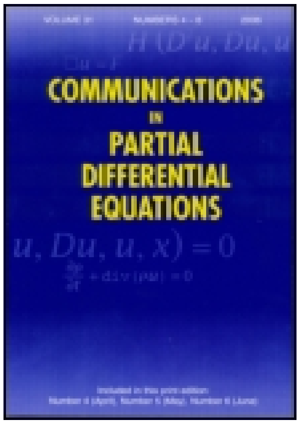

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Partial Regularity for Stationary Solutions to Liouville-Type Equation in Dimension 3

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In dimension $n = 3$, we prove that the singular set of any stationary solution to the Liouville equation $-\Delta u = e^u$, which belongs to $W^{1,2}$, has Hausdorff dimension at most 1.

Keywords Quasilinear equations; Regularity of generalized solutions.

Mathematics Subject Classification 35J60; 35D10.

1. Introduction

The regularity theory for nonlinear elliptic equations has a long history. It is beyond the scope of the present work to describe even part of it and we thus refer the reader to Chapter 14 in [16] for a presentation of this theory and further references.

Typical examples of nonlinear elliptic problems are the semilinear elliptic problems of the form

$$Lu = f(x, u, \ldots, \nabla^{m-1}u), \tag{1}$$

where the function u is defined on some open subset of $\mathbb{R}^n$, L is a linear elliptic operator of order m and where the nonlinear operator f involves derivatives of u up to order $m-1$.

Once we fix the dimension n of the underlying space and the function space $\mathcal{V}$, to which the solution u is assumed to belong, equations like (1) can be classified in three categories: the sub-critical, critical and super-critical equations.

These three categories of equations (which depend on the choice of n and $\mathcal{V}$) are characterized as follows. Starting from the fact that u belongs to $\mathcal{V}$, one can estimate the nonlinear part $f(x, u, \ldots, \nabla^{m-1}u)$, if in addition u is a solution to (1), this implies that Lu belongs to some function space $\mathcal{W}$ (which is usually larger than the space $\mathcal{V}$ itself). Sub-critical (respectively super-critical) equations, are the one

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for which the information $Lu \in \mathcal{W}$ implies, through elliptic regularity theory, that u belongs to a function space which is strictly smaller (respectively strictly larger) and which has different homogeneities than the original $\mathcal{V}$. In turn, critical equations are the one which are neither sub-critical nor supercritical in the above sense.

It is well known that solutions to subcritical equations of the form (1) for a smooth f and smooth L are in fact smooth. This is a consequence of the standard bootstrap argument. In contrast with the subcritical situation, solutions to a given critical equation either can all be proven to be smooth or can have non trivial singular sets (that is, non removable singularities). These results then depend on the nature of the nonlinearity f.

For example, in dimension $n = 2$, when $\mathcal{V} = W^{1,2}(B_1, \mathbb{R})$, the equation

$$-\Delta u = |\nabla u|^2 \tag{2}$$

is critical. Indeed, plugging the information $u \in W^{1,2}(B_1, \mathbb{R})$ into $f(\nabla u) = |\nabla u|^2$, one obtains that $\Delta u \in L^1$ which itself implies that ∇u is in $L^{2,\infty}$, the weak-L^2 space, which has the same homogeneity as L^2. Thus, in a some sense, we are back to the initial situation and this shows that the equation is critical. Observe that this critical equation, when $n = 2$ and $\mathcal{V} = W^{1,2}((B_1, \mathbb{R})$, admits singular solutions such as $\log\log \frac{1}{r}$.

In contrast to the above situation, one can consider the equation

$$-\Delta u = u_x \wedge u_y \tag{3}$$

which is again critical in dimension $n = 2$ when $\mathcal{V} = W^{1,2}(B_1, \mathbb{R}^3)$, but this time any solution can be shown to be smooth (see for instance [8]).

Finally, in dimension $n = 3$ and when $\mathcal{V} = W^{1,2}(B_1, \mathbb{R}^3)$, this equation is super-critical and the existence result of Rivière [12] of everywhere discontinuous harmonic maps in $W^{1,2}(B_1, S^2)$ has annihilated all hopes of having a partial regularity result for solution to this super-critical semilinear equation.

When the equation has a variational structure, namely when the equation is the Euler-Lagrange equation of a functional, it makes sense to restrict our attention to the subspace of solutions which are stationary. That is, one considers the critical points to the functional which are also critical with respect to perturbations of the domain (see Definition 1.1 below and see also [8]). A consequence of this stationarity assumption is that the solution satisfies an identity (which is in fact a conservation law) which, in the most studied cases, can be converted into a monotonicity formula. In most of the cases which have been studied so far, this monotonicity formula implies that the solution u belongs to some Morrey type space $\mathcal{M}$, which is much smaller than the original space $\mathcal{V}$. In the good cases, replacing $\mathcal{V}$ by $\mathcal{M}$ makes the problem critical and this allows one to obtain a partial regularity result for the stationary solutions (see for instance [3, 8] for harmonic maps, and [11] when the nonlinearity is u^α with α greater than the critical exponent).

The aim of this paper is to present an alternative approach to the partial regularity theory when the stationary assumption cannot be converted in a monotonicity formula. We illustrate this method by applying it to the famous Liouville equation in dimension $n = 3$

$$-\Delta u = e^u \quad \text{in } \Omega. \tag{4}$$

Throughout the paper, $\Omega \subseteq \mathbb{R}^3$ denotes an open set, u is a scalar function and ∇u and $\nabla^2 u$ denote respectively the gradient and Hessian matrix of u. In dimension $n = 2$, the geometric meaning of equation (4) is well known and it corresponds to the problem of finding metrics g, which are conformally equivalent to the flat metric, and which have constant Gauss curvature. In dimension $n \geq 3$, equation (4) arises in the modeling of several physical phenomena such as the theory of isothermal gas sphere and gas combustion.

A function u is said to be a weak solution of (4) in Ω if for all $\varphi \in C_c^\infty(\Omega)$ it satisfies

$$-\int_\Omega u \Delta \varphi \, dx = \int_\Omega e^u \varphi \, dx. \tag{5}$$

We now recall the definition of stationary solution.

Definition 1.1. A weak solution of (4) is said to be stationary if it satisfies

$$\frac{d}{dt} E(u(x + tX))|_{t=0} = 0, \tag{6}$$

for all smooth vector fields X with compact support in Ω, where

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega e^u dx.$$

Computing (6) for weak solutions in $W^{1,2}(\Omega)$ we find that for any smooth vector field X the following identity holds

$$\int_\Omega \left[\frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} \frac{\partial X^k}{\partial x_i} - \frac{1}{2} |\nabla u|^2 \frac{\partial X^i}{\partial x_i} + e^u \frac{\partial X^i}{\partial x_i} \right] dx = 0, \tag{7}$$

This identity can be also understood as a conservation law (see again [8]).

