ERGODIC PROBLEMS AND PERIODIC HOMOGENIZATION FOR FULLY NONLINEAR EQUATIONS IN HALF-SPACE TYPE DOMAINS WITH NEUMANN BOUNDARY CONDITIONS

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Abstract. We study periodic homogenization problems for second-order pde in half-space type domains with Neumann boundary conditions. In particular, we are interested in "singular problems" for which it is necessary to determine both the homogenized equation and boundary conditions. We provide new results for fully nonlinear equations and boundary conditions. Our results extend previous work of Tanaka in the linear, periodic setting in half-spaces parallel to the axes of the periodicity, and of Arisawa in a rather restrictive nonlinear periodic framework. The key step in our analysis is the study of associated ergodic problems in domains with similar structure.

1. Introduction

We study issues related to the homogenization and ergodic problems for fully nonlinear, non-divergence form, elliptic and parabolic boundary value problems in half-space type domains with possibly nonlinear Neumann boundary conditions. In particular, we are interested in problems for which it is necessary to identify both the homogenized equation and boundary conditions. Our results represent a first step towards the resolution of the problem in general domains, which remains open even for the linear non-divergence form problem. The situation is, of course, different for divergence form equations, since, in that case the boundary conditions are encoded in the variational formulation.

In order to be more specific and to describe the problem in a clear way, we first discuss heuristically two model cases involving linear elliptic equations which lead to different behaviors and difficulties. To simplify the presentation, we assume that all the functions appearing in the examples below have the needed regularity properties and are periodic with respect to the fast variable x/ε, which we denote throughout the paper by y.

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The first problem, which we call "regular", is

$$(1.1)\qquad \begin{cases} -\mathrm{tr}(A(x,\varepsilon^{-1}x)D^2u^\varepsilon) - b(x,\varepsilon^{-1}x)\cdot Du^\varepsilon + u^\varepsilon = f(x,\varepsilon^{-1}x) & \text{in } \mathcal{O}\,,\\ Du^\varepsilon - \gamma = g & \text{on } \partial\mathcal{O}\,, \end{cases}$$

where $\mathcal{O}$ is a domain in $\mathbb{R}^N$, not necessarily of half-space type, Du^ε and D^2u^ε denote respectively the gradient and Hessian matrix of the solution u^ε, and γ and g depend only on x.

The typical "singular" problem is

$$(1.2)\qquad \begin{cases} -\mathrm{tr}(A(x,\varepsilon^{-1}x)D^2u^\varepsilon) - \varepsilon^{-1}b(x,\varepsilon^{-1}x)\cdot Du^\varepsilon + u^\varepsilon = f(x,\varepsilon^{-1}x) & \text{in } \mathcal{O}\,,\\ Du^\varepsilon \cdot \gamma = g & \text{on } \partial\mathcal{O}\,, \end{cases}$$

which has as a particular case the divergence form equation

$$-\mathrm{div}(A(x,\varepsilon^{-1}x)Du^\varepsilon) + u^\varepsilon = f(x,\varepsilon^{-1}x) \quad \text{in } \mathcal{O}\,.$$

If we use the formal expansion

$$u^\varepsilon(x) = \bar{u}(x) + \varepsilon v(x,\varepsilon^{-1}x) + \varepsilon^2 w(x,\varepsilon^{-1}x) + O(\varepsilon^3)\ ,$$

then, in the "regular" case, we find that the leading (the ε^{-1}) term yields

$$-\mathrm{tr}(A(x,y)D^2_{yy}v) = 0 \quad \text{in } \mathbb{R}^N\ ;$$

a standard Liouville-type property implies that v is independent of y. Moreover, when γ and g depend only on x and not on y, the function $\bar{u}$ is expected to satisfy the same boundary condition as u^ε, since the formal expansion yields

$$Du^\varepsilon\cdot\gamma = D\bar{u}\cdot\gamma + \varepsilon D_x v\cdot\gamma + \varepsilon D_y w\cdot\gamma + O(\varepsilon^2)\ .$$

This problem can be treated, at least for uniformly elliptic A, using the perturbed test-function method of Evans [12]. If, however, either γ or g depend on y, different arguments are needed.

The same expansion, in the "singular" problem leads ot

$$\mathrm{tr}(A(x,y)D^2_{yy}v) + b(x,y)\cdot(D\bar{u} + D_y v) = 0\ ,$$

where now the function v actually depends on y in general. This interferes with the boundary condition, since the expansion is now

$$Du^\varepsilon\cdot\gamma = D\bar{u}\cdot\gamma + D_y v\cdot\gamma + O(\varepsilon)\ ,$$

and clearly v plays a role in determining the boundary condition for the limiting equation.

This is the main problem in the singular case. The key issues are the identification of the equation and the boundary condition for $\bar{u}$. In the "regular" case, the issue is the study of the asymptotic limit for y-dependent γ and g.

There exists an extensive body of work dealing with the homogenization of the Dirichlet problem for fully nonlinear first- and second-order partial differential equations in periodic, quasi-periodic, almost periodic and, more recently, stationary ergodic media. Listing references is beyond the scope of this paper.

Little is, however, known for the Neumann problem except for divergence form equations with the usual (co-normal) boundary conditions, which are treated in the classical book of Bensoussan, Lions and Papanicolaou [9].

In [19], Tanaka considered the two model problems discussed earlier by purely probabilistic methods in half-spaces. In [1], Arisawa studied special cases of homogenization problems again in half-space type domains under rather restrictive assumptions. Our methods are inspired from [1] but yield more general results.

To study the asymptotic behavior of the u^ε's, we first consider the usual cell (ergodic) problem which is supposed to give the equation inside $\mathcal{O}$. In the "singular" case, it is formulated as follows:

For each $x, p \in \mathbb{R}^N$, find a unique constant $\bar\lambda(p,x)$ such that there exists a bounded solution v of the equation

$$(1.3)\qquad -\operatorname{tr}(A(x,y)D^2_{yy}v) - b(x,y)\cdot(p + D_y v) = \bar\lambda(p,x) \quad \text{in } \mathbb{R}^N .$$

The map $p \mapsto \bar\lambda(p,x)$ is clearly linear. Moreover, the only interesting case is when $\bar\lambda(p,x) = 0$, otherwise we get a trivial first-order equation for $\bar u$. We do not discuss here the type of conditions on A and b which yield $\bar\lambda(p,x) \equiv 0$. We just point out, however, that, for divergence form equations, this is true, as it can be easily seen by integrating the equation over a period.

For the boundary behavior, we restrict, for simplicity, to the half-space case. To fix the notation, we assume that $\mathcal{O} = \{x_N > 0\}$, in which case $\mathcal{O} = \varepsilon^{-1}\mathcal{O}$. The fact that both $\mathcal{O}$ and $\partial\mathcal{O}$ are invariant under the scaling is a property which is needed, at least for our approach, in the study of general domains.

The natural ergodic problem on the boundary is to find, for each $p, x \in \mathbb{R}^N$, a unique constant $\bar\mu(x,p)$ such that there exists a bounded solution w of the boundary value problem

$$(1.4)\qquad \begin{cases} -\operatorname{tr}(A(x,y)D^2_{yy}w) - b(x,y)\cdot(p + D_y w) = 0 & \text{in } \mathcal{O}\,, \\ (p + D_y w)\cdot\gamma(x,y) = g(x,y) + \bar\mu(x,p) & \text{on } \partial\mathcal{O}\ . \end{cases}$$

This type of problem was first studied in [1] both in bounded and in half-space type domains for Hamilton-Jacobi-Bellman type equations with oblique Neumann boundary conditions but under rather restrictive assumptions. We provide here existence and uniqueness results for general nonlinear equations and some boundary conditions.

