e_j) + (q \cdot e_j)\bar{g}(\theta)(e_k, e_i). \end{aligned}$$

We are now able to compute the curl of A. To do this we consider several cases:

1. $e_k, e_i, e_j \parallel p$. Then $A_{ij}(p) = A_{kj}(p) = 0$ and so $\partial_k A_{ij} - \partial_i A_{kj} = 0$.
2. $e_k, e_i \parallel p$, $e_j \perp p$. In the same way, $\partial_k A_{ij} - \partial_i A_{kj} = 0$.
3. $e_k, e_j \parallel p$, $e_i \perp p$. Then $\partial_k A_{ij} = 0$ and $\partial_i A_{kj} = 0$ (since $\theta \cdot e_j = \theta \cdot e_k = 0$).
4. $e_k \perp p$, $e_i, e_j \parallel p$. The same as case 3.
5. $e_k \parallel p$, $e_i, e_j \perp p$. Then $\partial_k A_{ij} = -\frac{1}{|p|} A_{ij}$ (if $e_k = p/|p|$) and

$$\begin{aligned} \partial_i A_{kj} &= \frac{1}{|p|^2} \int_{\mathbf{S}^{n-1} \cap p^{\perp}} d\theta \, \frac{1}{2} q \cdot e_k \bar{g}(\theta)(e_i, e_j) \\ &= -\frac{1}{|p|^2} \int_{\mathbf{S}^{n-1} \cap p^{\perp}} d\theta \, \frac{1}{2} \bar{g}(\theta)(e_i, e_j) = -\frac{1}{|p|} A_{ij}(p). \end{aligned}$$

We have used the fact that $q \cdot e_k = \frac{-p}{|p|} \cdot \frac{p}{|p|} = -1$. So $\partial_k A_{ij} - \partial_i A_{kj} = 0$.

6. $e_k, e_j \perp p, e_i \parallel p$. The same as case 4.
7. $e_k, e_i \perp p, e_j \parallel p$. Then

$$\partial_k A_{ij}(p) = \frac{1}{|p|^2} \int_{\mathbf{S}^{n-1}\cap p^\perp} d\theta\, (q \cdot e_j)\bar{g}(\theta)(e_k, e_i) = \partial_i A_{kj}(p).$$

8. $e_k, e_i, e_j \perp p$. Then

$$\partial_k A_{ij}(p) = \frac{1}{|p|^2} \int_{\mathbf{S}^{n-1}\cap p^\perp} d\theta\, \frac{1}{2}(q \cdot \nabla g(\theta))(\theta \cdot e_k)(\theta \cdot e_i)(\theta \cdot e_j) = \partial_i A_{kj}(p).$$

We have used the fact that $q \cdot e_i = q \cdot e_j = q \cdot e_k = 0$.

Thus, $\mathrm{curl}(A) = 0$ on $\mathbb{R}^n \setminus \{0\}$. We now remark that

$$\begin{aligned}
\langle -(\mathrm{curl}\, A)_{i,j,k}, \varphi \rangle &= \int_{\mathbb{R}^n} (A_{ij}\partial_k\varphi - A_{kj}\partial_i\varphi) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n\setminus B_\varepsilon} (A_{ij}\partial_k\varphi - A_{kj}\partial_i\varphi) \\
&= \lim_{\varepsilon\to 0} \left(\int_{\mathbb{R}^n\setminus B_\varepsilon} -(\partial_k A_{ij} - \partial_i A_{kj})\varphi + \int_{\partial B_\varepsilon} (A_{ij} n_k - A_{kj} n_i)\varphi \right) \\
&= \lim_{\varepsilon\to 0} \varepsilon^{n-2} \int_{\partial B_1} (A_{ij}(\theta)\theta_k - A_{kj}(\theta)\theta_i)\varphi(\varepsilon\theta) d\theta \\
&= \begin{cases} \varphi(0) \int_{\mathbf{S}^1} (A_{ij}(\theta)\theta_k - A_{kj}(\theta)\theta_i)\, d\theta & \text{if } n = 2, \\ 0 & \text{if } n \neq 2. \end{cases}
\end{aligned}$$

In particular, we have used the fact that $A \equiv 0$ for $n = 1$. Now, using the symmetry of g, we deduce that $A(-\theta) = A(\theta)$ and then by antisymmetry the last integral on $\mathbf{S}^1$ vanishes. Therefore $\mathrm{curl}(A) = 0$ on $\mathbb{R}^n$. By a passage to the limit, this is still true if $g \in C^0$ (and not only $g \in C^1$).

To deduce that there exists Φ such that $A = D^2\Phi$, we use the following lemma:

Lemma 7.3 (Vector fields with zero curl are gradients). *Let* $f = (f_1, \ldots, f_n) \in \mathcal{D}'(\mathbb{R}^n)$ *be such that* $\mathrm{curl}(f) = (\partial_k f_i - \partial_i f_k)_{i,k} = 0$. *Then there is* $h \in \mathcal{D}'(\mathbb{R}^n)$ *such that* $f_i = \partial_i h$.

For the proof, we refer to Schwartz [38, Chapter II, Paragraph 6, Theorem VI, p. 59].

We write $f_j = (f_{j1}, \ldots, f_{jn}) = (A_{j1}, \ldots, A_{jn})$. Using the fact that $\mathrm{curl}(A) = 0$, we deduce that $\mathrm{curl}(f_j) = 0$ for all $j \in \{1, \ldots, n\}$. Then, by Lemma 7.3 there are h_j such that $f_j = \nabla h_j$. Using the fact that A is symmetric, we deduce that $\partial_j h_i - \partial_i h_j = 0$. Applying again Lemma 7.3, we deduce that there is Φ such that $h = \nabla\Phi$ and so $A = D^2\Phi$. Note that Φ is unique up to a polynomial of degree 1. Let $\Phi^s(p) = \frac{1}{2}(\Phi(p) + \Phi(-p))$. Then $A = D^2\Phi^s$ and hence Φ^s is unique up to a constant. Moreover, $D^2\Phi(p)$ behaves like $1/|p|$ for small p and so $D^2\Phi \in L^{n-\varepsilon}$ for every $\varepsilon > 0$. Therefore $\Phi \in W^{2,n-\varepsilon}_{\mathrm{loc}}$ and by Sobolev injections $\Phi \in C^0(\mathbb{R}^n)$. We deduce that there is a unique Φ such that

$$\Phi(-p) = \Phi(p) \quad \text{and} \quad \Phi(0) = 0. \tag{7.41}$$

Finally, we remark that

$$D^2(\Phi(\lambda p)/|\lambda|) = |\lambda|(D^2\Phi)(\lambda p) = |\lambda| A(\lambda p) = A(p) = D^2\Phi(p).$$

Therefore $\Phi(\lambda p) = |\lambda|\Phi(p)$ if Φ satisfies (7.41).