Arguing as in [7], we insert in (7) the vector field $X^\delta = x\varphi^\delta(|x|)$ where

$$\begin{cases} \varphi^\delta(s) = 1 & \text{if } s < r \\ \varphi^\delta(s) = 1 + \dfrac{r - s}{r\delta} & \text{if } r \leq s \leq r + r\delta \\ \varphi^\delta(s) = 0 & \text{if } s > r. \end{cases}$$

After some calculations, we let $\delta \to 0$ and deduce that, for almost every $r > 0$, the following formula holds

$$\frac{1}{r} \int_{B_r} \left(\frac{1}{2} |\nabla u|^2 - 3e^u \right) dx = \frac{1}{2} \int_{\partial B_r} |\nabla_T u|^2 dx - \frac{1}{2} \int_{\partial B_r} \left| \frac{\partial u}{\partial r} \right|^2 dx - \int_{\partial B_r} e^u dx. \tag{8}$$

This can also be written as follows

$$\frac{d}{dr} \left[\frac{1}{r} \int_{B_r} (|\nabla u|^2 - 6e^u) dx \right] = \frac{2}{r} \int_{\partial B_r} \left| \frac{\partial u}{\partial r} \right|^2 dx - \frac{4}{r} \int_{\partial B_r} e^u dx. \tag{9}$$

Unlike the cases of stationary solutions to super-critical semilinear equations which have mainly be considered so far, the formula (8) does not seem to provide any monotonicity information, any uniform bound neither for the term $\frac{1}{r}\int_{B_r}|\nabla u|^2 dx$ nor for $\frac{1}{r}\int_{B_r} e^u dx$. As already mentioned, the main contribution of the present work is to present an alternative approach to the partial regularity theory in absence of monotonicity and Morrey type estimates. Our approach is inspired by the technique introduced by Lin and Rivière in [10] in the context of Ginzburg–Landau equations. This technique is based on some kind of dimension reduction argument. More precisely, applying Fubini's Theorem one first extracts "good" 2 dimensional slices to get estimates of the some suitable quantities, then one restricts these quantities to these slices (whose dimension is such that the non-linearity e^u becomes critical for $W^{1,2}$) and obtains some estimates in interpolation spaces: the Lorentz spaces $L^{2,\infty} - L^{2,1}$. Finally, the stationarity condition (8) can be used to "propagate" these estimates from the slices (basically the boundary of balls) into the domain bounded by the slices (the balls themselves).

Now we state our main result.

Theorem 1.1. *Assume that* $u \in W^{1,2}(\Omega)$ *is a stationary solution of* (4), *such that* $e^u \in L^1(\Omega)$. *Then there exists an open set* $\mathcal{O} \subset \Omega$ *such that*

$$u \in C^\infty(\mathcal{O}) \quad \text{and} \quad \mathcal{H}_{\text{dim}}(\Omega \backslash \mathcal{O}) \leq 1$$

where $\mathcal{H}_{\text{dim}}$ *denotes the dimensional Hausdorff measure.*

It is an open question whether such a partial regularity result is optimal or not (the same question holds for instance also for stationary harmonic maps). What is known is that stationary solutions to (4) can have singularities. Indeed the function $u(x) = \log\left(\frac{2}{|x|^2}\right)$ satisfies $-\Delta u = e^u$ but is not bounded.

Our approach and the above result should also hold for the more general class of equations of the form $-\Delta u = V(x)e^u$ where $V(x)$ is some smooth given potential. For the sake of simplicity, we have chosen to focus our attention on the case where $V \equiv 1$ in order to the keep the technicalities as low as possible and make the paper more "readable". We recall that, in dimension $n = 2$, the regularity of weak solutions to the equation (4), starting from the hypothesis that u is in $W^{1,2}$, is a straightforward consequence of the Moser–Trudinger inequality (see [6]). Still in dimension $n = 2$, a L^∞ estimate for solutions in $L^1(\Omega)$ to the equation (4), starting from the hypothesis that $e^u \in L^1(\Omega)$, has been obtained by Brezis and Merle [2]. Finally, in [1] the authors prove some *a priori* estimates for solutions of (4) in any dimension but under the stronger assumption e^u is in some ad-hoc Morrey Space which makes the problem critical.

It is aim of future work to investigate compactness properties of stationary solutions to equation (4).

2. Preliminary Estimates of the Energy

In this section we are going to prove some preliminary estimates.

We first introduce some notations and recall the definition of Hausforff measure.

For $x_0 \in \Omega$, $r > 0$ we will denote by B_{r,x_0} or simply by B_r the ball $B(x_0, r)$ centered at x_0 and with radius r. Given $A \subset \mathbb{R}^3$ we denote by $|A|$ its Lebegue measure, by $\mathscr{H}^s(A)$ its s dimensional Hausdorff measure and by $\mathbb{1}_A(\cdot)$ the characteristic function of A.

We recall (see e.g., [4]) the definition of the s-dimensional Hausforff measure $\mathscr{H}^s$ in $\mathbb{R}^n$, with $0 \le s \le n$. For any $\delta > 0$ and for any $A \subseteq \mathbb{R}^n$ we set

$$\mathscr{H}^s_\delta(A) = \inf\left\{\sum_i w_s r_i^s : A \subseteq \bigcup_i B_{r_i}.\ r_i < \delta, \forall i\right\}$$

where $w_s = \frac{\pi^{s/2}}{\Gamma(1+s/2)}$ and the infimum is taken over all contable collections of ball $\{B_{r_i}\}$ covering the set A and having radii $r < \delta$. The s-dimensional Hausdorff measure is then defined as

$$\mathscr{H}^s(A) = \lim_{\delta\to 0} \mathscr{H}^s_\delta(A).$$

Given $x_0 \in \Omega$ and $0 < r < d(x_0, \partial\Omega)$ we introduce the following energy

$$\mathscr{E}_{x_0,r}(u) = \frac{1}{r}\int_{B_r} |\nabla u|^2 dx + \frac{1}{r}\int_{B_r} e^u dx \tag{10}$$

and set

$$(u)_{x_0,r} := \frac{1}{|\partial B_{r,x_0}|}\int_{\partial B_{r,x_0}} u(y)dy.$$

The key result to prove Theorem 1.1 is the following assertion about the energy (10).

Theorem 2.1. *There exist constants $\eta, \beta \in (0,1)$ such that for every $x_0 \in \Omega$ and $0 < r < d(x_0, \partial\Omega)/2$,*

$$\mathscr{E}_{x_0,2r}(u) \le \eta \quad\text{and}\quad (u)_{x_0,r}\mathscr{E}_{x_0,2r}(u) \le \eta \tag{11}$$

imply

$$\mathscr{E}_{y,s}(u) \le Cs^\beta$$

for all y in a neighborhood of x_0, for all $s \le r$, and for some C depending on r and independent on y, s.

In order to prove Theorem 2.1 we need to give some definitions and to show a series of preliminary results.

We decompose $u - (u)_{x_0,r}$ as the sum of two functions solving to different Dirichlet problems. More precisely we write $u - (u)_{x_0,r} = v + w$, with v and w satisfying respectively

$$\begin{cases} -\Delta v = e^u & \text{in } B_r \\ v = 0 & \text{on } \partial B_r; \end{cases} \tag{12}$$

$$\begin{cases} -\Delta w = 0 & \text{in } B_r \\ w = u - (u)_{x_0,r} & \text{on } \partial B_r. \end{cases} \tag{13}$$

We observe that up to a choice of smaller radius r and η we may suppose without loss of generality that

$$\int_{\partial B_r} e^u + |\nabla u|^2 dx \leq \frac{C}{2r}\int_{B_{2r}} e^u + |\nabla u|^2 dx \leq \eta. \tag{14}$$

Indeed by Mean Value Theorem we can find $\bar{\rho} \in [3/4, 4/5]$ such that

$$\int_{\partial B_{\bar{\rho}r}} e^u + |\nabla u|^2 dx \leq \frac{20}{r}\int_{B_r} e^u + |\nabla u|^2 dx.$$

Setting $r' = \bar{\rho}r$, $\eta' = 40\eta$, with $\eta < \frac{1}{120}$, we get

$$\int_{\partial B_{r'}} e^u + |\nabla u|^2 dx \leq \frac{40}{2r'}\int_{B_{2r'}} e^u + |\nabla u|^2 dx \leq \eta'.$$

The new r and η will be respectively r' and η'.