The paper is organized as follows: In Section 2 we introduce the hypotheses and state and prove the existence result for the nonlinear boundary ergodic problem. Section 3 is devoted to the study of the uniqueness properties of $\bar\mu(x,p)$. In Section 4, we consider homogenization problems. We conclude with some remarks about the non-periodic case.

2. Existence results for ergodic boundary problems in half-spaces

We study the following boundary ergodic problem : Find a constant μ such that there exists a bounded, continuous solution of

$$\begin{cases} F(D^2u, Du, x) & = \lambda \ \text{ in } \mathcal{O}, \\ L(Du, x) & = \mu \ \text{ on } \partial\mathcal{O}. \end{cases} \tag{2.1}$$

Here $\mathcal{O} \subset \mathbb{R}^N$ is a smooth, periodic, half-space-type domain (see (O1) below for the definition), F and L are real valued, at least, continuous functions defined on $S^N \times \mathbb{R}^N \times \mathcal{O}$ and $\mathbb{R}^N \times \partial\mathcal{O}$ respectively, where S^N is the space of $N \times N$ symmetric matrices, and λ is a constant which is chosen in a suitable way below.

We formulate next the assumptions on F, L and $\mathcal{O}$. Note that, for the sake of the clarity of the exposition, in this note, we do not attempt to state the most general conditions.

As far as F is concerned, we assume that:

(F0) $\qquad F : S^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a continuous, $\mathbb{Z}^N$-periodic function in x,

(F1)
$$\begin{cases} F \text{ is locally Lipschitz continuous and there exists a constant } K > 0 \\ \text{such that, for all } x, y \in \mathbb{R}^N,\ p, q \in \mathbb{R}^N, \text{ and } M, N \in S^N, \\ |F(M,p,x) - F(N,q,y)| \leq K\{|x-y|(1+|p|+|q|+||M||+||N||) \\ \qquad\qquad +|p-q| + ||M-N||\}\ , \end{cases}$$

(F2)
$$\begin{cases} \text{there exists } \kappa > 0 \text{ such that, for all } x \in \mathbb{R}^N,\ p \in \mathbb{R}^N \setminus \{0\}, \\ \text{and } M, N \in S^N \text{ with } N \geq 0, \\ F(M+N,p,x) - F(M,p,x) \leq -\kappa(N\hat{p},\hat{p}) + o(1)||N||\ , \\ \text{with } o(1) \to 0 \text{ as } |p| \to \infty, \end{cases}$$

where, for $p \in \mathbb{R}^N \setminus \{0\}$, $\hat{p} = p/|p|$.

As far as the domain is concerned, we assume that

$(O1)$ $\qquad \mathcal{O}$ is a smooth, periodic half-space type domain,

i.e., that:
(i) $\partial\mathcal{O}$ is a C^2–boundary,
(ii) there exists $(N-1)$ linearly independent vectors $f_1, \cdots f_{N-1} \in \mathbb{Z}^N$, such that, for all $x \in \partial\mathcal{O}$, $\bar{x} \in \mathcal{O}$ and $z_1, \cdots z_{N-1} \in \mathbb{Z}$,

$$x + z_1 f_1 + \cdots + z_{N-1} f_{N-1} \in \partial\mathcal{O} \quad \text{ and } \quad \bar{x} + z_1 f_1 + \cdots + z_{N-1} f_{N-1} \in \mathcal{O}\ ,$$

and
(iii) there exists a unit vector $f_N \in \mathbb{R}^N$, orthogonal to $f_1, \cdots, f_{N-1}$, and $\bar{R} > 0$ such that, for all $x \in \mathcal{O}$ and $R \geq \bar{R}$,

$$x + R f_N \in \mathcal{O}\ .$$

Examples of domains satisfying $(O1)$ are

$$\mathcal{O} = \{x = (x_1, \cdots, x_N) : \ x_N > 0\}$$

or, more generally,

$$\mathcal{O} = \{x = (x_1, \cdots, x_N) : \ x_N > \psi(x_1, \cdots, x_{N-1})\},$$

where $\psi \in C^2(\mathbb{R}^{N-1})$ is $\mathbb{Z}^{N-1}$–periodic .

We denote by d the sign-distance function to $\partial\mathcal{O}$, normalized to be positive in $\mathcal{O}$ and negative in $\mathbb{R}^N \setminus \overline{\mathcal{O}}$, and we recall that, for all $x \in \partial\mathcal{O}$, the outward normal $n(x)$ to $\partial\mathcal{O}$ at x is given by

$$-n(x) = Dd(x) \ .$$

A key ingredient in the existence proof are the up to the boundary $C^{0,\alpha}$–regularity results for Neumann boundary value problems which were obtained by Barles and Da Lio in [5]. To use them, it is necessary to introduce the following set of assumptions on L.

$$\text{(L1)} \qquad \begin{cases} \text{There exists } \nu > 0 \text{ such that, for all } (x,p) \in \partial\mathcal{O} \times \mathbb{R}^N \text{ and } t > 0, \\ \qquad L(p + tn(x), x) - L(p,x) \geq \nu t \, . \end{cases}$$

$$\text{(L2)} \qquad \begin{cases} \text{There exists constant } \overline{K} > 0 \text{ such that, for all } x, y \in \partial\mathcal{O} \text{ and } p, q \in \mathbb{R}^N, \\ \qquad |L(p,x) - L(q,y)| \leq \overline{K}\,[(1 + |p| + |q|)|x - y| + |p - q|] \ . \end{cases}$$

$$\text{(L3)} \qquad \begin{cases} \text{There exists a } L_\infty \in C(\mathbb{R}^N \times \overline{\mathcal{O}}) \text{ such that,} \\ \text{as } t \to +\infty \text{ and locally uniformly in } (p,x), \\ \qquad t^{-1}L(tp,x) \to L_\infty(p,x) \ . \end{cases}$$

The regularity results of [5] depend on whether L is linear or nonlinear and require some additional assumptions on F. In the nonlinear case, it is necessary to strengthen (F2) and to ask for F to be a uniformly elliptic, i.e., to assume that

$$\text{(F3)} \quad \begin{cases} \text{there exists } \kappa > 0 \text{ such that, for all } x \in \overline{\mathcal{O}},\ p \in \mathbb{R}^N, \text{ and } M, N \in S^N \text{ with } N \geq 0, \\ F(M + N, p, x) - F(M, p, x) \leq -\kappa \operatorname{tr}(N) \ . \end{cases}$$

Moreover, we need to assume that

$$\text{(F4)} \qquad \begin{cases} \text{there exists } F_\infty \in C(S^N \times \mathbb{R}^N \times \overline{\mathcal{O}}) \text{ such that, locally uniformly in } (M,p,x), \\ \text{as } t \to \infty, \quad t^{-1}F(tM, tp, x) \to F_\infty(M,p,x) \end{cases}$$