Proof of Proposition 7.1. We show that $\Phi = -(2\pi)^{-1}\mathcal{F}(L_g)$ (where Φ is defined in Lemma 7.2). Let $\varphi \in \mathcal{S}$. Then

$$\begin{aligned}\langle -D^2_{\xi\xi}\mathcal{F}(L_g)(\xi), \varphi\rangle(\zeta,\zeta) &= \langle \mathcal{F}(-ix\otimes ixL_g(x)), \varphi\rangle(\zeta,\zeta) = \langle L_g, (x\otimes x)\mathcal{F}(\varphi)\rangle(\zeta,\zeta)\\ &= \langle L_g, (x\cdot\zeta)^2\mathcal{F}(\varphi)(x)\rangle\\ &= \int_{\mathbb{R}^n} dx\, \frac{g(x/|x|)}{|x|^{n+1}}(x\cdot\zeta)^2\mathcal{F}(\varphi)(x)\\ &= \left\langle \mathcal{F}\left(\frac{g(x/|x|)}{|x|^{n+1}}(x\cdot\zeta)^2\right), \varphi\right\rangle.\end{aligned}$$

We next have the following lemma:

Lemma 7.4. *Let $n \geq 2$. Let $g \in C^0(\mathbb{R}^n \setminus \{0\})$ be such that $g(\lambda p) = g(p)/|\lambda|^{n+1}$. Then*

$$\mathcal{F}\left(\frac{g(x/|x|)}{|x|^{n+1}}(x\cdot\zeta)^2\right)(\xi) = 2\pi A(\xi)(\zeta,\zeta).$$

Here, we just give a formal proof. The complete proof is given in the Appendix.

By definition of Fourier transform, we have formally for $\xi \neq 0$, with $\theta = x/|x|$ and $r = |x|$,

$$\begin{aligned}\mathcal{F}\left(\frac{g(x/|x|)}{|x|^{n+1}}(x\cdot\zeta)^2\right) &= \int_{\mathbb{R}^n} \frac{g(x/|x|)(x\cdot\zeta)^2 e^{-i\xi\cdot x}}{|x|^{n+1}}\,dx = \int_{\mathbb{R}^n} \frac{g(\theta)(\theta\cdot\zeta)^2 e^{-i\xi\cdot x}}{|x|^{n-1}}\,dx\\ &= \int_{\mathbf{S}^{n-1}\times(0,\infty)} g(\theta)(\theta\cdot\zeta)^2 e^{-i\xi\cdot\theta r}\,d\theta\,dr\\ &= \int_{\mathbf{S}^{n-1}} d\theta\, g(\theta)(\theta\cdot\zeta)^2 \int_0^\infty dr\left(\frac{e^{i\xi\cdot\theta r}+e^{-i\xi\cdot\theta r}}{2}\right)\\ &= \int_{\mathbf{S}^{n-1}} d\theta\, g(\theta)(\theta\cdot\zeta)^2 \int_{-\infty}^\infty dr\,\frac{e^{i\xi\cdot\theta r}}{2}\\ &= \frac{2\pi}{|\xi|}\int_{\mathbf{S}^{n-1}\cap\xi^\perp} d\theta\,\frac{1}{2}g(\theta)(\theta\cdot\zeta)^2 = 2\pi A(\xi)(\zeta,\zeta),\end{aligned}$$

where we have used the fact that $\mathcal{F}(1) = 2\pi\delta_0$ in 1D, which formally gives

$$\int_{-\infty}^\infty dr\, e^{i\xi\cdot\theta r} = 2\pi\delta_0(\xi\cdot\theta) = \frac{2\pi}{|\xi|}\delta_0\left(\frac{\xi}{|\xi|}\cdot\theta\right).$$

This completes the formal proof of Lemma 7.4.

We then get

$$-D^2\mathcal{F}(L_g)(\xi) = 2\pi A(\xi) = 2\pi D^2\Phi.$$

Moreover, $\mathcal{F}(L_g)(-\xi) = \mathcal{F}(L_g)(\xi)$ and $\mathcal{F}(L_g)(0) = 0$. Therefore, by Lemma 7.2,

$$\Phi = -\frac{1}{2\pi}\mathcal{F}(L_g)$$

and $\Phi(\lambda p) = |\lambda|\Phi(p)$. This completes the proof of the proposition.

Proof of Theorem 1.7. Let us first compute div $\nabla G(Du/|Du|)$. We set $p = Du$. Then

$$\begin{aligned}
\operatorname{div}\nabla G\left(\frac{Du}{|Du|}\right) &= \sum_i \frac{\partial}{\partial x_i}\left(\frac{\partial G}{\partial x_i}\left(\frac{p}{|p|}\right)\right) = \sum_{i,j}\frac{\partial^2 G}{\partial x_i \partial x_j}\left(\frac{p}{|p|}\right)\frac{\partial}{\partial x_i}\left(\frac{D_j u}{|Du|}\right)\\
&= \frac{1}{|p|}\sum_{i,j}\frac{\partial^2 G}{\partial x_i\partial x_j}\left(\frac{p}{|p|}\right)\left(D^2_{ij}u - \frac{D^2_{\cdot i}u\cdot p\otimes p_j}{|p|^2}\right)\\
&= \frac{1}{|p|}\operatorname{trace}\left(D^2G\left(\frac{p}{|p|}\right)\left(\mathrm{Id} - \frac{p\otimes p}{|p|^2}\right)D^2u\right).
\end{aligned}$$

Moreover, for $\lambda > 0$, we have $G(\lambda p) = \lambda G(p)$. Then by differentiation we get

$$p\cdot\nabla G(\lambda p) = G(p).$$

Taking the gradient, we get

$$\nabla G(p) = \nabla G(\lambda p) + p\cdot D^2G(\lambda p)\lambda,$$

which implies, for $\lambda = 1$,

$$p\cdot D^2G(P) = 0.$$

This yields

$$D^2G\left(\frac{p}{|p|}\right)\left(\mathrm{Id} - \frac{p\otimes p}{p^2}\right) = D^2G\left(\frac{p}{|p|}\right).$$

Hence

$$\operatorname{div}\nabla G\left(\frac{Du}{|Du|}\right) = \frac{1}{|p|}\operatorname{trace}\left(A\left(\frac{p}{|p|}\right)\cdot D^2u\right).$$

This shows the first part of the theorem.

In the two-dimensional case, we simply remark that

$$g(\theta)\theta\otimes\theta = D^2G(\theta^\perp),$$

which implies the result. This ends the proof of Theorem 1.7.

Proof of Theorem 1.11. We can rewrite $A(p)$ as

$$\begin{aligned}
A(p) &= \int_{x\in p^\perp}\frac{1}{2}K_0(x)x\otimes x\,dx = \int_{\theta\in p^\perp\cap\mathbf{S}^{n-1}} d\theta\,\frac{1}{2}\left(\int_{(0,\infty)} dr\, r^n K_0(r\theta)\right)\theta\otimes\theta\\
&= \int_{\theta\in p^\perp\cap\mathbf{S}^{n-1}} d\theta\,\frac{1}{2}g(\theta)\theta\otimes\theta
\end{aligned}$$

with $g(\theta) = \int_{(0,\infty)} dr\, r^n K_0(r\theta)$. So, by applying Theorem 1.7, we see that the mean curvature motion defined by (1.5)–(1.6) using the matrix $A(p)$ is of variational type.

