Convex Hamilton-Jacobi equations under superlinear growth conditions on data

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March 29, 2010

Abstract

Unbounded stochastic control problems may lead to Hamilton-Jacobi-Bellman equations whose Hamiltonians are not always defined, especially when the diffusion term is unbounded with respect to the control. We obtain existence and uniqueness of viscosity solutions growing at most like $o(1+|x|^p)$ at infinity for such HJB equations and more generally for degenerate parabolic equations with a superlinear convex gradient nonlinearity. If the corresponding control problem has a bounded diffusion with respect to the control, then our results apply to a larger class of solutions, namely those growing like $O(1+|x|^p)$ at infinity. This latter case encompasses some equations related to backward stochastic differential equations.

Keywords. degenerate parabolic equations, Hamilton-Jacobi-Bellman equations, viscosity solutions, unbounded solutions, maximum principle, backward stochastic differential equations, unbounded stochastic control problems.

AMS subject classifications. 35K65, 49L25, 35B50, 35B37, 49N10, 60H35.

1 Introduction

In the joint paper [13] the authors obtain a comparison result between semicontinuous viscosity solutions, neither bounded from below nor from above, growing at most quadratically in the state variable, of second order degenerate parabolic equations of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} + H(x,t,Du,D^2u) = 0 & \text{in } I\!\!R^N \times (0,T), \\ u(x,0) = \psi(x) & \text{in } I\!\!R^N, \end{cases} \tag{1}$$

where

$$H(x,t,q,X) = \sup_{\alpha \in A} \left\{ -\langle b(x,t,\alpha), q\rangle - \ell(x,t,\alpha) - \mathrm{Trace}\left[\sigma(x,t,\alpha)\sigma^T(x,t,\alpha)X\right] \right\} \tag{2}$$

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is convex with respect to the gradient variable. The main point is that A is non compact and therefore (1) is the Hamilton-Jacobi-Bellman equation associated with stochastic control problems with unbounded controls.

In [13], the functions b and ℓ grow respectively at most linearly and quadratically with respect to both the control and the state. Instead, the diffusion σ is assumed to grow at most linearly with respect to the state but is bounded with respect to the control. One of the issue in this paper, which is the main contribution, is to be able to deal with (1) when σ is unbounded with respect to the control.

Let us explain what are the main difficulty and motivation through the example of the well-known stochastic linear quadratic problem (SLQ in short, see Example 2.1 for details). When trying to use a PDE approach for such kind of problem, take for simplicity $A = I\!R^N$, $b = \alpha$, $\sigma = |\alpha| I$ and $\ell = |\alpha|^2$, the Hamiltonian (2) becomes

$$\sup_{\alpha \in I\!R^N} \{-\langle \alpha, Du\rangle - |\alpha|^2 - \frac{|\alpha|^2}{2}\Delta u\}, \tag{3}$$

which is $+\infty$ as soon as $\Delta u < -2$. Therefore (2) is not always defined. In the literature, this difficulty is overcomed by using essentially a control approach, plugging into the equation value functions V of particular form (for instance quadratic in space) for which one knows that $H(x, t, V, DV)$ is defined. This procedure leads to some ordinary differential equations of Ricatti type which allow to determine explicitely the value function (see [25]). Up to our knowledges, the only work so far using directly (1) is the one of Krylov [21]. He succeeded in dealing with equations encompassing the classical SLQ problem but his assumptions are designed to handle exactly this case.

The main result of this paper is to propose a new way to deal with the PDE (1) with diffusion matrix σ which is unbounded with respect to the control. Our strategy consists in noticing that we can express the definition of viscosity solutions of (1) without writing the "sup" in (2) (see Definition 2.1). By this way, we can handle functions u which are not in the domain of H. We then obtain comparison, existence and uniqueness for viscosity solutions in the class of functions growing at most like $o(1 + |x|^p)$ for $p > 1$. Comparing to [21] (which addresses the very SLQ problem where $p = 2$), we are not able to treat the case $O(1 + |x|^2)$ but we stress that our results apply to very general nonlinear datas (not only polynomials of degree 1 or 2 in (x, α)), see Example 2.1 and the discussion therein. The question to know if it is possible to extend such study to $O(1 + |x|^p)$ is so far open.

The second issue of the present paper is to extend the results of [13] for p-growth type conditions on the datas and the solutions and for more general equations with an additional nonlinearity f which is convex with respect to the gradient and depends on u. The typical case is

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1}{2}\mathrm{Trace}(\sigma\sigma^T D^2 u) - \langle b(x,t), Du\rangle + f(x, t, u, s(x,t)Du) = 0 & \text{in } I\!R^N \times (0,T) \\ u(x, 0) = \psi(x) & \text{in } I\!R^N. \end{cases} \tag{4}$$

where

$$f(x, t, u, s(x,t)Du) = g(x, t, u) + |s(x,t)Du|^{p'}, \quad p' > 1,$$

see Examples 3.1 and 3.2. The presence of x in the power-p' term is delicate to treat. The u-dependence in g is Lipschitz continuous which is more classical but induces some technical difficulties.