For linear boundary condition, i.e., if

$$L(p,x) = p \cdot \gamma(x) - g(x) \ ,$$

with $\gamma \in C^{0,1}(\partial\mathcal{O}; \mathbb{R}^N)$ and $g \in C^{0,\beta}(\partial\mathcal{O})$ for some $\beta \in (0,1)$, instead of (F3) we assume that

(O2) $$\begin{cases} \text{there exists } A \in C^{0,1}(\overline{\mathcal{O}}; S(N)) \text{ and } c_0 > 0, \text{ such that} \\ A \geq c_0 Id \;, \quad \text{and} \quad A(x)\gamma(x) = n(x) \;\text{ on }\; \partial\mathcal{O} \;, \end{cases}$$

and, for all $R > 0$,

(F5) $$\begin{cases} \text{there exist } L_R, \lambda_R > 0 \text{ such that, for all } x \in \overline{\mathcal{O}},\ |u| \leq R,\ |p| > L_R \\ \text{and } M, \tilde{M} \in S^N \text{ with } \tilde{M} \geq 0, \\ F(M + \tilde{M}, p, x) - F(M, p, x) \leq -\lambda_R(\tilde{M}\widehat{A^{-1}(x)}p, \widehat{A^{-1}(x)}p) + o(1)\|\tilde{M}\| \;, \end{cases}$$

where $o(1) \to 0$ as $|p| \to \infty$.

We call (F5) "adapted ellipticity", since the ellipticity condition is adapted in the direction of the oblique derivative through A.

We remark that all the assumptions on F are satisfied by Hamilton-Jacobi-Bellman or Isaacs Equations under the usual conditions on the coefficients, while (L1), (L2), (L3) are natural conditions for (nonlinear) Neumann boundary conditions.

The constant λ in (2.1) is given by the next proposition. Since the proof follows the argument of Barles and Souganidis [8], we omit it.

Proposition 2.1. *Assume* (F0), (F1) *and* (F2). *There exists a unique constant* λ *such that there exists a periodic solution* $\bar{u} \in C^{0,1}(\mathbb{R}^N)$ *of*

(2.2) $$F(D^2\bar{u}, D\bar{u}, x) = \lambda \quad \textit{in } \mathbb{R}^N \;.$$

Our result is

Theorem 2.1. *Assume* (O1),(F0) *and either* (F1), (F3), (F4) *and* (L1), (L2), (L3) *if* L *is nonlinear or* (F1), (F2), (F5), (O2), $\gamma \in C^{0,1}(\partial\mathcal{O})$ *and* $g \in C^{0,\beta}(\partial\mathcal{O})$, *if* L *is linear. There exists* μ *such that* (2.1) *has a continuous bounded viscosity solution, which has the same periodicity property as the domain* $\mathcal{O}$.

Proof. We concentrate first on the nonlinear case. To simplify the exposition, we assume that $f_N = e_N$ and $\text{span}(f_1, \cdots, f_{N-1}) = \text{span}(e_1, \cdots, e_{N-1})$, where $(e_1, \cdots, e_N)$ is the standard orthonormal basis of $\mathbb{R}^N$. In addition, we assume that $0 \in \partial\mathcal{O}$.

For $0 < \varepsilon < \alpha < 1$, we introduce the approximate problem

(2.3) $$\begin{cases} F(D^2\tilde{u}, D\tilde{u}, x) + \varepsilon\tilde{u} = \lambda + \varepsilon\bar{u} \text{ in } \mathcal{O}, \\ L(D\tilde{u}, x) + \alpha\tilde{u} = 0 \text{ on } \partial\mathcal{O}, \end{cases}$$

where F and $\bar{u}$ are given by Proposition 2.1.

The existence and uniqueness of $\tilde{u}$ follows from classical arguments from the theory of viscosity solutions, and in particular, the Perron's method [15] and the comparison results [3]. Moreover, in view of its uniqueness, the solution has the same periodicity properties as the domain, i.e., for all $x \in \overline{\mathcal{O}}$ and $k_1 \cdots k_{N-1} \in \mathbb{Z}$,

$$\tilde{u}(x + k_1e_1 + \cdots + k_{N-1}e_{N-1}) = \tilde{u}(x) \,.$$

Finally standard comparison arguments (see [3]) yield that

$$\max_{\overline{\mathcal{O}}}(\tilde{u}-\bar{u}) \le \alpha^{-1} \sup_{x\in\mathbb{R}^N, |e|\le ||D\bar{u}||_\infty} |L(e,x)|\ ,$$

which, of course, implies that

$$\alpha\tilde{u} \text{ is uniformly bounded in } x, \alpha \text{ and } \varepsilon\ ;$$

notice that the ε-term in (2.3) is introduced just in order to prove this estimate.

For $R>0$ sufficiently large, we consider next the domain

$$\mathcal{O}_R = \{x\in\mathcal{O}:\ x_N < R\}$$

and solve the new problem

$$\begin{cases} F(D^2\tilde{u}_R, D\tilde{u}_R, x) + \varepsilon\tilde{u}_R = \lambda + \varepsilon\bar{u} & \text{in } \mathcal{O}_R, \\ D\tilde{u}_R\cdot n_R = 0 & \text{on } \{x_N = R\}, \\ L(D\tilde{u}_R, x) + \alpha\tilde{u}_R = 0 & \text{on } \partial\mathcal{O}, \end{cases} \tag{2.4}$$

where n_R is the external normal to $\mathcal{O}_R$ on $\{x_N = R\}$.

Again it is a standard fact that (2.4) admits a unique continuous viscosity solution $\tilde{u}_R$ with the same periodicity properties as the domain. Moreover, the maximum principle yields that, for all $x\in\overline{\mathcal{O}}$,

$$|\tilde{u}_R(x) - \tilde{u}_R(0)| \le ||\tilde{u}_R - \tilde{u}_R(0)||_{L^\infty(\partial\mathcal{O})}\ ,$$

and since $\partial\mathcal{O}$ is periodic, we may assume that all the $u_R - \tilde{u}_R(0)$'s attain their maximum and minimum values at points which remain in a compact subset of $\mathbb{R}^N$.

Letting $R\to\infty$ and using the standard fact that, as $R\to\infty$, $\tilde{u}_R\to\tilde{u}$ in $C(\mathcal{O})$, we find that, for all $x\in\overline{\mathcal{O}}$,

$$|\tilde{u}(x) - \tilde{u}(0)| \le ||\tilde{u} - \tilde{u}(0)||_{L^\infty(\partial\mathcal{O})}\ .$$

We claim next that

$$||\tilde{u} - \tilde{u}(0)||_{L^\infty(\overline{\mathcal{O}})} = ||\tilde{u} - \tilde{u}(0)||_{L^\infty(\partial\mathcal{O})}$$

remains bounded as first $\varepsilon\to 0$ and then $\alpha\to 0$. To this end, we argue by contradiction and introduce the function $\tilde{w}:\overline{\mathcal{O}}\to\mathbb{R}$ given by

$$\tilde{w} = \frac{\tilde{u}-\tilde{u}(0)}{||\tilde{u}-\tilde{u}(0)||_{L^\infty(\overline{\mathcal{O}})}}\ .$$

It is immediate that $|\tilde{w}|\le 1$, $\tilde{w}(0) = 0$ and w is a solution of a nonlinear Neumann type problem with nonlinearities satisfying assumptions (F1), (F4), (F5), (L1), (L2) and (L3) with uniform constants. The $C^{0,\alpha}$–estimates of [5] yield that the $\tilde{w}$'s are locally uniformly bounded in $C^{0,\alpha}$. Extracting a subsequence, we may assume that, as $\varepsilon\to 0$ and $\alpha\to 0$, the $\tilde{w}$'s converge to $\bar{w}$, which solves

$$\begin{cases} F_\infty(D^2\bar{w}, D\bar{w}, x) = 0 \text{ in } \mathcal{O}, \\ L_\infty(D\bar{w}, x) = 0 \text{ on } \partial\mathcal{O}, \end{cases} \tag{2.5}$$

and, moreover,

$$\bar{w}(0) = 0, \quad \text{and} \quad ||\bar{w}||_\infty = 1$$

with the sup–norm achieved at some point of $\partial\mathcal{O}$, a fact which contradicts the strong maximum principle of [4].