Given a constant $C > 1$ and a positive function $f \in L^1(B_{2r})$, by a *C-good slice* in $[\rho_1, \rho_2] \subset (0, 1/2)$, we mean any sphere of radius $\rho \in [\rho_1, \rho_2]$ such that

$$\int_{\partial_{B_{\rho r}}} f\, dx \leq \frac{C}{r(\rho_2 - \rho_1)}\int_{B_r} f\, dx.$$

We observe that the existence of a good slice is a consequence of Fubini Theorem. Moreover one can check that for every $\delta > 0$ there exists $C_\delta > 0$ such the set of the C_δ-good slices has Lebesgue measure bounded from below by $\rho_2 - \rho_1 - \delta$. This property will be frequently used throughout the paper.

We now prove two Lemmae which will be useful in the next subsections. In the first one we show that $\mathscr{E}_{x_0,r}(w)$ is nonincreasing with respect to the radius r and in the second one we show that that ∇v and ∇w are orthogonal in B_r.

Lemma 2.1. *The functions e^w and $|\nabla w|^2$ satisfy respectively*

$$\frac{1}{\rho r}\int_{B_{\rho r}} e^w dx \leq \rho^2 \frac{1}{r}\int_{B_r} e^w dx. \tag{15}$$

and

$$\frac{1}{\rho r}\int_{B_{\rho r}} |\nabla w|^2 dx \leq \rho^2 \frac{1}{r}\int_{B_r} |\nabla w|^2 dx. \tag{16}$$

for every $\rho \in (0, 1)$.

Proof. We observe that $w \in C^\infty(B_r)$ and both e^w and $|\nabla w|^2$ are sub-harmonic in B_r. Indeed we have

$$-\Delta e^w = -|\nabla w|^2 e^w \leq 0 \quad \text{in } B_r$$

and

$$-\Delta|\nabla w|^2 = -\sum_{i,j}\left(\frac{\partial^2 w}{\partial x_i \partial x_j}\right)^2 \le 0 \quad \text{in } B_r.$$

A well-known fact of sub-harmonic functions is that their mean value on a ball is a non increasing function with respect to radius of the ball, namely for every $\rho \in (0, 1)$ we have

$$\frac{1}{|B_{\rho r}|}\int_{B_{\rho r}} e^w dx \le \frac{1}{|B_r|}\int_{B_r} e^w dx$$

$$\frac{1}{|B_{\rho r}|}\int_{B_{\rho r}} |\nabla w|^2 dx \le \frac{1}{|B_r|}\int_{B_r} |\nabla w|^2 dx.$$

This clearly implies

$$\frac{1}{\rho}\int_{B_{\rho r}} e^w dx \le \rho^2 \frac{1}{r}\int_{B_r} e^w dx$$

$$\frac{1}{\rho}\int_{B_{\rho r}} |\nabla w|^2 dx \le \rho^2 \frac{1}{r}\int_{B_r} |\nabla w|^2 dx$$

and we conclude. □

Lemma 2.2. *The following estimate holds*

$$\int_{B_r} |\nabla u|^2 dx = \int_{B_r} |\nabla v|^2 dx + \int_{B_r} |\nabla w|^2 dx.$$

Proof. Let $\nu(x)$ denote the exterior normal versor to ∂B_r at the point $x \in \partial B_r$. We have

$$\int_{B_r} \nabla w \cdot \nabla v \, dx = \int_{\partial B_r} v \nabla w \cdot \nu - \int_{B_r} v \, \Delta w = 0.$$ □

Now we recall the definition of the weak L^2 space (or Marcinkievicz space $L^{2,\infty}$, see [13]).

The space $L^{2,\infty}(\Omega)$ is defined as the space of functions $f: \Omega \to \mathbb{R}$ such that

$$\sup_{\lambda \in \mathbb{R}} \lambda |\{x : |f|(x) \ge \lambda\}|^{1/2} < +\infty.$$

The dual space of $L^{2,\infty}(\Omega)$ is the Lorentz space $L^{2,1}(\Omega)$ whose norm is equivalent

$$||f||_{2,1} \simeq \int_0^\infty 2|\{x : |f|(x) \ge s\}| ds.$$

In dimension 2 we have the following property: $W^{1,1}(\Omega)$ continuously embeds in $L^{2,1}(\Omega)$, (see e.g. [9, 14, 15]).

We next recall a result obtained by Lin and Riviere in [10] in the framework of Ginzurg–Landau functionals, which will play a crucial role to get estimates of the energy (10).

Lemma 2.3 (Lemma A.2, [10]). *For any $g \in L^1(B_1)$, if we denote*

$$f(x) = \int_{B_1} \frac{g(y)}{|y-x|^2}\, dy,$$

then for every $\delta > 0$ there exists a subset $E_\delta \subseteq (0,1)$ and $|E_\delta| > 1-\delta$ such that for all $\rho \in E_\delta$ we have

$$\|f\|_{L^{2,\infty}(\partial B_\rho)} \le C_\delta \|g\|_{L^1(B_1)},$$

where C_δ depends only on δ.

In the next two subsections we are going to estimate $\frac{1}{\rho r}\int_{B_{\rho r}} e^u dx$ and $\frac{1}{\rho r}\int_{B_{\rho r}} |\nabla u|^2 dx$ (with $0 < \rho < 1/2$) in function of the energy $\mathscr{E}_{x_0,2r}(u)$.

2.1. *Estimate of e^u*

In this subsection we are going to estimate $\frac{1}{\rho r}\int_{B_{\rho r}} e^u dx$ in function of the energy $\mathscr{E}_{x_0,2r}(u)$. More precisely we prove the following theorem.

Theorem 2.2. *For all $\alpha \in (0,1)$, there exist constants $\eta \in (0,1)$, $0 < \rho_1 < \rho_2 < 1/2$ such that*

$$\mathscr{E}_{x_0,2r}(u) \le \eta, \tag{17}$$

implies

$$\frac{1}{\rho r}\int_{B_{\rho r}} e^u dx \le \alpha\, \mathscr{E}_{x_0,2r}(u), \tag{18}$$

for every $\rho \in [\rho_1, \rho_2]$.

Proof. We split the proof in several steps.

Step 1. We start by estimating the oscillation of w in $B_{\rho r}$ with $0 < \rho < 1/2$ and the mean value of v^2 in B_r.

Proposition 2.1. *For all $\rho \in (0, 1/2)$ and $x, y \in B_{\rho r}$ we have*

$$|w(x) - w(y)| \le C\eta^{1/2}(2\rho r),$$

for some C depending only on the dimension of the space.