Proof of Proposition 1.8. The idea of building a function g which changes its sign and satisfies

$$\int_{\mathbf{S}^{n-1}\cap p^\perp} d\theta\, \frac{1}{2} g(\theta)\, \theta \otimes \theta \geq 0$$

for all $p \in \mathbb{R}^n \setminus \{0\}$ is simple. First, we consider the set

$$\mathbf{S} = \bigcup_{i=1}^{n} (\mathbf{S}^{n-1} \cap \{x_i = 0\})$$

and we remark that any hyperplane Π which contains the origin intersects $\mathbf{S}$ at an angle $\alpha \geq \alpha_0$ with $\alpha_0 > 0$ independent of Π. We then define g on $\mathbf{S}^{n-1}$ as a mollification of $\delta_{\mathbf{S}} - \eta$ for η small enough where $\delta_{\mathbf{S}}$ is the Dirac mass on $\mathbf{S}^{n-1}$ with support $\mathbf{S}$.

We now make a rigorous construction. We denote by $(e_i)_{i=1,\dots,n}$ an orthonormal basis of $\mathbb{R}^n$. We use the following lemma:

Lemma 7.5. *For $\varepsilon \in (0,1]$ and $i = 1,\dots,n$, there exist $g_i^\varepsilon \in C^\infty(\mathbf{S}^{n-1})$ such that for all $\Psi^\varepsilon \in C^\infty(\mathbf{S}^{n-1})$ and $p_\varepsilon \in \mathbf{S}^{n-1}$, if $p_\varepsilon \to p_0$ and $\|\Psi^\varepsilon - \Psi^0\|_{L^\infty(\mathbf{S}^{n-1})} \to 0$, then*

$$\int_{\mathbf{S}^{n-1}\cap p_\varepsilon^\perp \simeq \mathbf{S}^{n-2}} d\theta\, g_i^\varepsilon(\theta)\Psi^\varepsilon(\theta) \to \frac{1}{\sin\widehat{(p_0, e_i)}} \int_{\mathbf{S}^{n-1}\cap p_0^\perp \cap e_i^\perp \simeq \mathbf{S}^{n-3}} d\theta\, \Psi^0(\theta) \quad \text{as } \varepsilon \to 0$$

provided p_0 is not parallel to e_i, where $\widehat{(p_0, e_i)} \in [0, \pi/2]$ denotes the angle between p_0 and e_i. Moreover,

$$g_i^\varepsilon(\theta) = 0 \quad \text{if } |\langle\theta, e_i\rangle| \geq \varepsilon. \tag{7.42}$$

The proof is postponed.

We set

$$g^\varepsilon = \sum_{i=1}^{n} g_i^\varepsilon - \eta$$

with η a small parameter to be specified. We remark that by (7.42) for ε small enough, g^ε is not non-negative. We want to show that there exists ε_0 such that for all $0 < \varepsilon < \varepsilon_0$ and all $p, \xi \in \mathbf{S}^{n-1}$,

$$\int_{\mathbf{S}^{n-1}\cap p^\perp \simeq \mathbf{S}^{n-2}} d\theta\, g^\varepsilon(\theta)\langle\theta, \xi\rangle^2 \geq 0. \tag{7.43}$$

We will prove (7.43) by contradiction, using the following lemma:

Lemma 7.6. *There exists $C_0 > 0$ such that for all $p \in \mathbf{S}^{n-1}$ and $\xi \in \mathbf{S}^{n-1} \cap p^\perp$, there exists $i_0 \in \{1,\dots,n\}$ such that*

$$\int_{\mathbf{S}^{n-1}\cap p^\perp \cap e_{i_0}^\perp \simeq \mathbf{S}^{n-3}} d\theta\, \langle\theta, \xi\rangle^2 \geq C_0 \quad \text{and} \quad \widehat{(p, e_{i_0})} \geq C_0$$

where $\widehat{(p, e_{i_0})} \in [0, \pi/2]$.

The proof is postponed.

We now prove (7.43) by contradiction. Suppose that there exists a subsequence $\varepsilon_k \to 0$ for which there exist $p_k \in \mathbf{S}^{n-1}$ and $\xi_k \in \mathbf{S}^{n-1} \cap p_k^\perp$ such that

$$\int_{\mathbf{S}^{n-1}\cap p_k^\perp \simeq \mathbf{S}^{n-2}} d\theta \, g^{\varepsilon_k}(\theta)\langle \theta, \xi_k\rangle^2 \leq 0.$$

Up to extracting a subsequence, we can assume that $p_k \to p_\infty$ and $\xi_k \to \xi_\infty$ with $p_\infty, \xi_\infty \in \mathbf{S}^{n-1}$. We then have, with the index i_0 given by Lemma 7.6 for $p = p_\infty$ and $\xi = \xi_\infty$,

$$\begin{aligned} 0 \geq \int_{\mathbf{S}^{n-1}\cap p_k^\perp} d\theta \, g^{\varepsilon_k}(\theta)\langle \theta, \xi_k\rangle^2 &\geq \int_{\mathbf{S}^{n-1}\cap p_k^\perp} d\theta \, g_{i_0}^{\varepsilon_k}(\theta)\langle \theta, \xi_k\rangle^2 - \eta \int_{\mathbf{S}^{n-1}\cap p_k^\perp} d\theta \\ &\geq \int_{\mathbf{S}^{n-1}\cap p_k^\perp} d\theta \, g_{i_0}^{\varepsilon_k}(\theta)\langle \theta, \xi_k\rangle^2 - \eta|\mathbf{S}^{n-2}|. \end{aligned}$$

By passing to the limit, using Lemma 7.5, we obtain

$$0 \geq \frac{1}{\sin(\widehat{p_\infty, e_{i_0}})} \int_{\mathbf{S}^{n-1}\cap p_\infty^\perp \cap e_{i_0}^\perp} d\theta \, \langle \theta, \xi_\infty\rangle^2 - \eta|\mathbf{S}^{n-2}| \geq \frac{C_0}{\sin C_0} - \eta|\mathbf{S}^{n-2}|$$

where C_0 is given in Lemma 7.6. This is a contradiction for η small enough.