Since, by the previous step, $\tilde{u} - \tilde{u}(0)$ remains bounded, we can use again the $C^{0,\alpha}$-local estimates of [5] and pass to the limit, letting first $\varepsilon \to 0$ and then $\alpha \to 0$. It follows that, up to subsequences,

$$-\alpha\tilde{u}(0) \to \mu \ .$$

The proof of the linear case follows the same arguments using a different $C^{0,\alpha}$-local estimate in [5]. It is worth pointing out that the strong maximum principle of [4] is still valid under assumption (F5). □

3. Uniqueness results for boundary ergodic problems for Hamilton-Jacobi-Bellman type equations in half-space type domains

We present here two proofs for the uniqueness of the boundary ergodic cost μ. The first one assumes that F is convex and uses the stochastic control interpretation of the problem. It turns out, however, that it is also possible to present a proof which is based entirely on pde-type arguments and does not utilize convexity and stochastic control.

3.1. The stochastic control proof.

We consider the solution $((X_t)_{t\geq 0}, (k_t)_{t\geq 0})$ of the stochastic differential equations

$$\left\{ \begin{array}{ll} dX_t = b(X_t, \alpha_t)dt + \sqrt{2}\sigma(X_t, \alpha_t)dW_t - dk_t \ , & X_0 = x \in \overline{\mathcal{O}}, \\ k_t = \int_0^t 1_{\partial\mathcal{O}}(X_s)\gamma(X_s)d|k|_s \ , & X_t \in \overline{\mathcal{O}} \ , \quad t \geq 0 \ , \end{array} \right. \tag{3.1}$$

where $(X_t)_{t\geq 0}$ is a continuous process in $\mathbb{R}^N$ and $(k_t)_{t\geq 0}$ is a process with bounded variation. Here $(W_t)_{t\geq 0}$ is an N-dimensional Brownian motion, the control $(\alpha_t)_{t\geq 0}$ is a progressively measurable process with respect to the filtration associated to the Brownian motion with values in a compact metric space $\mathcal{A}$ and the drift b and the diffusion matrix σ satisfy the classical assumptions

$$\left\{ \begin{array}{l} b \in C(\overline{\mathcal{O}} \times \mathcal{A}; \mathbb{R}^N),\ \sigma \in C(\overline{\mathcal{O}} \times \mathcal{A}; S(N)) \text{ and, for each } \alpha \in \mathcal{A}, \\ b(\cdot, \alpha) \in C^{0,1}(\overline{\mathcal{O}}; \mathbb{R}^N),\ \sigma(\cdot, \alpha) \in C^{0,1}(\overline{\mathcal{O}} \times S(N)) \text{ and } a = \sigma\sigma^T \\ \text{is uniformly elliptic, with all the constants uniform in } \alpha. \end{array} \right. \tag{C1}$$

It is known that (C1) yields the existence of a unique solution $((X_t)_{t\geq 0}, (k_t)_{t\geq 0})$ of (3.1). (See Lions and Sznitman [18] for the existence and Barles and Lions [6] for the uniqueness.)

The nonlinearity F of the associated Bellman equation and the boundary condition for the ergodic boundary problem are given respectively by

$$F(M, p, x) = \sup_{\alpha \in \mathcal{A}} \Big\{ - \mathrm{tr}[a(x, \alpha)M] - b(x, \alpha) \cdot p - f(x, \alpha) \Big\} \tag{3.2}$$

and

$$L(Du, u, x) = Du \cdot \gamma - g + \mu = 0 \quad \text{on } \partial\mathcal{O}. \tag{3.3}$$

We assume that

$$\text{(C2)} \quad \begin{cases} \text{there exist } \beta \in (0,1) \text{ and } \nu > 0 \text{ such that, for all } \alpha \in \mathcal{A},\ f(\cdot,\alpha), \\ \gamma \text{ and } g \text{ satisfy : } f(\cdot,\alpha) \in BUC(\mathbb{R}^N),\ g \in C^{0,\beta}(\partial\mathcal{O}), \\ \ \gamma \in C^{0,1}(\partial\mathcal{O};\mathbb{R}^N) \text{ and } \gamma(x)\cdot n(x) \geqq 0 \text{ on } \partial\mathcal{O}. \end{cases}$$

Finally, it is necessary to make the additional "ergodic"-type assumption that

$$\text{(E1)} \quad \begin{cases} \text{there exists a bounded, Lipschitz continuous subsolution } \hat{w} \text{ of} \\ \qquad \hat{F}(D^2\hat{w}, x, -f_N + D\hat{w}, x) = 0 \quad \text{in } \mathbb{R}^N\ , \end{cases}$$

where f_N appears in $(O1)$ and

$$\hat{F}(X,p,x) = \sup_{\alpha\in\mathcal{A}} \Big\{ -\frac{1}{2}\mathrm{tr}\,[a(x,\alpha)X] - b(x,\alpha)\cdot p \Big\}.$$

We remark that, for linear equations, (E1) is not a real restriction. Indeed, in the Introduction, we explain that $\bar{\lambda}(p,x) = 0$ for all p is natural in our framework and the condition (E1) reads $\bar{\lambda}(-f_N, x) = 0$.

The result is

Theorem 3.1. *Assume* (C1), (C2) *and* (E1)*. There exists at most one constant μ which solves the boundary ergodic control problem* (2.1) *with λ and F and L given by Proposition 2.1 and* (3.2) *and* (3.3) *respectively.*

The key step in the proof of the uniqueness is the following lemma.