Proof. We set $\bar{w}_r = w(rx + x_0)$, $\bar{w}_r$ satisfies

$$\begin{cases} -\Delta \bar{w}_r = 0 & \text{in } B_1 \\ \bar{w}_r = u_r - (u_r)_{1,x_0} & \text{on } \partial B_1. \end{cases} \tag{19}$$

Standard elliptic estimates and Poincaŕe–Wirtinger Inequality imply that for all $\rho \in (0, 1/2)$ and for some $C > 0$ we have

$$\|\bar{w}_r\|_{C^1(B_\rho)} \leq C\left(\int_{\partial B_1} (u_r - (u_r)_{1,x_0})^2 dx\right)^{1/2} \leq C\|\nabla u_r\|_{L^2(\partial B_1)}.$$

Thus for every $x, y \in B_{\rho r}$ we have

$$|w(x) - w(y)|^2 \leq C|x-y|^2 \int_{\partial B_r} |\nabla u(z)|^2 dz \leq C|x-y|^2 \eta. \tag{20}$$

where in the last inequality of (20) we use assumption (17) on the energy and (14). Hence

$$|w(x) - w(y)| \leq C(2\rho r)\eta^{1/2} \tag{21}$$

and we can conclude. □

Proposition 2.2. *The function v satisfies*

$$\left(\frac{1}{r^3}\int_{B_r} v^2(x)dx\right)^{1/2} \leq C\eta. \tag{22}$$

Proof of Proposition 2.2. We recall that

$$v(x) = \int_{B_r} e^{u(y)} G(x, y) dy,$$

where $G(x, y)$ is the Green function on the ball which satisfies

$$|G(x, y)| \leq C\frac{1}{|x-y|} \quad \text{and} \quad |\nabla_x G(x, y)| \leq C\frac{1}{|x-y|^2},$$

(see for instance [6]).

Thus

$$\left(\int_{B_1} v^2(rx)dx\right)^{1/2} \leq Cr^2 \int_{B_1} e^{u(rx)} dx. \tag{23}$$

Setting $y = rx$ we get

$$\left(\frac{1}{r^3}\int_{B_r} v^2 dy\right)^{1/2} \leq C\frac{1}{r}\int_{B_r} e^u dy.$$

By assumption (17) we get (22) and we conclude. □

Step 2. From (22) it follows

$$\begin{aligned}(|\{x \in B_{\rho r} : v \geq \eta^{1/2}\}|\eta)^{1/2} &\leq \left(\int_{B_r} v^2(x)dx\right)^{1/2} \\ &\leq C\eta r^{3/2}. \end{aligned} \tag{24}$$

Hence

$$|\{x \in B_{\rho r} : v \geq \eta^{1/2}\}| \leq C\eta r^3. \tag{25}$$

Now let us take $\lambda > 0$ (that we will determine later) and set

$$\begin{aligned} I^1_\eta &:= \{x : v(x) \leq \eta^{1/2}\} \\ I^1_\lambda &:= \{x : v(x) \geq \lambda\} \\ I^2_{\lambda,\eta} &= \{x : \eta^{1/2} \leq v(x) \leq \lambda\}. \end{aligned}$$

By using the fact that v solves (12) and Lemma 2.2, we obtain

$$\begin{aligned} \frac{1}{r}\int_{B_r} v e^u dx &= \frac{1}{r}\int_{B_r} v(-\Delta v)dx \\ &= \frac{1}{r}\int_{B_r} |\nabla v|^2 dx \leq \frac{1}{r}\int_{B_r} |\nabla u|^2 dx. \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{\rho r}\int_{B_{\rho r}\cap I^1_\lambda} e^u dx &\leq \frac{1}{\rho r\lambda}\int_{B_{\rho r}\cap I^1_\lambda} v e^u dx \\ &\leq \frac{1}{\rho r\lambda}\int_{B_r} |\nabla v|^2 dx \leq \frac{1}{\rho r\lambda}\int_{B_r} |\nabla u|^2 dx. \end{aligned} \tag{26}$$

We also have

$$\frac{1}{\rho r}\int_{B_{\rho r}\cap I^1_\eta} e^u dx \leq e^{\eta^{1/2}} e^{(u)_{x_0,r}} \frac{1}{\rho r}\int_{B_{\rho r}} e^w dx. \tag{27}$$

By Proposition 2.1, estimate (25) and the fact that $e^{w(0)} \leq \frac{1}{|B_{\rho r}|}\int_{B_{\rho r}} e^{w(x)}dx$, being e^w sub-harmonic, we get

$$\begin{aligned} \frac{1}{\rho r}\int_{B_{\rho r}\cap I^2_{\lambda,\eta}} e^u dx &\leq e^\lambda e^{(u)_{x_0,r}} \frac{1}{\rho r}\int_{B_{\rho r}\cap I^2_{\lambda,\eta}} e^w dx \\ &= e^\lambda e^{(u)_{x_0,r}} e^{w(0)} \frac{1}{\rho r}\int_{B_{\rho r}\cap I^2_{\lambda,\eta}} e^{w(x)-w(0)} dx \\ &\leq e^\lambda e^{(u)_{x_0,r}} e^{C\eta^{1/2}} \big|\{x : v \geq \eta^{1/2}\}\big| \frac{1}{\rho r}\frac{1}{(\rho r)^3}\int_{B_{\rho r}} e^w dx \\ &\leq C e^\lambda e^{(u)_{x_0,r}} e^{C\eta^{1/2}} \eta \frac{1}{\rho^3}\frac{1}{\rho r}\int_{B_{\rho r}} e^w dx. \end{aligned} \tag{28}$$

By combining estimates (26), (27) and (28) and Lemma 2.1 we finally get

$$\begin{aligned} \frac{1}{\rho r}\int_{B_{\rho r}} e^u dx &= \frac{1}{\rho r}\int_{B_{\rho r}\cap I^1_\eta} e^u dx + \frac{1}{\rho r}\int_{B_{\rho r}\cap I^1_\lambda} e^u dx + \frac{1}{\rho r}\int_{B_{\rho r}\cap I^2_{\lambda,\eta}} e^u dx \\ &\leq e^{\eta^{1/2}} e^{(u)_{x_0,r}} \frac{1}{\rho r}\int_{B_{\rho r}} e^w dx + \frac{1}{\rho r\lambda}\int_{B_{\rho r}} |\nabla u|^2 dx + C e^\lambda e^{(u)_{x_0,r}} e^{C\eta^{1/2}} \eta \frac{1}{\rho^3}\frac{1}{\rho r}\int_{B_{\rho r}} e^w dx \end{aligned}$$

$$\leq e^{\eta^{1/2}} e^{(u)_{x_0,r}} \rho^2 \frac{1}{r}\int_{B_r} e^w dx + \frac{1}{\rho\lambda}\frac{1}{r}\int_{B_r} |\nabla u|^2 dx + \frac{Ce^{\lambda} e^{(u)_{x_0,r}} e^{C\eta^{1/2}}\eta}{\rho}\frac{1}{r}\int_{B_r} e^w dx$$
$$= \left[\frac{Ce^{\lambda}e^{C\eta^{1/2}}\eta}{\rho} + \rho^2 e^{\eta^{1/2}}\right] e^{(u)_{x_0,r}} \frac{1}{r}\int_{B_r} e^w dx + \frac{1}{\rho\lambda}\frac{1}{r}\int_{B_r} |\nabla u|^2\, dx.$$

Now we first fix the interval $[\rho_1, \rho_2] \subseteq (0, 1/2)$ where we make ρ vary, and then the constants λ, η.