Proof of Lemma 7.6. The proof is by contradiction. If the result is false, then there are $C_k \to 0$, $p_k \in \mathbf{S}^{n-1}$ and $\xi_k \in \mathbf{S}^{n-1} \cap p_k^\perp$ such that for all $i \in \{1, \ldots, n\}$,

$$0 \leq (\widehat{p_k, e_i}) \leq C_k \tag{7.44}$$

or

$$\int_{\mathbf{S}^{n-1}\cap p_k^\perp \cap e_i^\perp} d\theta \, (\xi_k \cdot \theta)^2 \leq C_k \quad \text{if } (\widehat{p_k, e_i}) \neq 0. \tag{7.45}$$

We distinguish two cases:

1. There exist two indices i such that (7.44) holds. Up to relabelling the indices, we can assume that (7.44) holds for $i = 1, 2$. We deduce by extracting a subsequence and passing to the limit that $p_\infty = \lim p_k$ exists and $(\widehat{p_\infty, e_i}) = 0$ for $i = 1, 2$, which is a contradiction.
2. There exist two indices i such that (7.45) holds. Up to relabelling the indices, we can assume that (7.45) holds for $i = 1, 2$. In this case, by passing to the limit, up to extracting a subsequence, we obtain

$$\int_{\mathbf{S}^{n-1}\cap p_\infty^\perp \cap e_i^\perp} d\theta \, (\xi_\infty \cdot \theta)^2 = 0, \qquad \forall i = 1, 2.$$

We then deduce that $\xi_\infty \in \mathbf{S}^{n-1} \cap p_\infty^\perp$ is parallel to e_i for $i = 1, 2$, which is a contradiction.

Finally, in dimension $n \geq 3$, we are either in case 1 or case 2, so we have obtained a contradiction.

Proof of Lemma 7.5. We set $\tilde{g}_i^\varepsilon(x) = (1/\varepsilon)\rho(x \cdot e_i/\varepsilon)$ where $\rho \in C_c^\infty(\mathbb{R}, \mathbb{R})$ and

$$\rho \geq 0, \quad \mathrm{supp}(\rho) \subset [-1, 1], \quad \int_{\mathbb{R}} \rho(x)\, dx = 1.$$

We then set

$$g_i^\varepsilon(\theta) = \int_0^\infty r^{n-1} \tilde{g}_i^\varepsilon(r\theta) f(r)\, dr$$

with $f \in C_c^\infty((0,\infty), \mathbb{R})$ satisfying $\int_0^\infty f(r) r^{n-2}\, dr = 1$. For all $\Psi^\varepsilon \in C^\infty(\mathbf{S}^{n-1})$ and $p_\varepsilon \in \mathbf{S}^{n-1}$, define

$$\mathcal{I}^\varepsilon = \int_{\mathbf{S}^{n-1} \cap p_\varepsilon^\perp \simeq \mathbf{S}^{n-2}} d\theta\, g_i^\varepsilon(\theta) \Psi^\varepsilon(\theta).$$

To simplify notation, set $\Psi = \Psi^\varepsilon$ and $p = p_\varepsilon$. We then have, if p is not parallel to e_i,

$$\mathcal{I}^\varepsilon = \int_{\mathbf{S}^{n-1} \cap p^\perp \simeq \mathbf{S}^{n-2}} d\theta\, \Psi(\theta) \int_0^\infty dr\, r^{n-1} \tilde{g}_i^\varepsilon(r\theta) f(r) = \int_{p^\perp} dx\, \tilde{g}_i^\varepsilon(x) \tilde{\Psi}(x)$$

where

$$\tilde{\Psi}(x) = f(|x|)\Psi(x/|x|)|x|.$$

Using the definition of $\tilde{g}_i^\varepsilon$, we then find, by setting $\alpha_i = \widehat{(p, e_i)}$ and using the change of coordinates $x = (y', y_n)$ with $y' \in p^\perp$ and $y_n \in \mathbb{R}$, that

$$\begin{aligned}
\mathcal{I}^\varepsilon &= \int_{p^\perp} dx\, \frac{1}{\varepsilon} \rho\left(\frac{x \cdot e_i}{\varepsilon}\right) \tilde{\Psi}(x) \\
&= \int_{p^\perp} dy'\, \frac{1}{\sin\alpha_i} \frac{\sin\alpha_i}{\varepsilon} \rho\left(\frac{y'.e_i'}{\frac{\varepsilon}{\sin\alpha_i} \sin\alpha_i}\right) \tilde{\Psi}(y', 0) \\
&= \frac{1}{\sin\alpha_i} \int_{p^\perp} dy'\, \frac{1}{\varepsilon'} \rho\left(\frac{y' \cdot e_i'/|e_i'|}{\varepsilon'}\right) \tilde{\Psi}(y', 0)
\end{aligned}$$

where $\varepsilon' = \varepsilon/\sin\hat{\theta}_i$ and e_i' is the orthogonal projection of e_i onto the hyperplane $p^\perp$. In particular, $|e_i'| = \sin\alpha_i$. Passing to the limit in ε, with $p_\varepsilon \to p_0$, $\Psi^\varepsilon \to \Psi^0$, $\alpha_i = \alpha_i^\varepsilon = \widehat{(p_\varepsilon, e_i)} \to \alpha_i^0 = \widehat{(p_0, e_i)}$ and $\tilde{\Psi}^\varepsilon = f(|x|)\Psi^\varepsilon(x/|x|)|x| \to \tilde{\Psi}^0 = f(|x|)\Psi^0(x/|x|)|x|$, yields

$$\begin{aligned}
\mathcal{I}^\varepsilon \to \frac{1}{\sin\alpha_i^0} \int_{p^\perp \cap e_i'^\perp} dy'\, \tilde{\Psi}^0(y', 0) &= \frac{1}{\sin\alpha_i^0} \int_{p^\perp \cap e_i^\perp} dy'\, \tilde{\Psi}^0(y', 0) \\
= \frac{1}{\sin\alpha_i^0} \int_{\mathbf{S}^{n-3} \simeq \mathbf{S}^{n-1} \cap p^\perp \cap e_i^\perp} d\theta \left(\int_0^\infty dr\, r^{n-3} f(r) r\right) \Psi^0(\theta) \\
= \frac{1}{\sin\alpha_i^0} \int_{\mathbf{S}^{n-3} \simeq \mathbf{S}^{n-1} \cap p^\perp \cap e_i^\perp} d\theta\, \Psi^0(\theta).
\end{aligned}$$

This ends the proof of the lemma.

8. Heuristical convergence and properties of the energies

8.1. Monotonicity of the energy

We begin by showing that the energy associated to (1.4) is non-increasing in time. We recall that (1.4) is formally associated to the energy

$$\mathcal{E}^\varepsilon(u^\varepsilon) = \int_\lambda \overline{\mathcal{E}^\varepsilon}(\lambda)\, d\lambda \tag{8.46}$$

where

$$\overline{\mathcal{E}^\varepsilon}(\lambda) = \int_{\mathbb{R}^n} -\frac{1}{2}(\overline{c}_0^\varepsilon \star \rho_\lambda^\varepsilon)\rho_\lambda^\varepsilon$$

with

$$\rho_\lambda^\varepsilon = 1_{\{u^\varepsilon > \lambda\}}, \qquad \overline{c}_0^\varepsilon = c_0^\varepsilon - \left(\int_{\mathbb{R}^n} c_0^\varepsilon\right)\delta_0.$$

Formally, we have

$$\frac{d\overline{\mathcal{E}^\varepsilon}(\lambda)}{dt} = \int_{\mathbb{R}^n} -(\overline{c}_0^\varepsilon \star \rho_\lambda^\varepsilon)(\rho_\lambda^\varepsilon)_t,$$

which is defined only on the support of $|D\rho_\lambda^\varepsilon|$ (since (1.4) formally implies $(\rho_\lambda^\varepsilon)_t = (\overline{c}_0^\varepsilon \star \rho_\lambda^\varepsilon)|D\rho_\lambda^\varepsilon|$). Moreover, $\overline{c}_0^\varepsilon \star \rho_\lambda^\varepsilon = c_0^\varepsilon \star \rho_\lambda^\varepsilon - (\int c_0^\varepsilon)\delta_0 \star \rho_\lambda^\varepsilon$. If we let η_n be a regularisation of the Dirac mass, then $\eta_n \star \rho_\lambda^\varepsilon = 1/2$ on the support of $|D\rho_\lambda^\varepsilon|$ (see Figure 2).