Lemma 3.1. *Assume* (C1), (C2) *and* (E1)*. There exists $x \in \mathcal{O}$ such that*

$$\inf_{(\alpha_t)_t} \mathbb{E}_x \left[\int_0^{+\infty} d|k|_s \right] = +\infty\ . \tag{3.4}$$

Proof. Following the proof of Theorem 2.1, we assume that $f_N = e_N$, and we work in

$$\mathcal{O}_R = \{x \in \mathcal{O}\ :\ x_N < R\}\ ,$$

Define

$$v(x) = -x_N + w(x)$$

where w is given by Proposition 2.1. Then

$$\hat{F}(D^2v, Dv, x) = 0 \quad \text{in } \mathbb{R}^N\ ,$$

Since v is Lipschitz continuous and w is bounded, the exists a $C > 0$ such that v

$$Dv\cdot\gamma \le C \quad \text{on } \partial\mathcal{O} \quad \text{and} \quad v \le -R + C \quad \text{on } x_N = R.$$

Let τ_x be the first exit time through the boundary $\{x_N = R\}$ of the process $(X_t)_{t\geqq 0}$ starting at x. Using the dynamic programming principle for the stopping times τ_x we find that, for all $t > 0$,

$$v(x) \leq C \inf_{(\alpha_t)_t} I\!E_x \left[\int_0^{t\wedge\tau_x} d|k|_s \right] - (R - C)\chi(x,t) \ ,$$

where

$$\chi(x,t) = \inf_{(\alpha_t)_{t\geqq 0}} I\!E_x \left[1\!\!1_{\{\tau_x \leq t\}} \right] \ .$$

Since σ is nondegenerate, it follows that, for each fixed R, as $t \to \infty$ and locally uniformly in x,

$$\chi(\cdot, t) \to 1 \ ,$$

which, in turn, implies that, locally uniformly in x,

$$\liminf_{t\to\infty} \inf_{(\alpha_t)_{t\geqq 0}} I\!E_x \left[\int_0^{t\wedge\tau_x} d|k|_s \right] \geq C^{-1}(R - C + v(x)) \ .$$

Since R is arbitrary, the result now follows for all $x \in \mathcal{O}$. □

We return now to the proof of Theorem 3 which is based on the intuitive idea that, to have a unique μ, the boundary needs to be seen in a "sufficient way". This last fact which is quantified in a precise way by (3.4). A similar fact is also a key point for the uniqueness of μ in bounded domain.

Proof of Theorem 3.1. Assume that there exist two constants μ_1 and μ_2 such that $\mu_1 < \mu_2$ with corresponding solutions u_1 and u_2. The dynamic programming principle or the uniqueness for the associated time dependent problem yield, for all $t > 0$ and $i = 1, 2$,

$$u_1(x) = \inf_{(\alpha_t)_{t\geqq 0}} I\!E_x \left[\int_0^t [f(X_s, \alpha_s) + \lambda] ds + \int_0^t [g(X_s) + \mu_1] d|k|_s + u_1(X_t) \right] \ .$$

Choosing an ε-optimal control for u_2 and subtracting the inequalities for u_1 and u_2 we find

$$u_1(x) - u_2(x) \leq (\mu_1 - \mu_1) I\!E_x \left[\int_0^t d|k|_s \right] + u_1(X_t) - u_2(X_t) + \varepsilon \ ,$$

which yields the estimate

$$(\mu_2 - \mu_1) I\!E_x \left[\int_0^t d|k|_s \right] \leq 2(||u||_\infty + ||\tilde{u}||_\infty) + \varepsilon \ .$$

If $\mu_2 > \mu_1$ this last inequality contradicts Lemma 3.1. □

It turns out that, for μ to be unique, it is necessary to assume (E1). Indeed, consider the simple Neumann problem

$$-\varphi'' - \varphi' = 0 \quad \text{in } (0, +\infty) \quad \text{and} \quad -\varphi'(0) = \mu \ .$$

Since $\phi(x) = \mu \exp(-x)$ solves this problem for all $\mu \geqq 0$, it turns out that since $\varphi(x) = \mu \exp(-x)$, any $\mu \geqq 0$ is a solution of the associated ergodic problem.

On the other hand, a direct integration yields that there does not exist a bounded subsolution of

$$-\psi'' - \psi' + 1 = 0 \quad \text{in } (0, +\infty)\ .$$

Finally, we remark that in [1] the uniqueness is proved in the uniformly elliptic case under the additional assumption that, for all x and α,

$$b_N(x, \alpha) \le 0\ ,$$

a fact which allows to choose $w \equiv 0$ in (E1).

3.2. A pde proof and a more general result.

We state and prove using pde techniques a stronger uniqueness result than the one asserted in Theorem 3.1. To this end, we need to assume that there exist

$$\text{(F6)} \qquad \begin{cases} \text{a homogeneous of degree one, uniformly elliptic, } \hat{F} \in C(S^N \times \mathbb{R}^N \times \mathbb{R}^N) \\ \text{such that, for all } x \in \mathbb{R}^N,\ p, q \in \mathbb{R}^N \text{ and } M_1, M_2 \in S^N, \\ \qquad F(M_1, p, x) - F(M_2, q, x) \le \hat{F}(M_1 - M_2, p - q, x)\ , \end{cases}$$

and

$$\text{(E2)} \qquad \begin{cases} \text{a Lipschitz continuous subsolution } v \text{ of} \\ \qquad \qquad \hat{F}(D^2 v, Dv, x) = 0 \quad \text{in } \mathcal{O}\ , \\ \text{such that, uniformly with respect to } x',\ v(x', x_N) \to -\infty \text{ as } x_N \to +\infty, \end{cases}$$

where, for $x \in \mathbb{R}^N$, we write $x = (x', x_N)$ with $x' = (x_1, \ldots, x_N)$.

It is immediate that the convex function F given by (3.2) satisfies (F6), while (E2) is a weaker version of (E1). We also remark that, although all these assumptions appear a bit artificial, [1] provides some counterexamples.

The result is

Theorem 3.2. *Assume* (F6) *and* (E2). *If* $\hat{F}$ *satisfies* (F1), *there exists at most one constant* μ *solving the boundary ergodic control problem.*

Proof of Theorem 3.2. 1. Let μ_1, μ_2, u_1, u_2 and $\mathcal{O}_R$ be as in the proof of Theorem 3.1 and assume that $\mu_2 > \mu_1$. Then $w = \tilde{u} - u$ is a bounded supersolution of

$$\begin{cases} \hat{F}(D^2 w, Dw, x) \ge 0 \quad \text{in } \mathcal{O}\ , \\ Dw \cdot \gamma \ge \mu_2 - \mu_1 \quad \text{on } \partial\mathcal{O}. \end{cases}$$

For some $c > 0$, assume that there exists $\psi^R \in C(\mathcal{O} \times [0, \infty))$ such that

$$\text{(3.5)} \qquad \psi^R \leqq c \text{ on } \{x_N = \mathbb{R}\} \text{ and } \psi_R(\cdot, 0) \leqq w \text{ on } \partial\mathcal{O}\ ,$$

and

$$\text{(3.6)} \qquad \begin{cases} \psi^R_t + \hat{F}(D^2\psi^R, D\psi^R, x) \leqq 0 \text{ on } \mathcal{O}_R\ , \\ \dfrac{\partial \psi^R}{\partial \gamma} \leqq \tilde{\mu} - \mu \text{ on } \partial\mathcal{O}_R\ . \end{cases}$$

The comparison principle of viscosity solutions and the fact that w is bounded yield the existence of another constant $C > 0$ such that

$$\psi^R \leqq w + C \quad \text{in} \quad \mathcal{O} \times (0, \infty) \ .$$

This last inequality leads to a contradiction, if it is possible to choose the ψ^R's so that, in addition to (3.5) and (3.6), they also satisfy,

$$\lim_{R\to\infty} \lim_{t\to\infty} \psi^R(\cdot, t) = +\infty \quad \text{locally uniformly in } \mathcal{O}. \tag{3.7}$$

For each $R > 0$, we define ψ^R by

$$\psi^R = \delta v + \delta m_R \chi^R \ ,$$

where $\delta > 0$ is chosen below sufficiently small enough,

$$m_R = \inf_{x_N = R} (-v(x', x_N)) \to \infty \quad \text{as} \quad R \to \infty \ ,$$

and $\chi^R \in C(\mathcal{O} \times (0, \infty))$ is such

$$0 \leqq \chi^R \leqq 1 \text{ and } \lim_{t\to\infty} \chi^R(\cdot, t) = 1 \text{ locally uniformly in } \mathcal{O} \ . \tag{3.8}$$

Such ψ^R clearly satisfies (3.7) and the first inequality in (11). To prove that ψ^R is a subsolution we compute

$$\begin{aligned} \psi^R_t + \hat{F}(D^2\psi^R, D\psi^R, x) &= \delta m_R \chi^R_t + \hat{F}(\delta D^2 v + \delta m_R D^2\chi^R, \delta Dv + \delta m_R D\chi^R, x) \\ &\leq \hat{F}(\delta D^2 v, \delta Dv, x) + \delta m_R \left(\chi^R_t - \mathcal{M}_-(D^2\chi^R) - C|D\chi^R|\right) \ , \end{aligned}$$

where $\mathcal{M}_-$ is the minimal Pucci's operator associated with $\hat{F}$.