Consider any $0 < \alpha' < 1/4$. We choose ρ such that

$$e\rho^2 < \frac{\alpha'}{3}.$$

Hence we take ρ_1, ρ_2 satsfying $0 < \frac{\sqrt{\alpha'}}{3\sqrt{3e}} < \rho_1 < \rho_2 < \frac{\sqrt{\alpha'}}{\sqrt{3e}}$. Then we choose λ large enough so that

$$\frac{1}{\rho_1\lambda} < \frac{\alpha'}{3}$$

and finally we choose η small enough so that

$$Ce^{C\eta^{1/2}} e^{\lambda}\eta\frac{1}{\rho_1} < \frac{\alpha'}{3}.$$

We observe that $\int_{B_r} e^w dx \leq \int_{B_r} e^{u-u_{x_0,r}} dx$, being v nonnegative by the Maximum Principle. Thus by these choices of the constants ρ_1, ρ_2, η and λ we have

$$\frac{1}{\rho r}\int_{B_{\rho r}} e^u dx \leq \alpha'\left[\frac{1}{r}\int_{B_r} e^u dx + \frac{1}{r}\int_{B_r} |\nabla u|^2 dx\right] \leq 2\alpha' \mathscr{E}_{x_0,2r},$$

for all $\rho \in [\rho_1, \rho_2]$. Setting, $\alpha = 2\alpha'$ we can conclude. □

2.2. *Estimate of* ∇u

In this subsection we are going to estimate $\frac{1}{\rho r}\int_{B_{\rho r}} |\nabla u|^2 dx$ in function of $\mathscr{E}_{x_0,2r}(u)$, with $\rho \in [\rho_1, \rho_2] \subseteq (0, 1/2)$, ρ_1, ρ_2 being the constants determined in Theorem 2.2.

Theorem 2.3 (Estimate Gradient of u). *For every $\delta > 0$ there exists a subset $E_\delta \subseteq [\rho_1, \rho_2]$ and $|E_\delta| > \rho_2 - \rho_1 - 3\delta$ such that for all $\rho \in E_\delta$ we have*

$$\frac{1}{\rho r}\int_{B_{\rho r}} |\nabla u|^2 dx \leq \gamma(\mathscr{E}_{x_0,2r}(u)) + C(\mathscr{E}_{x_0,2r}(u))\big(\mathscr{E}_{x_0,2r}(u) + (u)^+_{x_0,r}\mathscr{E}_{x_0,2r}(u)\big), \tag{29}$$

for some $C > 0$ and $0 < \gamma < 1$ independent on r.

Proof. We split the proof in several steps.

Step 1. Estimate of ∇v.

Proposition 2.3. *For some $C > 0$ (independent on r) we have*

$$\int_{B_r} |\Delta v| \log(2+|\Delta v|)dx \leq C\bigg(\int_{B_r} (e^u + |\nabla u|^2 dx + (u)^+_{r,x_0}\int_{B_r} e^u dx + 4r\int_{\partial B_r} |\nabla u|^2 dx\bigg). \tag{30}$$

Proof of Proposition 2.3. We have

$$\begin{aligned}\int_{B_r} |\Delta v| \log(2+|\Delta v|)dx &= \int_{B_r} e^u \log(2+e^u)dx \\ &\leq C\int_{B_r} e^u(1+u^+)dx \qquad (31)\\ &\leq C\bigg[\int_{B_r} e^u dx + (u)^+_{r,x_0}\int_{B_r} e^u dx + \int_{B_r} e^u v^+ dx + \int_{B_r} e^u w^+ dx\bigg]\end{aligned}$$

We estimate $\int_{B_r} e^u v^+ dx$ and $\int_{B_r} e^u w^+ dx$. We have

$$\begin{aligned}\int_{B_r} e^u v^+ dx &= \int_{B_r} -\Delta v v\, dx = \int_{B_r} |\nabla v|^2 dx \\ &\leq \int_{B_r} |\nabla u|^2 dx. \qquad (32)\\ \int_{B_r} e^u w^+ dx &= \int_{B_r} -\Delta u w^+ dx \\ &= -\int_{B_r} div(\nabla u w^+)dx + \int_{B_r} \nabla u \cdot \nabla w^+ \\ &\leq -\int_{\partial B_r} \frac{\partial u}{\partial \nu} w^+ dx + 2\bigg(\int_{B_r} |\nabla u|^2 dx + \int_{B_r} |\nabla w|^2 dx\bigg) \\ &\leq 2r\int_{\partial B_r} |\nabla u|^2 dx + \frac{2}{r}\int_{\partial B_r} [(u-(u)_{r,x_0})^+]^2 dx + 4\int_{B_r} |\nabla u|^2 dx \\ &\leq 2r\int_{\partial B_r} |\nabla u|^2 dx + 2r\int_{\partial B_r} |\nabla u|^2 dx + 4\int_{B_r} |\nabla u|^2 dx. \\ &\leq 4\int_{B_r} |\nabla u|^2 dx + 4r\int_{\partial B_r} |\nabla u|^2 dx. \qquad (33)\end{aligned}$$

By combining (31), (32), (33) we finally get

$$\begin{aligned}&\int_{B_r} |\Delta v| \log(2+|\Delta v|)dx \\ &\quad \leq C\bigg[\int_{B_r} e^u dx + (u)^+_{r,x_0}\int_{B_r} e^u dx + 4\int_{B_r} |\nabla u|^2 dx + 4r\int_{\partial B_r} |\nabla u|^2 dx\bigg] \\ &\quad \leq C\bigg(\int_{B_r} e^u + |\nabla u|^2 dx + (u)^+_{r,x_0}\int_{B_r} e^u dx + 4r\int_{\partial B_r} |\nabla u|^2 dx\bigg).\end{aligned}$$

Thus we can conclude. □

Corollary 2.1. *We have* $\nabla^2 v \in L^1(B_r)$ *and*

$$\|\nabla^2 v\|_{L^1(B_r)} \leq C\bigg(\int_{B_r}(e^u + |\nabla u|^2 dx + (u)^+_{r,x_0}\int_{B_r} e^u dx + 4r\int_{\partial B_r}|\nabla u|^2 dx\bigg).$$

Proof. Calderon–Zygmund theory (see e.g., [14]) yields that if $\int_{B_r}|\Delta v|\log(2 + |\Delta v|)dx < +\infty$ then $\nabla^2 v \in L^1(B_r)$ and

$$\|\nabla^2 v\|_{L^1(B_r)} \leq C\int_{B_r}|\Delta v|\log(2 + |\Delta v|)dx.$$

Thus the result follows directly from Proposition 2.3 and we conclude. □

We can now use Lemma 2.3 to prove the following result.

Proposition 2.4. *For every* $\delta > 0$ *small enough, there exists a subset* $E^1_\delta \subseteq [\rho_1, \rho_2]$ *and* $|E^1_\delta| > \rho_2 - \rho_1 - \delta$ *such that for all* $\rho \in E^1_\delta$ *we have*

$$\|\nabla v\|_{L^{2,\infty}(\partial B_{\rho r})} \leq \frac{C_\delta}{r}\int_{B_r} e^u dx,$$

where C_δ *depends only on* δ.