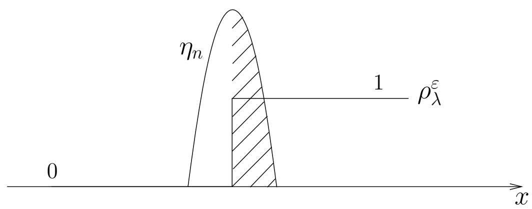

Fig. 2. The convolution of ρ_λ^ε with the Dirac mass.

So, we can assume that $\overline{c}_0^\varepsilon \star \rho_\lambda^\varepsilon = c_0^\varepsilon \star \rho_\lambda^\varepsilon - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon$ on the support of $|D\rho_\lambda^\varepsilon|$. Then

$$\frac{d\overline{\mathcal{E}^\varepsilon}(\lambda)}{dt} = \int_{\mathbb{R}^n} -\left(c_0^\varepsilon \star \rho_\lambda^\varepsilon - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon\right)^2 |D\rho_\lambda^\varepsilon|.$$

This implies

$$\frac{d\mathcal{E}^\varepsilon(u^\varepsilon)}{dt} = \int d\lambda \int_{\mathbb{R}^n} -\left(c_0^\varepsilon \star \rho_\lambda^\varepsilon - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon\right)^2 |D\rho_\lambda^\varepsilon| \le 0.$$

So the energy is non-increasing in time.

8.2. Formal convergence of the energy

We let $\mathcal{E}(u^0) = \int G(Du^0)$ be the energy associated to the mean curvature motion. We have formally

$$\frac{d}{dt}\mathcal{E}(u^0) = \int \nabla G\left(\frac{Du^0}{|Du^0|}\right) \cdot Du^0_t = \int -\left(\mathrm{div}\nabla G\left(\frac{Du^0}{|Du^0|}\right)\right)^2 |Du^0|.$$

Moreover, still formally we have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}^\varepsilon(u^\varepsilon) &= \int d\lambda \int -\left(c_0^\varepsilon \star \rho_\lambda^\varepsilon - \frac{1}{2}\int c_0^\varepsilon\right)^2 |D\rho_\lambda^\varepsilon| \\ &\to \int d\lambda \int -\left(\mathrm{trace}\left(A\left(\frac{Du^0}{|Du^0|}\right)D^2u^0\right)\right)^2 |D\rho_\lambda^0| \\ &= \int d\lambda \int -\left(\mathrm{div}\left(\nabla G\left(\frac{Du^0}{|Du^0|}\right)\right)\right)^2 |D\rho_\lambda^0| \\ &= \int -\left(\mathrm{div}\left(\nabla G\left(\frac{Du^0}{|Du^0|}\right)\right)\right)^2 |Du^0|.\end{aligned}$$

So, formally,

$$\frac{d}{dt}\mathcal{E}^\varepsilon(u^\varepsilon) \to \frac{d}{dt}\mathcal{E}(u^0).$$

The work of Garroni and Müller [28] suggests that we should have $\int_{x,\lambda} \frac{1}{2}(c_0^\varepsilon \star \rho_\lambda^\varepsilon)\rho_\lambda^\varepsilon \to \int d\lambda \int_{\Gamma_\lambda} G(Du^0/|Du^0|)$, where Γ_λ is the λ level set of u^0. We deduce that (using formally the coarea formula for BV functions)

$$\int_{x,\lambda} \frac{1}{2}(c_0^\varepsilon \star \rho_\lambda^\varepsilon)\rho_\lambda^\varepsilon \to \int d\lambda \int_x G\left(\frac{Du^0}{|Du^0|}\right)|D\rho_\lambda^0| = \int_x G\left(\frac{Du^0}{|Du^0|}\right)|Du^0| = \int G(Du^0)$$

and so, formally, $\mathcal{E}^\varepsilon(u^\varepsilon) \to \mathcal{E}(u^0)$.

9. Appendix: some lemmata on Fourier transform

Lemma 9.1. *The distribution L_g associated to g (see Definition 1.6) has the following properties:*

$$L_g(\lambda\,\cdot) = \frac{1}{\lambda^{n+1}}L_g \qquad \forall\lambda > 0, \tag{9.47}$$

$$\mathcal{F}(L_g)(\lambda\,\cdot) = \lambda\mathcal{F}(L_g) \qquad \forall\lambda > 0, \tag{9.48}$$

where $\mathcal{F}(L_g)$ is the Fourier transform of L_g defined by

$$\forall\varphi \in \mathcal{S}, \qquad \langle\mathcal{F}(L_g), \varphi\rangle = \langle L_g, \mathcal{F}(\varphi)\rangle.$$

Proof. Equality (9.47) results from the definition of L_g which by construction is homogeneous of degree $-(n+1)$. This can be rigorously shown using the general definition for a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$,

$$\forall \varphi \in C_c^\infty(\mathbb{R}^n), \qquad \langle u(\lambda\,\cdot), \varphi\rangle := \frac{1}{\lambda^n}\left\langle u, \varphi\left(\frac{\cdot}{\lambda}\right)\right\rangle. \tag{9.49}$$

We now prove (9.48). A straightforward computation for $\varphi \in \mathcal{S}$ gives

$$\mathcal{F}(\varphi)(\lambda\,\cdot) = \mathcal{F}\left(\frac{1}{\lambda^n}\varphi\left(\frac{\cdot}{\lambda}\right)\right)(\cdot). \tag{9.50}$$

Using the definition (9.49), one can show that (9.50) is still true for elements of $\mathcal{S}'$. Hence,

$$\mathcal{F}(L_g)(\lambda\,\cdot) = \mathcal{F}\left(\frac{1}{\lambda^n}L_g\left(\frac{\cdot}{\lambda}\right)\right)(\cdot) = \mathcal{F}\left(\frac{\lambda^{n+1}}{\lambda^n}L_g(\cdot)\right)(\cdot) = \lambda\mathcal{F}(L_g(\cdot))(\cdot)$$

where we have used (9.47). This ends the proof of the lemma.