This computation shows that it is enough to choose χ^R to be the unique solution of

$$\begin{cases} \chi^R_t - \mathcal{M}_-(D^2\chi^R) - C|D\chi_R| = 0 \quad \text{in} \quad \mathcal{O}_R \times (0, +\infty) \ , \\ D\chi^R \cdot \gamma = 0 \quad \text{on} \quad \partial\mathcal{O} \times (0, +\infty) \ , \\ \chi^R = 1 \quad \text{on} \quad \{x_N = R\} \times (0, +\infty), \\ \chi^R = 0 \quad \text{on} \quad \overline{\mathcal{O}} \times \{0\} \ , \end{cases}$$

which satisfies (3.8).

With this choice of χ^R, ψ^R satisfies the subsolution inequality. The fact that v is bounded yields, if δ is chosen sufficiently small, that the boundary condition in (3.5) and the second inequality in (3.4) are also satisfied.

□

We conclude this section emphasizing that an import issue is to understand the way on which μ depends on F and L. This question was addressed in [4] in bounded domains. In the context of this paper, the dependence on L is an easy consequence of the existence proof, while the dependence on F is not so obvious.

4. The homogenization problem in half-space type domains with oscillating boundaries

To make the main ideas of our approach clear, we begin with the linear problem and we consider domains of the form

$$\mathcal{O}_\varepsilon = \left\{x \in \mathbb{R}^N \,:\, x_N > \varepsilon\psi(\varepsilon^{-1}x)\right\} = \varepsilon^{-1}\mathcal{O} = \left\{y \in \mathbb{R}^N \,:\, y_N > \psi(y')\right\}\ ,$$

where, as before, for $x \in \mathbb{R}^N$, $x' = (x_1, \cdots, x_{N-1})$ and ψ is a smooth (at least $C^{2,1}$) $\mathbb{Z}^{N-1}$-periodic function.

The model homogenization problem we are interested in is

$$\text{(4.1)}\quad \begin{cases} -\mathrm{tr}(A(x,\varepsilon^{-1}x)D^2u^\varepsilon) - \varepsilon^{-1}b(x,\varepsilon^{-1}x)\cdot Du^\varepsilon + u^\varepsilon &= f(x,\varepsilon^{-1}x) \quad \text{in } \mathcal{O}_\varepsilon\,,\\ Du^\varepsilon\cdot\gamma &= g(x,\varepsilon^{-1}x) \quad \text{on } \partial\mathcal{O}_\varepsilon\,. \end{cases}$$

As far as the coefficients are concerned, we assume that

$$\text{(H1)}\quad \begin{cases} A,\ b,\ f,\ \gamma \text{ and } g \text{ are bounded, Lipschitz continuous, } \mathbb{Z}^N\text{-periodic with}\\ \text{respect to the fast variable and there exists } \nu > 0 \text{ such that } A \geq \nu Id. \end{cases}$$

In order to formulate the result, we introduce first the associated cell problem which is used for the equation inside the domain. We assume that

$$\text{(H2)}\quad \begin{cases} \text{for all } x, p \in \mathbb{R}^N\text{, there exists a bounded, } \mathbb{Z}^N\text{-periodic in } y\text{, solution}\\ v(p,x,y) \text{ of}\\ \qquad -\mathrm{tr}(A(x,y)D^2_{yy}v) + b(x,y)\cdot(p + D_yv) = 0 \quad \text{in } \mathbb{R}^N\ ,\\ \text{which depends smoothly on } (p,x,y). \end{cases}$$

The cell problem for the second corrector is

$$\text{(H3)}\quad \begin{cases} \text{for all } x, p \in \mathbb{R}^N \text{ and } M \in S^N\text{, there exists a unique constant } \bar{F}(M,p,x)\\ \text{such that the equation}\\ \qquad -\mathrm{tr}(A(x,y)\left(M + 2(D^2_{xy}v + MD^2_{yp}v) + D^2_{yy}w\right) - b(x,y)\cdot\\ \qquad (D^2_x v + MD^2_p v + D_yw) = f(x,y) - \bar{F}(M,p,x) \quad \text{in } \mathbb{R}^N\\ \text{has a bounded solution } w\text{, which depends smoothly on } (M,p,x,y). \end{cases}$$

We remark that in this linear context the assumed uniqueness yields that $\bar{F}$ is an affine function of p and M.

As it was mentioned in the Introduction, (H2) and (H3) contain two assumptions. One is the fact that the cell problems have, for all x, p and M, bounded solutions. The second is, of course, the smoothness of v and w with respect to all the variables. This is a technical assumption, which is difficult to avoid and, most probably, is true for smoother coefficients.

We turn next to the boundary condition. The results of Sections 2 and 3 yield, for each $p, x \in \mathbb{R}^N$, the existence of a unique constant $\bar{\mu}(p,x)$ such that the boundary ergodic problem

$$(4.2)\qquad \begin{cases} -\mathrm{tr}(A(x,y)D^2_{yy}z) - b(x,y)\cdot(p + D_y z) &= 0 \quad \text{in } \mathcal{O}\,, \\ (Dz+p)\cdot\gamma(x,y) &= g(x,y) - \bar{\mu}(p,x) \quad \text{on } \partial\mathcal{O}\,, \end{cases}$$

has a bounded, periodic in y', solution z.

Now we are in position to state our result.

Theorem 4.1. *Assume* (H1), (H2), (H3) *and that* $\bar{\mu}$ *satisfies* (L1) *and* (L2). *The family* $(u^\varepsilon)_{\varepsilon>0}$ *converges locally uniformly, as* $\varepsilon \to 0$, *to the unique solution* $\bar{u}$ *of*

$$(4.3)\qquad \begin{cases} \bar{F}(D^2\bar{u}, D\bar{u}, x) + \bar{u} = 0 \quad \text{in } \{x_N > 0\}\,, \\ \bar{\mu}(D\bar{u}, x) = 0 \quad \text{on } \{x_N = 0\}\,. \end{cases}$$

The uniqueness of $\bar{\mu}(p,x)$ yields that $\bar{\mu}$ is an affine function of p and, hence, of the form

$$\bar{\mu}(x,p) = \bar{\gamma}(x)\cdot p - \bar{g}(x)\,,$$

for some $\bar{\gamma} \in C(\partial\mathcal{O}, \mathbb{R}^N)$ and $\bar{g} \in C(\partial\mathcal{O})$.