The motivation comes from PDEs arising in the context of backward stochastic differential equations (BSDEs in short). We prove the analytical counterpart of some results of Briand and Hu [9] in the framework of BDSEs (see Example 3.4). The proof is in the spirit of the one of [13] but, since it is not straightforward extension, we provide a readable self-contained proof and we hope that people will be able to adapt the techniques to different particular cases.

Let us mention that the convexity of the operator with respect to the gradient is crucial in our proofs. The case of unbounded solutions to equations with superlinear Hamiltonians which are neither convex nor concave (which, in the case of Equations (13), amounts to take both the control sets A and B unbounded) is also of interest and it is a widely open subject. Some results in this direction were obtained in [13, Section 4] and we want to investigate this issue in a forthcoming paper.

The paper is organized as follows. Section 2 is devoted to the study (1)-(2) with diffusion matrices σ which depend on the control in an unbounded way. We prove the comparison result Theorem 2.1 which is the main result of this part. As by-product, we obtain the existence and uniqueness of a continuous solution which grows at most as $o(1+|x|^p)$ at infinity for all $p>1$ (Theorem 2.2). Applications to SLQ type problems are discussed in Example 2.1. In Section 3, we consider (4). As above, we prove comparison (Theorem 3.1), existence and uniqueness (Theorem 3.2) for solutions growing at most $O(1+|x|^p)$ (which is a better growth condition than in the first case). Application to BDSEs is addressed in Example 3.4.

We conclude this Section by mentioning some results in literature in the case of unbounded Hamiltonians and solutions. Uniqueness and existence problems for a class of first-order Hamiltonians corresponding to unbounded control sets and under assumptions including deterministic linear quadratic problems have been addressed by several authors, see, e.g. the book of Bensoussan [8], the papers of Alvarez [2], Bardi and Da Lio [4], Cannarsa and Da Prato [10], Rampazzo and Sartori [24] in the case of convex operators, and the papers of Da Lio and McEneaney [14] and Ishii [18] for more general operators. As for second-order Hamiltonians under quadratic growth assumptions, Ito [19] obtained the existence of locally Lipschitz solutions to particular equations of the form (1) under more regularity conditions on the data, by establishing a priori estimates on the solutions. Whereas Crandall and Lions in [12] proved a uniqueness result for very particular operators depending only on the Hessian matrix of the solution. In the case of quasilinear degenerate parabolic equations, existence and uniqueness results for viscosity solutions which may have a quadratic growth are proved in [7].

Throughout the paper we will use the following notations. For all integer $N, M \geq 1$ we denote by $\mathcal{M}_{N,M}(I\!\!R)$ (respectively $\mathcal{S}_N(I\!\!R)$, $\mathcal{S}_N^+(I\!\!R)$) the set of real $N\times M$ matrices (respectively real symmetric matrices, real symmetric nonnegative $N\times N$ matrices). For the sake of notations, all the norms which appear in the sequel are denoting by $|\cdot|$. The standard Euclidean inner product in $I\!\!R^N$ is written $\langle\cdot,\cdot\rangle$. We recall that a modulus of continuity $m: I\!\!R \to I\!\!R^+$ is a nondecreasing continuous function such that $m(0)=0$. We set $B(0,R)=\{x\in I\!\!R^N : |x|<R\}$. Finally for any $O\subseteq I\!\!R^K$, we denote by $USC(O)$ the set of upper semicontinuous functions in O and by $LSC(O)$ the set of lower semicontinuous functions in O. Given $p>1$ we will denote by p' its conjugate, namely

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Acknowledgments. Part of this work was done while the second author was a visitor at the FIM at the ETH in Zürich in January 2007. He would like to thank the Department of Mathematics for his support. We thank Guy Barles for useful comments on the first version of this paper.

2 Hamilton-Jacobi-Bellman equations with unbounded diffusion in the control

In this Section we prove a comparison result for second-order fully nonlinear partial differential equations of the form (1) with an additional nonlinearity, that is

$$\begin{cases} \dfrac{\partial u}{\partial t} + H(x,t,Du,D^2u) + f(x,t,u,s(x,t)Du) = 0 & \text{in } I\!\!R^N \times (0,T), \\ u(x,0) = \psi(x) & \text{in } I\!\!R^N, \end{cases} \tag{5}$$

where H is given by (2). The main difference with respect to the result in [13] is that here we suppose that the diffusion matrix σ depends in an unbounded way in the control (see condition (6)). The compensation to the condition (6) is that we are able to get the uniqueness result in the smaller class of functions which are $o(1+|x|^p)$, $p>1$ as $|x| \to \infty$ (see (9)) instead of $O(1+|x|^2)$. Note that the case $p \le 1$ enters in the by now classical theory of viscosity solution.

We list below the main assumptions on H and f $(p>1)$.

(A) (Assumption on H) :

(i) A is a subset of a separable complete normed space. The main point here is the possible unboundedness of A.