Proof of Lemma 7.4. Let $R_0 > r_0 > 0$ and $\varphi \in C^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subset B_{R_0}(0) \setminus B_{r_0}(0)$. Let $\Psi_\lambda(y) = \Psi(\lambda y)$ for $y \in \mathbb{R}$ with $\Psi \in C_c^\infty(\mathbb{R})$ such that

$$\operatorname{supp}\Psi \subset [-1,1], \qquad \Psi \equiv 1 \text{ on } [-1/2, 1/2], \qquad 0 \le \Psi \le 1, \qquad \Psi(-y) = \Psi(y).$$

Consider $f \in C_c^\infty([0,\infty))$ with $\operatorname{supp} f \subset [r_0, R_0]$ and

$$\int_0^\infty f(\bar r)\bar r^n\, d\bar r = 1.$$

Assume first that $g \in C^\infty(\mathbf{S}^{n-1})$. Then for $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}\mathcal{I} &= \left\langle \mathcal{F}\left(\frac{g(x/|x|)}{|x|^{n+1}}(x\cdot\zeta)^2\right), \varphi\right\rangle = \left\langle \frac{g(x/|x|)}{|x|^{n+1}}(x\cdot\zeta)^2, \mathcal{F}(\varphi)\right\rangle \\ &= \int_{\mathbb{R}^n} dx\, \frac{g(x/|x|)}{|x|^{n+1}}(x\cdot\zeta)^2\left(\int_{\mathbb{R}^n} d\xi\, e^{-i\xi\cdot x}\varphi(\xi)\right)\left(\int_0^\infty f(\bar r)\bar r^n d\bar r\right).\end{aligned}$$

Since $|\Psi_\lambda(|x|/\bar r)| \le 1$ and $\Psi_\lambda(|x|/\bar r) \to 1$ as $\lambda \to 0$, we deduce by the dominated convergence theorem that

$$\begin{aligned}\mathcal{I} &= \lim_{\lambda\to 0}\int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+} dx\, d\xi\, d\bar r\, \frac{g(x/|x|)}{|x|^{n+1}}(x\cdot\zeta)^2\Psi_\lambda\left(\frac{|x|}{\bar r}\right)e^{-i\xi\cdot x} f(\bar r)\bar r^n\varphi(\xi) \\ &= \lim_{\lambda\to 0}\int_{\mathbf{S}^{n-1}\times\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{R}_+} d\theta\, d\xi\, d\bar r\, dr\, g(\theta)(\theta\cdot\zeta)^2\Psi_\lambda\left(\frac{r}{\bar r}\right)e^{-i\xi\cdot\theta r} f(\bar r)\bar r^n\varphi(\xi)\end{aligned}$$

where $\theta = x/|x|$ and $r = |x|$. We set $r = \bar{r}s$, $\bar{x} = \theta\bar{r}$ and $\bar{s} = |\xi|s$ to obtain

$$\begin{aligned}
\mathcal{I} &= \lim_{\lambda\to 0}\int_{\mathbf{S}^{n-1}\times\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{R}_+} d\theta\, d\xi\, d\bar{r}\, \bar{r}\, ds\, g(\theta)(\theta\cdot\zeta)^2\Psi_\lambda(s)e^{-i\xi\cdot\theta\bar{r}s} f(\bar{r})\bar{r}^n\varphi(\xi)\\
&= \lim_{\lambda\to 0}\int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+} d\bar{x}\, d\xi\, ds\, f(|\bar{x}|)g\left(\frac{\bar{x}}{|\bar{x}|}\right)(\bar{x}\cdot\zeta)^2\Psi_\lambda(s)e^{-i\xi\cdot\bar{x}s}\varphi(\xi)\\
&= \lim_{\lambda\to 0}\int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+} d\bar{x}\, d\xi\, d\bar{s}\, f(|\bar{x}|)g\left(\frac{\bar{x}}{|\bar{x}|}\right)(\bar{x}\cdot\zeta)^2\Psi_{\lambda/|\xi|}(\bar{s})e^{-i\frac{\xi}{|\xi|}\cdot\bar{x}\bar{s}}\frac{\varphi(\xi)}{|\xi|}\\
&= \lim_{\lambda\to 0}\int_{\mathbb{R}^n} d\xi\, \frac{\varphi(\xi)}{|\xi|}\int_{\mathbb{R}^n} d\bar{x}\, \Phi(\bar{x})\int_{R_+} d\bar{s}\, \Psi_{\lambda/|\xi|}(\bar{s})e^{-i\frac{\xi}{|\xi|}\cdot\bar{x}\bar{s}}
\end{aligned}$$

where $\Phi(\bar{x}) = f(|\bar{x}|)g(\bar{x}/|\bar{x}|)(\bar{x}\cdot\zeta)^2 \in C_c^\infty(\mathbb{R})$ and $\operatorname{supp}\Phi \subset B_{R_0}(0)\setminus B_{r_0}(0)$. Using the fact that $\Phi(-\bar{x}) = \Phi(\bar{x})$, we deduce that

$$\begin{aligned}
\mathcal{I} &= \lim_{\lambda\to 0}\int_{\mathbb{R}^n} d\xi\, \frac{\varphi(\xi)}{|\xi|}\int_{\mathbb{R}^n} d\bar{x}\, \Phi(\bar{x})\int_{R_+} d\bar{s}\, \Psi_{\lambda/|\xi|}(\bar{s})\frac{e^{-i\frac{\xi}{|\xi|}\cdot\bar{x}\bar{s}} + e^{i\frac{\xi}{|\xi|}\cdot\bar{x}\bar{s}}}{2}\\
&= \lim_{\lambda\to 0}\int_{\mathbb{R}^n} d\xi\, \frac{\varphi(\xi)}{2|\xi|}\int_{\mathbb{R}^n} d\bar{x}\, \Phi(\bar{x})\int_{R} d\bar{s}\, \Psi_{\lambda/|\xi|}(\bar{s})e^{-i\frac{\xi}{|\xi|}\cdot\bar{x}\bar{s}}.
\end{aligned}$$

We set $\bar{x} = \bar{x}' + \bar{y}e_\xi$ with $\bar{x}' \in e_\xi^\perp$, $\bar{y} \in \mathbb{R}$ and $e_\xi = \xi/|\xi|$ to obtain

$$\begin{aligned}
\mathcal{I} &= \lim_{\lambda\to 0}\int_{\mathbb{R}^n} d\xi\, \frac{\varphi(\xi)}{2|\xi|}\int_{\mathbb{R}^{n-1}} d\bar{x}'\int_{\mathbb{R}} d\bar{y}\, \Phi(\bar{x}',\bar{y})\int_{\mathbb{R}} d\bar{s}\, \Psi_{\lambda/|\xi|}(\bar{s})e^{-i\bar{y}\bar{s}}\\
&= \lim_{\lambda\to 0}\int_{\mathbb{R}^n} d\xi\, \frac{\varphi(\xi)}{2|\xi|}\int_{\mathbb{R}^{n-1}} d\bar{x}'\, J_{\lambda/|\xi|}(\bar{x}')
\end{aligned}$$

where

$$J_{\lambda/|\xi|}(\bar{x}') = \int_{\mathbb{R}} d\bar{y}\, \Phi(\bar{x}',\bar{y})\mathcal{F}(\Psi_{\lambda/|\xi|})(\bar{y}) = \langle\mathcal{F}(\Psi_{\lambda/|\xi|}), \Phi(\bar{x}',\cdot)\rangle.$$