Proof of Proposition 2.4. As we observe in the proof of Proposition 2.2, we can write

$$v(x) = \int_{B_r} e^{u(y)} G(x, y) dy,$$

where $G(x, y)$ is the Green function on B_r. Since $|\nabla_x G(x, y)| \leq C\frac{1}{|x-y|^2}$ we have

$$|\nabla v(x)| \leq C\int_{B_r}\frac{e^{u(y)}}{|x - y|^2}dy.$$

Lemma 2.3 yields that for every $\delta > 0$ there exists a subset $E^1_\delta \subseteq [\rho_1, \rho_2]$ and $|E^1_\delta| > \rho_2 - \rho_1 - \delta$ such that for almost very $\rho \in E^1_\delta$ we have

$$\|\nabla v\|_{L^{2,\infty}(\partial B_{\rho r})} \leq \frac{C_\delta}{r}\int_{B_r} e^u dx,$$

and we conclude. □

Proposition 2.5. *For every* $\delta > 0$ *there exists a subset* $E_\delta \subseteq [\rho_1, \rho_2]$ *and* $|E_\delta| > \rho_2 - \rho_1 - 3\delta$ *such that for all* $\rho \in E_\delta$ *we have*

$$\int_{\partial B_{\rho r}}|\nabla v|^2 dx \leq C\big(\mathscr{E}_{x_0,r}(u)\big)\big(\mathscr{E}_{x_0,r}(u) + (u)^+_{r,x_0}\mathscr{E}_{x_0,r}(u) + \mathscr{E}_{x_0,2r}(u)\big),$$

with C *depending on* δ *and the dimension.*

Proof. Since $\nabla^2 v \in L^1(B_r)$, from Fubini Theorem it follows that for all $\delta > 0$ there is $E_\delta^2 \subseteq [\rho_1, \rho_2]$, $|E_\delta^2| > \rho_2 - \rho_1 - \delta$ and C_δ such that for all $\rho \in E_\delta^2$ we have

$$\int_{\partial B_{\rho r}} |\nabla^2 v|\, dx \leq C_\delta \frac{1}{r(\rho_2 - \rho_1)} \int_{B_r} |\nabla^2 v|\, dx.$$

We set $C = \frac{C_\delta}{(\rho_2 - \rho_1)}$. By the embedding of the space $W^{1,1}(\partial B_{\rho r})$ into $L^{2,1}(\partial B_{\rho r})$, we have $\nabla v \in L^{2,1}(\partial B_{\rho r})$ as well and thus the following estimate holds

$$\begin{aligned}
\|\nabla v\|_{L^{2,1}(\partial B_{\rho r})} &\leq C\|\nabla^2 v\|_{L^1(\partial B_{\rho r})} \\
&\leq C\frac{1}{r}\|\nabla^2 v\|_{L^1(B_r)} \\
&\leq C\left(\frac{1}{r}\int_{B_r} |\nabla u|^2 + e^u dx + (u)^+_{r,x_0}\frac{1}{r}\int_{B_r} e^u dx + 4\int_{\partial B_r} |\nabla u|^2 dx\right).
\end{aligned}$$

Now set $E_\delta := E_\delta^1 \cap E_\delta^2$. We have $|E_\delta| > \rho_2 - \rho_1 - 3\delta$.

By using the duality between $L^{2,\infty}$ and $L^{2,1}$ and Proposition 2.4, for all $\rho \in E_\delta$ we obtain

$$\int_{\partial B_{\rho r}} |\nabla v|^2 dx \leq \|\nabla v\|_{L^{2,\infty}(\partial B_{\rho r})}\|\nabla v\|_{L^{2,1}(\partial B_{\rho r})} \tag{34}$$

$$\begin{aligned}
&\leq C\left(\frac{1}{r}\int_{B_r} e^u dx\right)\left(\frac{1}{r}\int_{B_r} |\nabla u|^2 + e^u dx + (u)^+_{r,x_0}\frac{1}{r}\int_{B_r} e^u dx\right. \\
&\qquad \left. + 4\int_{\partial B_r} |\nabla u|^2 dx\right).
\end{aligned} \tag{35}$$

Since we may suppose without restriction that

$$\int_{\partial B_r} |\nabla u|^2 dx \leq \frac{C}{2r}\int_{B_{2r}} |\nabla u|^2 dx,$$

(see (14)), we have

$$\begin{aligned}
\int_{\partial B_{\rho r}} |\nabla v|^2 dx &\leq C\left(\frac{1}{r}\int_{B_r} e^u dx\right) \\
&\quad \times \left(\frac{1}{r}\int_{B_r} |\nabla u|^2 + e^u dx + (u)^+_{r,x_0}\frac{1}{r}\int_{B_r} e^u dx + 4\int_{\partial B_r} |\nabla u|^2\, dx\right) \\
&\leq C\big(\mathscr{E}_{x_0,r}(u)\big)\big(\mathscr{E}_{x_0,r}(u) + (u)^+_{r,x_0}\mathscr{E}_{x_0,r}(u) + \mathscr{E}_{x_0,2r}(u)\big),
\end{aligned}$$

and we can conclude. □

Step 2. We premise two remarks.

1. Since w solves (13), then

$$\int_{\partial B_r} \left|\frac{\partial w}{\partial \nu}\right|^2 dx \leq \int_{\partial B_r} |\nabla_T w|^2\, dx = \int_{\partial B_r} |\nabla_T u|^2\, dx. \tag{36}$$

To show the first inequality in (36), one can use for instance the *monotonicity formula* for harmonic funtions.

Thus

$$\int_{\partial B_{\rho r}} |\nabla w|^2 dx \le 2\int_{\partial B_r} |\nabla u|^2 dx. \tag{37}$$

2. Since $|\nabla w|^2$ is sub-harmonic, for every $\rho \in E_\delta$ we have

$$\int_{\partial B_{\rho r}} |\nabla w|^2 dx \le \rho^2 \int_{\partial B_r} |\nabla w|^2 dx. \tag{38}$$

The proof of (38) is similar to the one of Lemma 2.1.

From (37), (38) and (14) it follows

$$\begin{aligned}\int_{\partial B_{\rho r}} |\nabla w|^2 dx &\le C\rho^2 \int_{\partial B_r} |\nabla u|^2 dx \\ &\le \frac{C\rho^2}{2r}\int_{B_{2r}} |\nabla u|^2 dx.\end{aligned} \tag{39}$$

From Proposition 2.5 and (39) it follows that for all every $\rho \in E_\delta$ we have

$$\begin{aligned}\int_{\partial B_{\rho r}} |\nabla u|^2 dx &\le 2\int_{\partial B_{\rho r}} |\nabla w|^2 dx + 2\int_{\partial B_{\rho r}} |\nabla v|^2 dx \\ &\le C\frac{\rho^2}{2r}\int_{B_{2r}} |\nabla u|^2 dx + C(\mathscr{E}_{x_0,r}(u))(\mathscr{E}_{x_0,r}(u) + (u)^+_{r,x_0}\mathscr{E}_{x_0,r}(u) + \mathscr{E}_{x_0,2r}(u)) \\ &\le C\rho^2\mathscr{E}_{x_0,2r}(u) + C(\mathscr{E}_{x_0,r}(u))(\mathscr{E}_{x_0,r}(u) + (u)^+_{r,x_0}\mathscr{E}_{x_0,r}(u) + \mathscr{E}_{x_0,2r}(u)).\end{aligned}$$