The stability and the uniform $C^{0,\alpha}$-estimates also yield that the boundary ergodic problem has a solution. To prove, however, that $\bar{\gamma}$ and $\bar{g}$ are Lipschitz continuous is more difficult, since it is necessary to take into account the dependence an x of A, b and f, i.e., the equation inside the domain. This is why we assume that $\bar{\mu}(p,x)$ satisfies (L1) and (L2) in order to have a uniqueness result for (4.3).

Proof. Since the proofs that the u_ε's converge to viscosity subsolutions and supersolutions of (4.3) are similar, here we present the argument only for the subsolution case.

To this end, we introduce the half-relaxed limit

$$\bar{u}(x) = \limsup{}^* u^\varepsilon(x) = \limsup_{x'\to x,\,\varepsilon\to 0} u^\varepsilon(x')$$

and assume that $\bar{x}$ is a strict maximum of $\bar{u} - \phi$, where ϕ is a smooth test-function.

If $\bar{x} \in \{x_N > 0\}$, the conclusion follows easily using the perturbed test-function. Indeed consider maximum points x_ε of

$$u^\varepsilon(x) - \big(\phi(x) + \varepsilon v(D\phi(x), x, \varepsilon^{-1}x) + \varepsilon^2 w(D^2\phi(\bar{x}), D\phi(\bar{x}), \bar{x}, \varepsilon^{-1}x)\big)\ .$$

The perturbed test-function is smooth and the proof follows as in the formal asymptotic expansion. It is worth pointing out that technically the smoothness of v is required because of its dependence on $D\phi(x)$ and x is important in the ε^{-1}-term. On the contrary, for w, the terms are less sensitive and the dependence on $\bar{x}, D\phi(\bar{x})$ and $D^2\phi(\bar{x})$ is enough.

This is exactly the difficulty we face, when $\bar{x} \in \{x_N = 0\}$. Assuming that the solution z of (4.2) is smooth with respect to all variables is too restrictive and, most probably, uncheckable. We avoid this in the following way. First we write

$$z(y; D\phi(x), x) = v(x; D\phi(x), y) + \tilde{z}(y, x)\ .$$

It follows that $\tilde z$ solves the problem

$$
(4.4) \qquad \begin{cases} -\mathrm{tr}(A(x,y)D^2_{yy}\tilde z) - b(x,y)\cdot D_y \tilde z = 0 & \text{in } \mathcal{O}\,, \\ D\tilde z\cdot\gamma = \tilde g(x,y) - \bar\mu(D\phi(x),x) & \text{on } \partial\mathcal{O}\;, \end{cases}
$$

where

$$
\tilde g(x,y) = g(x,y) - D\phi(x)\cdot\gamma(x,y) - D_y v(x,D\phi(x),y)\cdot\gamma(x,y)\;.
$$

Since $\tilde z$ is not a smooth function of x and, perhaps, y, we use Theorem 2.1 to solve the boundary value problem

$$
(4.5) \qquad \begin{cases} \max\limits_{|x'-x|\le\delta} \big(-\mathrm{tr}(A(x',y)D^2_{yy}\tilde z^\delta) - b(x',y)\cdot D_y\tilde z^\delta\big) &=& 0 \quad \text{in } \mathcal{O}\,, \\ \max\limits_{|x'-x|\le\delta} \big(D\tilde z^\delta\cdot\gamma - \tilde g(x',y)\big) &=& -\bar\mu^\delta \quad \text{on } \partial\mathcal{O}\;. \end{cases}
$$

Using the available estimates on $\tilde z^\delta$ and $\bar\mu^\delta$, it is then easy to prove that, as $\delta\to 0$, $\tilde z^\delta$ converges locally uniformly to a solution of (4.4) and, more importantly, using the uniqueness of $\bar\mu(D\phi(x),x)$, $\bar\mu^\delta$ that converges to $\bar\mu(D\phi(x),x)$.

Next we apply the perturbed test-function method in the following way. We look at maximum points x_ε of

$$
u^\varepsilon(x) - \phi(x) - \varepsilon v(x,D\phi(x),\varepsilon^{-1}x) - \tilde z^\delta(\varepsilon^{-1}x) - \varepsilon^2 w(\bar x, D\phi(\bar x), D^2\phi(\bar x),\varepsilon^{-1}x)\;.
$$

To conclude, it suffices to remark that, since $x_\varepsilon\to\bar x$ as $\varepsilon\to 0$, we have $|x_\varepsilon-\bar x|\le\delta$ for ε small enough and (4.5) provides the right inequalities to conclude, since the maximum is bigger than the value at $x=x_\varepsilon$. The fact that $\tilde z^\delta$ may be a non-smooth viscosity solution of (4.5) creates no real difficulty in the argument. Indeed, it suffices to double variables using the test-function associated to (4.5). □

We remark that v carries all the information to build the second corrector w. On the other hand, $\tilde z$ appears to be necessary to treat the boundary condition. For this reason, our decomposition of the first corrector appears natural.

We turn next to the nonlinear problem. We use the same notations and, in particular, the same domain as in the previous section, and, to simplify the presentation, we consider a model problem of the form

$$
(4.6) \qquad \begin{cases} F(\varepsilon D^2u^\varepsilon, Du^\varepsilon, x, \varepsilon^{-1}x) + \varepsilon u^\varepsilon &=& \varepsilon f(x,\varepsilon^{-1}x) \quad \text{in } \mathcal{O}_\varepsilon\,, \\ L(Du^\varepsilon, x,\varepsilon^{-1}x) &=& 0 \quad \text{on } \partial\mathcal{O}_\varepsilon\;, \end{cases}
$$

and assume that F and L satisfy (F0), (F1), (F2), (F3), (L1) and (L3). Note that, compared to the linear setting here, we have multiplied the equation by ε here to simplify the presentation.

Our assumption on the first cell problem is that

$$\text{(H4)}\quad \begin{cases} \text{for all } x,p\in\mathbb{R}^N\text{, there exists a bounded, periodic in } y \text{ solution } \bar v(p,x,y) \\ \text{of} \\ \qquad F(D^2_{yy}\bar v, p+D_y\bar v, x,y)=0 \quad \text{in } \mathbb{R}^N\ , \\ \text{which is a smooth function with respect to all its variable.}\end{cases}$$

Next we assume that F is smooth in M and p. Using the regularity of $\bar v$ in (H4) we set

$$A(p,x,y)=F_M(D^2_{yy}\bar v, p+D_y\bar v,x,y) \ \text{ and } \ b(p,x,y)=F_p(D^2_{yy}\bar v,p+D_y\bar v,x,y)\ .$$

The cell problem for the second corrector is:

$$\text{(H5)}\quad \begin{cases} \text{For all } x,\ p\ \in\ \mathbb{R}^N \text{ and } M\ \in\ S^N\text{, there exists a unique constant} \\ \bar F(M,p,x) \text{ such that the equation} \\ \qquad -\mathrm{tr}(A(x,p,y)\left(M+2(D^2_{xy}\bar v+MD^2_{yp}\bar v)+D^2_{yy}\bar w\right) \\ -b(x,p,y)\cdot(D^2_x\bar v+MD^2_p\bar v+D_y\bar w)=f(x,y)-\bar F(M,p,x) \quad \text{in } \mathbb{R}^N \\ \text{has a bounded, periodic in } y \text{ solution } \bar w\text{, which depends smoothly on} \\ M,\ p,\ x \text{ and } y.\end{cases}$$

Finally, to specify the boundary condition, we assume that (E1) and (F6) are also satisfied, and we use the results of Section 2 and 3, which yield the existence of a unique constant $\bar\mu(p,x)$ such that the boundary ergodic problem

$$\text{(4.7)}\quad \begin{cases} F(D^2_{yy}\tilde v, p+D_y\tilde v,x,y) &=& 0 \quad \text{in } \mathcal{O}\,, \\ L(p+D_y\tilde v,x,y) &=& -\bar\mu(p,x) \quad \text{on } \partial\mathcal{O}\ .\end{cases}$$

has a bounded, periodic in y', solution $\tilde v$.