We claim the following (the proof is postponed):

Lemma 9.2. *We have $\mathcal{F}(\Psi_\mu) \to 2\pi\delta_0$ in $\mathcal{S}'(\mathbb{R})$ as $\mu \to 0$.*

Using this result and the fact that $|J_{\lambda/|\xi|}(\bar{x}')| \le |\mathcal{F}(\Phi(x',\cdot))|_{L^1(\mathbb{R})}$, we deduce that

$$\begin{aligned}
\mathcal{I} &= 2\pi\int_{\mathbb{R}} d\xi\, \frac{\varphi(\xi)}{2|\xi|}\int_{\mathbb{R}^{n-1}} d\bar{x}'\, \Phi(\bar{x}',0)\\
&= 2\pi\int_{\mathbb{R}} d\xi\, \frac{\varphi(\xi)}{2|\xi|}\int_{\mathbb{R}^{n-1}} d\bar{x}'\, f(|\bar{x}|)g\left(\frac{\bar{x}}{|\bar{x}|}\right)(\bar{x}\cdot\zeta)^2\\
&= 2\pi\int_{\mathbb{R}} d\xi\, \frac{\varphi(\xi)}{2|\xi|}\int_{\mathbf{S}^{n-1}\cap\xi^\perp} d\theta\, g(\theta)(\theta\cdot\zeta)^2\int_{\mathbb{R}_+} d\bar{r}\, f(\bar{r})\bar{r}^n\\
&= 2\pi\int_{\mathbb{R}} d\xi\, \frac{\varphi(\xi)}{|\xi|}A\left(\frac{\xi}{|\xi|}\right)(\zeta,\zeta) = 2\pi\int_{\mathbb{R}} d\xi\, \varphi(\xi)A(\xi)(\zeta,\zeta) = 2\pi\langle A(\xi)(\zeta,\zeta),\varphi\rangle.
\end{aligned}$$

We have thus shown that

$$\left\langle \mathcal{F}\left(\frac{g(x/|x|)}{|x|^{n+1}} (x \cdot \zeta)^2 \right), \varphi \right\rangle = 2\pi \langle A(\xi)(\zeta, \zeta), \varphi \rangle.$$

By a passage to the limit, this is still true if $g \in C^0$ (and not only C^∞) and for all $\varphi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$. Hence

$$\mathcal{F}\left(\frac{g(x/|x|)}{|x|^{n+1}} (x \cdot \zeta)^2 \right) - 2\pi A(\xi)(\zeta, \zeta) = T$$

with $\operatorname{supp} T \subset \{0\}$, and so the distribution T is a finite sum of derivatives of the Dirac mass: $T = \sum a_\alpha \delta_0^{(\alpha)}$. Using the fact that $\delta_0^{(\alpha)}(\lambda\xi) = \lambda^{-n-|\alpha|}\delta_0^{(\alpha)}(\xi)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and the homogeneity of degree -1 of $D^2\mathcal{F}(L_g)$, we deduce that for $n \geq 2$, we have $T = 0$ and

$$\mathcal{F}\left(\frac{g(x/|x|)}{|x|^{n+1}} (x \cdot \zeta)^2 \right) = 2\pi A(\xi)(\zeta, \zeta).$$

This ends the proof of the lemma.

Proof of Lemma 9.2. Let $\varphi_1 \in \mathcal{S}(\mathbb{R})$. Then

$$\langle \mathcal{F}(\Psi_\mu) - 2\pi\delta_0, \varphi_1 \rangle = \langle \mathcal{F}(\Psi_\mu) - \mathcal{F}(1), \varphi_1 \rangle = \langle \Psi_\mu - 1, \mathcal{F}(\varphi_1) \rangle.$$

So, it just remains to show that $\Psi_\mu \to 1$ in $\mathcal{S}'(\mathbb{R})$ as $\mu \to 0$. Let $\varphi \in \mathcal{S}(\mathbb{R})$. Then

$$\begin{aligned}
\langle \Psi_\mu - 1, \varphi \rangle &= \int_{\mathbb{R}} dx\, (\Psi_\mu(x) - 1)\varphi(x) = \int_{\mathbb{R}} dx\, (\Psi(\mu x) - 1)\varphi(x) \\
&= \int_{|x| \geq 1/2\mu} dx\, (\Psi(\mu x) - 1)\varphi(x) \leq \int_{|x| \geq 1/2\mu} dx\, |\varphi(x)| \\
&\leq C\mathcal{N}_2(\varphi) \int_{|x| \geq 1/2\lambda} dx\, \frac{1}{1 + x^2} \to 0 \quad \text{as } \mu \to 0
\end{aligned}$$

where we have used the definition of $\mathcal{N}_p(\varphi) = \sup_{|\alpha|, |\beta| \leq p} \left| |x|^\alpha \frac{d^\beta \varphi(x)}{dx^\beta} \right|$ and the fact that

$$(1 + x^2)|\varphi(x)| \leq C\mathcal{N}_2(\varphi).$$

This ends the proof of the lemma.

Acknowledgments. The authors would like to thank G. Barles, G. Bellettini, A. Bossavit, A. Chambolle, B. Devincre, C. Imbert, A. Finel, Y. Le Bouar, M. Novaga, F. Murat and P. E. Souganidis for fruitful discussions during the preparation of this paper. The authors also thank an unknown referee for his suggestions and for correcting a mistake in a previous version of the paper. The last two authors were supported by the contract JC 1025 called "ACI jeunes chercheuses et jeunes chercheurs" of the French Ministry of Research (2003-2005).

References

[1] Alvarez, O., Cardaliaguet, P., Monneau, R.: Existence and uniqueness for dislocation dynamics with nonnegative velocity. Interfaces Free Bound. **7**, 415–434 (2005) Zbl 1099.35148 MR 2191694

[2] Alvarez, O., Carlini, E., Monneau, R., Rouy, E.: Convergence of a first order scheme for a nonlocal eikonal equation. Appl. Numer. Math. **56**, 1136–1146 (2006) Zbl 1100.65072 MR 2244967

[3] Alvarez, O., Carlini, E., Monneau, R., Rouy, E.: A convergent scheme for a nonlocal Hamilton–Jacobi equation, modeling dislocation dynamics. Numer. Math. **104**, 413–572 (2006) Zbl 1103.65100 MR 2249672

[4] Alvarez, O., Hoch, P., Le Bouar, Y., Monneau, R.: Résolution en temps court d'une équation de Hamilton–Jacobi non locale décrivant la dynamique d'une dislocation. C. R. Math. Acad. Sci. Paris **338**, 679–684 (2004) Zbl 1049.35005 MR 2065373