Finally by applying formula (8) to u in the ball $B_{\rho r}$ and Theorem 2.2, we obtain the following estimate

$$\begin{aligned}\frac{1}{\rho r}\int_{B_{\rho r}} |\nabla u|^2 dx &\le \int_{\partial B_{\rho r}} |\nabla u|^2 dx + \frac{6}{\rho r}\int_{B_{\rho r}} e^u dx \\ &\le C\rho^2\mathscr{E}_{x_0,2r}(u) + C\mathscr{E}_{x_0,2r}(u)(\mathscr{E}_{x_0,2r}(u) + (u)^+_{r,x_0}\mathscr{E}_{x_0,2r}(u) + \mathscr{E}_{x_0,2r}(u)) \\ &\quad + \frac{6}{\rho r}\int_{B_{\rho r}} e^u dx \\ &\le C\rho^2\mathscr{E}_{x_0,2r}(u) + 2C\mathscr{E}_{x_0,2r}(u)(\mathscr{E}_{x_0,2r}(u) + (u)^+_{r,x_0}\mathscr{E}_{x_0,2r}(u) + \mathscr{E}_{x_0,2r}(u)) \\ &\quad + 6\alpha\mathscr{E}_{x_0,2r}(u) \\ &\le (C\rho^2 + 6\alpha)\mathscr{E}_{x_0,2r}(u) + 2C\mathscr{E}_{x_0,2r}(u) \\ &\quad \times (\mathscr{E}_{x_0,2r}(u) + (u)^+_{r,x_0}\mathscr{E}_{x_0,2r}(u) + \mathscr{E}_{x_0,2r}(u))\end{aligned}$$

where α is the constant appearing in Theorem 2.2. We remark that we can always choose ρ_2 and α in Theorem 2.2 in such a way that $C\rho_2^2 + 6\alpha < \gamma < 1$. With this choice we also get

$$\frac{1}{\rho r}\int_{B_{\rho r}}|\nabla u|^2dx \le \gamma\mathscr{E}_{x_0,2r}(u)+C\mathscr{E}_{x_0,2r}(u)(\mathscr{E}_{x_0,2r}(u)+(u)^+_{r,x_0}\mathscr{E}_{x_0,2r}(u))$$

and we can conclude the proof of Theorem 2.3. □

3. Proofs of Theorems 2.1 and 1.1

In this section we give the proof of Theorems 2.1 and 1.1. We start by giving an estimate of the mean value $(u)_{r,x_0}$.

Lemma 3.1. *For all* $0 < r < s \le 1$ *the following estimate holds*

$$(u)_{r,x_0} \le (u)_{s,x_0} + \frac{1}{r}\int_{B_r} e^{u(x)}dx - \int_{B_s\setminus B_r}\frac{e^{u(x)}}{|x-x_0|}dx. \tag{40}$$

Proof. One can check that in the sense of distribution the following estimate holds

$$\frac{d}{dr}(u)_{r,x_0} = -\frac{1}{r^2}\int_{B_r} e^{u(x)}dx < 0. \tag{41}$$

Integrating (41) between r and $s > r$ we get

$$(u)_{r,x_0} = (u)_{s,x_0} + \frac{1}{r}\int_{B_r} e^{u(x)}dx - \frac{1}{s}\int_{B_s} e^u dx + \int_{B_s\setminus B_r}\frac{e^u}{|x-x_0|}dx, \tag{42}$$

and we conclude. □

Proof of Theorem 2.1. We split the proof in several steps.

Step 1. By combining Theorems 2.2 and 2.3 we can find $\rho\in[\rho_1,\rho_2]$ (independent on r) such that

$$\mathscr{E}_{x_0,\rho r}(u) \le \gamma\mathscr{E}_{x_0,2r}(u)+C\mathscr{E}_{x_0,2r}(u)\big(\mathscr{E}_{x_0,2r}(u)+(u)^+_{r,x_0}\mathscr{E}_{x_0,2r}(u)\big). \tag{43}$$

Indeed we observe that up to choosing η, γ and ρ_1,ρ_2 smaller, the constant ρ_1 always satisfies (43).

Up to replace ρ by $\rho/2$ and $2r$ by r we can rewrite (43) as follows

$$\mathscr{E}_{x_0,\rho r}(u) \le \gamma\mathscr{E}_{x_0,r}(u)+C\mathscr{E}_{x_0,r}(u)\big(\mathscr{E}_{x_0,r}(u)+(u)^+_{x_0,r/2}\mathscr{E}_{x_0,r}(u)\big). \tag{44}$$

We set $\tau_j=\rho^j r$, $a_j=\mathscr{E}_{x_0,\tau_j}(u)$ and $u_j=(u)_{x_0,\tau_j/2}$. We first estimate u_j. First of all we have

$$u_j \le u_0 + \frac{2}{\tau_0}\int_{B_{\tau_0/2}} e^u dx + \sum_{k=0}^{j}\frac{2}{\tau_{k+1}}\int_{B_{\tau_k}} e^u dx \tag{45}$$

$$\leq u_0 + \frac{2}{\tau_0}\int_{B_{\tau_0}} e^u dx + \frac{2}{\rho_1}\sum_{k=0}^{j}\mathscr{E}_{x_0,\tau_k}(u)$$

$$\leq u_0 + \frac{8}{\rho_1}\sum_{k=0}^{j}\mathscr{E}_{x_0,\tau_k}(u) \tag{46}$$

By plugging (45) in (44) we get

$$a_{j+1} \leq \gamma a_j + C a_j\left(a_j + a_j u_0 + \frac{8}{\rho_1} a_j\left(\sum_0^j a_k\right)\right) \tag{47}$$

The recursive formula (47) implies that if η is small enough then

$$a_j \leq a_0 \bar{\beta}^j,$$

for some $0 < \bar{\beta} < 1$. We deduce that for all $0 \leq s \leq r$ we also have

$$\mathscr{E}_{x_0,s}(u) \leq C s^{\beta},$$

with $0 < \beta < 1$ and C is a positive constant that may depend on r.

Step 2. *Claim*: The maps

$$\Omega \times (0,1] \to \mathbb{R} \quad (x_0, r) \mapsto \mathscr{E}_{x_0,r}(u)$$

and

$$\Omega \times (0,1] \to \mathbb{R}, \quad (x_0, r) \mapsto (u)_{r,x_0}$$

are continuous.

Proof of the Claim. The continuity of $(x_0, r) \mapsto \mathscr{E}_{x_0,2r}(u)$ follows from the fact that e^u and $|\nabla u|^2$ are in $L^1(\Omega)$.

As far as the continuity of $(x_0, r) \mapsto (u)_{r,x_0}$ is concerned, we observe that for $0 < r < 1$ we have

$$(u)_{r,x_0} = (u)_{1,x_0} + \frac{1}{r}\int_{B_r} e^{u(x)} dx - \int_{B_1} e^{u(x)} dx + \int_{B_1\setminus B_r} \frac{e^{u(x)}}{|x - x_0|} dx. \tag{48}$$

Therefore we need only to show the continuity of the map $x_0 \mapsto (u)_{1,x_0}$, being the other terms in the right hand side of (48) continuous with respect to (x_0, r).

The continuity of the map $x_0 \mapsto (u)_{1,x_0}$, follows from the fact it can be represented as the composition of the following three continuous maps.

The first map is: $\Omega \to W^{1,2}(B_{1,0})$, $x_0 \mapsto u(x - x_0)$.