The result is

Theorem 4.2. *Assume* (F0), (F1), (F2), (F3), (F6), (E1), (L1), (L3), (H4), (H5) *and that F is a smooth function of M and p. If $\bar\mu$ satisfies* (L1) *and* (L2), *the sequence* $(u^\varepsilon)_{\varepsilon>0}$ *converges, locally uniformly, as $\varepsilon\to 0$, to the unique solution $\bar u$ of*

$$\begin{cases} \bar F(D^2\bar u, D\bar u, x)+\bar u=0 \quad & in \ \ \{x_N>0\}\,, \\ \bar\mu(D\bar u,x)=0 \quad & on \ \ \{x_N=0\}\ .\end{cases}$$

We omit the proof since it follows very closely the proof of Theorem 4.1. We remark that the nonlinear boundary condition does not introduce any additional difficulties. The most restrictive assumption concerns the smoothness of F in M and p, which is not easily satisfied by Hamilton-Jacobi-Bellman equations. At least Theorem 4.2 applies to cases where F is of the form $F=F_1+H$, where F_1 a linear second order operator and H is nonlinear in p and smooth in p and x.

5. Open Problems and Remarks

The periodicity of the equation and the boundary is a key assumption in our approach. The existence of the boundary cost and of the associated solution of the Neumann problem uses this property in a very strong way.

It would be interesting to know whether approximate correctors exists when the equation and the boundary condition are periodic but the half-space domain does not have the right periodicity property, i.e., it is typically a hyperplane with an irrational slope. This seems to lead to a framework rather close to the one considered by Ishii [14], but at the moment we are unable to obtain results in this direction. Of course the ultimate goal is to consider the homogenization problem in random environments.

The following remark illustrates one of the difficulties to obtain such type of result. For $q \in \mathbb{R}^N \setminus \{0\}$, consider the half-space

$$H_q = \{x \in \mathbb{R}^N \ : q \cdot x > 0\}$$

and assume that the problem

$$\begin{cases} -\Delta u = 0 \quad \text{in } H_q \ , \\ Du \cdot n = g(x) + \mu \quad \text{on } \partial H_q \ , \end{cases}$$

has a bounded solution u for $\hat{a}$ continuous, $\mathbb{Z}^N$-periodic. Then it is easy to see that we must have

$$\mu(q) = - \lim_{R \to +\infty} |B(0,R) \cap H_q|^{-1} \int_{B(0,R) \cap H_q} g(x) dx \, .$$

Now, in $\mathbb{R}^2$, choose $g(x_1, x_2) = \tilde{g}(x_2)$ where $\tilde{g}$ is a $\mathbb{Z}$-periodic function. It follows $\mu(e_2) = \tilde{g}(0)$, while, if $q = e_2 + \alpha e_1$ with α is a small parameter, $\mu(e_2) = \int_0^1 \tilde{g}(s) ds$. This shows that a priori $\mu(q)$ is not a continuous function of q. Therefore an argument by approximation of a non-periodic situation by a periodic one cannot be used to prove the existence of μ and of the approximate corrector in the non-periodic framework.

References

[1] Arisawa, M.: *Long time averaged reflection force and homogenization of oscillating Neumann boundary conditions,* Ann. Inst. H. Poincaré Anal. Linéaire 20 (2003) 293–332.

[2] Barles, G.: Solutions de viscosité des équations de Hamilton-Jacobi. Collection "Mathématiques et Applications" de la SMAI, n°17, Springer-Verlag (1994).

[3] Barles, G.: *Nonlinear Neumann boundary conditions for quasilinear, degenerate elliptic equations and applications*, Journal of Diff. Eqs. 154 (1999) 191–224.

[4] Barles, G. and Da Lio, F.: *On the boundary ergodic problem for fully nonlinear equations in bounded domains with general nonlinear Neumann boundary conditions*, Ann. IHP, Analyse non linéaire, to appear.

[5] Barles, G. and Da Lio, F.: *Local $C^{0,\alpha}$ estimates for viscosity solutions of Neumann-type boundary value problems*, preprint.

[6] Barles, G. and Lions, P.-L.: *Remarques sur les problèmes de reflexion obliques*, C. R. Acad. Sci. Paris, t. 320 Serie I (1995) 69–74.

[7] Barles, G. and Souganidis, P.E.: *On the large time behaviour of solutions of Hamilton-Jacobi equations,* SIAM J. Math. Anal. 31 (2000) 925–939.

[8] Barles, G. and Souganidis, P.E.: *Space-time periodic solutions and long-time behavior of solutions to quasi-linear parabolic equations,* SIAM J. Math. Anal. 32 (2001) 1311–1323.

[9] Bensoussan, A., Lions, J.-L., and Papanicolaou, G.: `Cited on page 3.`

[10] Bensoussan, A.: PERTURBATION METHODS IN OPTIMAL CONTROL, Translated from the French by C. Tomson. Wiley/Gauthier-Villars Series in Modern Applied Mathematics. John Wiley & Sons, Ltd., Chichester; Gauthier-Villars, Montrouge, 1988.

[11] Crandall M.G., Ishii, H. and Lions, P.-L.: *User's guide to viscosity solutions of second order Partial differential equations*, Bull. Amer. Soc. 27 (1992) 1–67.

[12] Evans, L.C.: *The perturbed test function method for viscosity solutions of nonlinear PDE,* Proc. Roy. Soc. Edinburgh Sect. A 111 (1989) 359–375.

[13] Evans, L.C.: *Periodic homogenisation of certain fully nonlinear partial differential equations*, Proc. Roy. Soc. Edinburgh Sect. A 120 (1992) 245–265.

[14] Ishii, H.: *Almost periodic homogenization of Hamilton-Jacobi equations,* International Conference on Differential Equations, Vol. 1, 2 (Berlin, 1999), 600-605, World Sci. Publishing, River Edge, NJ, 2000.

[15] Ishii, H.: *Perron's method for Hamilton-Jacobi Equations,* Duke Math. J. 55 (1987) 369–384.

[16] Lions P.-L and Souganidis, P.E.: viscous

[17] Lions P.-L and Souganidis, P.E.: periodic

[18] Lions P.-L and Sznitman, A.S.: *Stochastic differential equations with reflecting boundary conditions,* Comm. Pure and Applied Math. 37 (1984) 511–537.

[19] Tanaka, H.: Homogenization of diffusion processes with boundary conditions. Stochastic analysis and applications, 411–437, Adv. Probab. Related Topics 7 Dekker, New York, 1984.