[5] Alvarez, O., Hoch, P., Le Bouar, Y., Monneau, R.: Dislocation dynamics: short time existence and uniqueness of the solution. Arch. Ration. Mech. Anal. **181**, 449–504 (2006) Zbl 05051264 MR 2231781

[6] Bardi, M., Capuzzo-Dolcetta, I.: Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations. Systems Control Found. Appl., Birkhäuser Boston (1997) Zbl 0890.49011 MR 1484411

[7] Barles, G.: Solutions de viscosité des équations de Hamilton–Jacobi. Math. Appl. 17, Springer, Paris (1994) Zbl 0819.35002 MR 1613876

[8] Barles, G., Cardaliaguet, P., Ley, O., Monneau, R.: General results for dislocation type equation. Preprint (2007)

[9] Barles, G., Georgelin, C.: A simple proof of convergence for an approximation scheme for computing motions by mean curvature. SIAM J. Numer. Anal. **32**, 484–500 (1995) Zbl 0831.65138 MR 1324298

[10] Barles, G., Ley, O.: Nonlocal first-order Hamilton–Jacobi equations modelling dislocations dynamics. Comm. Partial Differential Equations **31**, 1191–1208 (2006) Zbl 1109.35026 MR 2254611

[11] Barles, G., Perthame, B.: Exit time problems in optimal control and vanishing viscosity method. SIAM J. Control Optim. **26**, 1133–1148 (1988) Zbl 0674.49027 MR 0957658

[12] Barles, G., Soner, H. M., Souganidis, P. E.: Front propagation and phase field theory. SIAM J. Control Optim. **31**, 439–469 (1993) Zbl 0785.35049 MR 1205984

[13] Barles, G., Souganidis, P. E.: A new approach to front propagation problems: theory and applications. Arch. Ration. Mech. Anal. **141**, 237–296 (1998) Zbl 0904.35034 MR 1617291

[14] Barnett, D. M., Gavazza, S. D.: The self-force on a planar dislocation loop in an anisotropic linear-elastic medium. J. Mech. Phys. Solids **24**, 171–185 (1976) Zbl 0368.73094 MR 0521388

[15] Bellettini, G., Buttà, P., Presutti, E.: Sharp interface limits for non-local anisotropic interactions. Arch. Ration. Mech. Anal. **159**, 109–135 (2001) Zbl 0994.82057 MR 1857376

[16] Bellettini, G., Novaga, M.: Minimal barriers for geometric evolutions. J. Differential Equations **139**, 76–103 (1997) Zbl 0882.35028 MR 1467354

[17] Bellettini, G.: Novaga, M.: Some aspects of De Giorgi's barriers for geometric evolutions. In: Calculus of Variations and Partial Differential Equations (Pisa, 1996), Springer, Berlin, 115–151 (2000) Zbl 0957.35080 MR 1757698

[18] Brown, L. M.: The self-stress of dislocations and the shape of extended nodes. Philos. Mag. **10**, 441–466 (1964) Zbl 0125.25701

[19] Chambolle, A., Novaga, M.: Approximation of the anisotropic mean curvature flow. Math. Models Methods Appl. Sci. **17**, 833–844 (2007) Zbl 1120.35054

[20] Chen, Y. G., Giga, Y., Goto, S.: Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. J. Differential Geom. **33**, 749–786 (1991) Zbl 0696.35087 MR 1100211

[21] Crandall, M. G., Ishii, H., Lions, P.-L.: User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. (N.S.) **27**, 1–67 (1992) Zbl 0755.35015 MR 1118699

[22] Da Lio, F., Forcadel, N., Monneau, R.: A numerical scheme for anisotropic mean curvature motion. Work in progress (2005)

[23] Da Lio, F., Kim, C. I., Slepčev, D.: Nonlocal front propagation problems in bounded domains with Neumann-type boundary conditions and applications. Asymptot. Anal. **37**, 257–292 (2004) Zbl 1063.35022 MR 2047742

[24] De Masi, A., Orlandi, E., Presutti, E., Triolo, L.: Motion by curvature by scaling nonlocal evolution equations. J. Statist. Phys. **73**, 543–570 (1993) Zbl 1102.82323 MR 1251654

[25] Evans, L. C.: Convergence of an algorithm for mean curvature motion. Indiana Univ. Math. J. **42**, 533–557 (1993) Zbl 0802.65098

[26] Evans, L. C., Spruck, J.: Motion of level sets by mean curvature. I. J. Differential Geom. **33**, 635–681 (1991) Zbl 0726.53029 MR 1100206

[27] Forcadel, N.: Dislocations dynamics with a mean curvature term: short time existence and uniqueness. Preprint (2005)

[28] Garroni, A., Müller, S.: A variational model for dislocations in the line tension limit. Arch. Ration. Mech. Anal. **181**, 535–578 (2006) Zbl pre05051266 MR 2231783

[29] Garroni, A., Müller, S.: Γ-limit of a phase-field model of dislocations. SIAM J. Math. Anal. **36**, 1943–1964 (2005) Zbl 1094.82008 MR 2178227

[30] Head, A.: Unstable dislocations in anisotropic crystals. Phys. Stat. Sol. **19**, 185–192 (1967)

[31] Hirth, J. P., Lothe, J.: Theory of Dislocations. 2nd ed., Krieger, Malabar, FL (1992)

[32] Ishii, H.: Hamilton–Jacobi equations with discontinuous Hamiltonians on arbitrary open sets. Bull. Fac. Sci. Engrg. Chuo Univ. **28**, 33–77 (1985) Zbl 0937.35505 MR 0845397

[33] Ishii, H.: A generalization of the Bence, Merriman and Osher algorithm for motion by mean curvature. In: Curvature Flows and Related Topics (Levico, 1994), GAKUTO Int. Ser. Math. Sci. Appl. 5, Gakkōtosho, Tokyo, 111–127 (1995) Zbl 0844.35043 MR 1365304

[34] Ishii, H., Pires, G. E., Souganidis, P. E.: Threshold dynamics type approximation schemes for propagating fronts. J. Math. Soc. Japan **51**, 267–308 (1999) Zbl 0935.53006 MR 1674750

[35] Katsoulakis, M. A., Souganidis, P. E.: Stochastic Ising models and anisotropic front propagation. J. Statist. Phys. **87**, 63–89 (1997) Zbl 0937.82034 MR 1453736

[36] Merriman, B., Bence, J., Osher, S.: Diffusion generated motion by mean curvature. In: J. Taylor (ed.), Computational Crystal Growers Workshop, Amer. Math. Soc., Providence, RI, 73–83 (1992)

[37] Rodney, D., Le Bouar, Y., Finel, A.: Phase field methods and dislocations. Acta Mater. **51**, 17–30 (2003)

[38] Schwartz, L.: Théorie des distributions. Hermann, Paris (1966)

[39] Slepčev, D.: Approximation schemes for propagation of fronts with nonlocal velocities and Neumann boundary conditions. Nonlinear Anal. **52**, 79–115 (2003) Zbl 1028.35068 MR 1938652

[40] Souganidis, P.: Work in progress