The second map is the trace operator:

$$W^{1,2}(B_{1,0}) \to W^{1/2,2}(\partial B_{1,0}), \quad u \mapsto u|_{\partial B_{r,0}}.$$

The third one is the bounded linear operator

$$H^{1/2}(\partial B_{1,0}) \to \mathbb{R}, \quad w \mapsto \frac{1}{|\partial B_{1,0}|}\int_{\partial B_{1,0}} w\,dx.$$

Step 3. By the continuity of $(x,r) \mapsto \mathscr{E}_{x,2r}(u)$ and $(x,r) \mapsto (u)_{x,r}$ we can conclude that up to the choice of a smaller η, we can find $\varepsilon > 0$ such that for all $y \in B_{\varepsilon,x_0}$ and for $s \in (r-\varepsilon, r+\varepsilon)$ we have

$$\mathscr{E}_{y,2s}(u) < \eta, \quad \text{and} \quad (u)_{y,s}\mathscr{E}_{y,2s}(u) < \eta.$$

Finally by Theorems 2.2, 2.3 and Step 1 we get

$$\mathscr{E}_{s,y}(u) \le Cs^{\beta},$$

for all $y \in B_{\varepsilon,x_0}$ and for $s \le s_0$, s_0 being a constant independent on y (actually by changing C we could choose $s_0 = r$). Thus we can conclude the proof of Theorem 2.1. □

Proof of Theorem 1.1. Set

$$\mathscr{O} = \{x \in \Omega : \mathscr{E}_{x,2r}(u) < \eta \text{ and } (u)_{x,r}\mathscr{E}_{x,2r}(u) < \eta \text{ for some } 0 < r < d(x,\Omega)/2\}.$$

From Theorem 2.1 it follows that $\mathscr{O}$ is open. Moreover $u \in C^{\beta/2}(\mathscr{O})$, (see e.g., Giaquinta [5]), and routine elliptic regularity theory then proves that $u \in C^{\infty}(\mathscr{O})$.

We set

$$A_1 = \{x \in \Omega : \mathscr{E}_{x,r}(u) \ge \eta \text{ for all } 0 < r < d(x,\Omega)\}$$

and

$$A_2 = \{x \in \Omega : (u)_{x,r/2}\mathscr{E}_{x,r}(u) \ge \eta \text{ for all } 0 < r < d(x,\Omega)\}.$$

We have

$$V = \mathscr{O}^c = A_1 \cup \mathscr{A}_2.$$

Next we show that $\mathscr{H}^1(A_1) = 0$ and $\mathscr{H}^{1+\alpha}(A_2) = 0$ for any $\alpha > 0$.

1. $\mathscr{H}^1(A_1) = 0$.

Let $x \in A_1$. By definition we have

$$\mathscr{E}_{r,x}(u) \ge \eta \tag{49}$$

for all $0 < r < d(x,\Omega)$. Now fix $\delta > 0$ and define

$$\mathscr{F} = \left\{B_{r,x} : x \in A_1, 0 < r < \delta, B_{r,x} \subseteq \Omega, \text{ and } \int_{B_{r,x}} |\nabla u|^2 + e^u dx \ge \eta(r)\right\}$$

By Vitali–Besicovitch Covering Theorem (see for instance [4]), we can find an at most countable family of points $(x_0^i)_{i\in I}$, $x_0^i \in A_1$ and $0 < r_i < \delta$ such that $B_{r_i}(x_0^i) \in \mathscr{F}$

and $A_1 \subseteq \bigcup_{i\in I} B_{r_i}(x_0^i)$. Moreover every $x \in A_1$ is contained in at most N balls, N being a number depending only on the dimension of the space.

The following estimates holds

$$\eta \sum_{i\in I} r_i \leq \sum_{i\in I} \int_{B_{r_i}(x_0^i)} |\nabla u|^2 + e^u dx. \tag{50}$$

$$\begin{aligned}\sum_{i\in I} \int_{B_{r_i}(x_0^i)} |\nabla u|^2 + e^u dx &\leq \int_\Omega \sum_{i\in I} \mathbb{1}_{B_{r_i}(x_0^i)}(x)(|\nabla u|^2 + e^u) dx \\ &\leq N \int_{\{x: d(x,A_1)<\delta\}} (|\nabla u|^2 + e^u) dx \leq C, \end{aligned}\tag{51}$$

where the constant C does not depend on δ.

Claim: The estimates (50) and (51) imply that $\mathscr{H}^1(A_1) = 0$.

Proof of the Claim. By combining (50) and (51) and sending $\delta \to 0$, we get a priori that $\mathscr{H}^1(A_1) < +\infty$. As a consequence $|A_1| = \mathscr{H}^3(A_1) = 0$ and in particular $\lim_{\delta\to 0} |\{x : d(x, A_1) < \delta\}| = 0$. Hence since $(|\nabla u|^2 + e^u) \in L^1(\Omega)$, we also have

$$\lim_{\delta\to 0} \int_{\{x: d(x,A_1)<\delta\}} (|\nabla u|^2 + e^u) dx = 0.$$

Therefore by sending $\delta \to 0$ in (50) and (51) we actually get that that $\mathscr{H}^1(A_1) = 0$ and we conclude the proof of the claim.

2. $\mathscr{H}^{1+\alpha}(A_2) = 0$.

Let $x \in A_2$. By definition

$$(u)_{x,r} \mathscr{E}_{x,r}(u) \geq \eta \tag{52}$$

for all $0 < r < d(x, \Omega)$.

Let us consider $r > 0$ such that

$$\int_{\partial B_{r/2}} u\, dx \leq \frac{C}{r} \int_{B_r} u\, dx \leq Cr^{1/2}.$$

For such a r, Jensen's Inequality implies that

$$e^{(u)_{r,x}} \leq \frac{1}{|\partial B_{r,x}|} \int_{\partial B_{r,x}} e^u dx \leq \frac{C}{r^{3/2}}.$$

Thus

$$(u)_{r,x} \leq -C \log(r^{3/2}).$$

Therefore if $x \in A_2$, we can find $0 < r < d(x, \Omega)$ such that $-C \log(r^{3/2}) \mathscr{E}_{r,x}(u) \geq \eta$.

Now fix $\delta \in (0, 1)$ and define

$$\mathscr{F} = \left\{ B_{r,x} : x \in A_2, 0 < r < \delta, B_{r,x} \subseteq \Omega, \text{ and } \int_{B_{r,x}} |\nabla u|^2 + e^u dx \geq -C\eta r \log(r^{3/2}) \right\}$$

By Vitali–Besicovitch Covering Theorem we can find an at most countable family of points $(x_0^i)_{i\in I}$, $x_0^i \in A_2$ and $0 < r_i < \delta$ such that $B_{r_i}(x_0^i) \in \mathscr{F}$ and $A_2 \subseteq \bigcup_{i\in I} B_{r_i}(x_0^i)$. Moreover every $x \in A_2$ is contained in at most N balls, N being a number depending only on the dimension of the space.

We have

$$\begin{aligned} N\int_\Omega (e^u + |\nabla u|^2)dx &\geq N\int_{\{x:d(x,A_2)<\delta\}} (e^u + |\nabla u|^2)dx \\ &\leq \sum_{i\in I}\int_{B_{r_i}(x_0^i)} (e^u + |\nabla u|^2)dx \\ &\geq -\eta C\sum_{i\in I}(r_i)(\log(r_i))^{-1}. \end{aligned} \tag{53}$$

If δ is small enough, then (53) implies that for all $\theta > 1\ \sum_{i\in I}(r_i^\theta) < +\infty$ as well. This implies by definition $\mathscr{H}_{\dim}(A_2) \leq 1$.

It follows that $\mathscr{H}_{\dim}(V) \leq 1$ too and we conclude. □

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