(ii) $b \in C(I\!\!R^N \times [0,T] \times A; I\!\!R^N)$ and there exists $C_b > 0$ such that, for all $x,y \in I\!\!R^N$, $t \in [0,T]$, $\alpha \in A$,

$$\begin{aligned} |b(x,t,\alpha) - b(y,t,\alpha)| &\le C_b(1+|\alpha|)|x-y|, \\ |b(x,t,\alpha)| &\le C_b(1+|x|+|\alpha|)\ ; \end{aligned}$$

(iii) $\ell \in C(I\!\!R^N \times [0,T] \times A; I\!\!R)$ and, there exist $p>1$ and $C_\ell, \nu > 0$ such that, for all $x \in I\!\!R^N$, $t \in [0,T]$, $\alpha \in A$,

$$C_\ell(1+|x|^p+|\alpha|^p) \ge \ell(x,t,\alpha) \ge \nu|\alpha|^p - C_\ell(1+|x|^p)$$

and for every $R>0$, there exists a modulus of continuity m_R such that for all $x,y \in B(0,R)$, $t \in [0,T]$, $\alpha \in A$,

$$|\ell(x,t,\alpha) - \ell(y,t,\alpha)| \le (1+|\alpha|^p)\, m_R(|x-y|)\ ;$$

(iv) $\sigma \in C(I\!R^N \times [0,T] \times A; \mathcal{M}_{N,M}(I\!R))$ is Lipschitz continuous with respect to x with a constant independent of (t,α): namely, there exists $C_\sigma > 0$ such that, for all $x, y \in I\!R^N$ and $(t,\alpha) \in [0,T] \times A$,

$$|\sigma(x,t,\alpha) - \sigma(y,t,\alpha)| \le C_\sigma |x-y|,$$

and satisfies for every $x \in I\!R^N$, $t \in [0,T]$, $\alpha \in A$,

$$|\sigma(x,t,\alpha)| \le C_\sigma(1+|x|+|\alpha|). \tag{6}$$

(B) (Assumption on f)
$f \in C([0,T] \times I\!R^N \times I\!R \times I\!R^N; I\!R)$ and, for all $R>0$, there exist a modulus of continuity m_R and $C_s, \hat{C} > 0$ such that, for all $t \in [0,T]$, $x, y \in I\!R^N$, $u, v \in I\!R$, $z \in I\!R^N$,

(i) $|f(x,t,u,z)| \le C_f(1+|x|^p+|u|+|z|^{p'})$,
(ii) $|f(x,t,u,z) - f(y,t,u,z)| \le m_R((1+|u|+|z|)|x-y|)$ if $|x|+|y| \le R$,
(iii) $z \mapsto f(x,t,u,z)$ is convex,
(iv) $s \in C(I\!R^N \times [0,T]; \mathcal{M}_N)$, $|s(x,t) - s(y,t)| \le C_s|x-y|$, $|s(x,t)| \le C_s$,
(v) $|f(x,t,u,z) - f(x,t,v,z)| \le \hat{C}|u-v|$.

The typical case we have in mind in the context of **(A)**(iv) (σ not bounded with respect to the control) is

$$\sigma(x,t,\alpha) = Q(t)x + R(t)\alpha,$$

where $Q(t)$ and $R(t)$ are matrices of suitable sizes. This case includes Linear Quadratic control problems, see Example 2.1.

Under the current hypotheses, the Hamiltonian H may be infinite (see Example 2.1) and for this reason we re-formulate the definition of viscosity solution in the following way.

Definition 2.1
(i) A function $u \in USC(I\!R^N \times [0,T])$ is a viscosity subsolution of (5) *if for all $(x,t) \in I\!R^N \times [0,T]$ and $\varphi \in C^2(I\!R^N \times [0,T])$ such that $u-\varphi$ has a maximum at (x,t), we have $u(x,t) \le \psi(x)$ if $t=0$ and, if $t>0$, then*

$$\frac{\partial \varphi}{\partial t}(x,t) + H(x,t,D\varphi(x,t),D^2\varphi(x,t)) + f(x,t,u(x,t),s(x,t)D\varphi(x,t)) \le 0,$$

which is equivalent to: for all $\alpha \in A$,

$$\frac{\partial \varphi}{\partial t}(x,t) - \langle b(x,t,\alpha), D\varphi(x,t)\rangle - \ell(x,t,\alpha) - \mathrm{Trace}\left[\sigma(x,t,\alpha)\sigma^T(x,t,\alpha)D^2\varphi(x,t)\right]$$
$$+ f(x,t,u(x,t),s(x,t)D\varphi(x,t)) \le 0. \tag{7}$$

(ii) A function $u \in USC(I\!R^N \times [0,T])$ is a viscosity supersolution of (5) *if for all $(x,t) \in I\!R^N \times [0,T]$ and $\varphi \in C^2(I\!R^N \times [0,T])$ such that $u-\varphi$ has a minimum at (x,t), we have*

$u(x,t) \geq \psi(x)$ if $t = 0$ and, if $t > 0$, then for all $\eta > 0$, there exists $\alpha_\eta = \alpha(\eta, x, t) \in A$, such that

$$\begin{aligned}\frac{\partial \varphi}{\partial t}(x,t) - \langle b(x,t,\alpha_\eta), D\varphi(x,t)\rangle - \ell(x,t,\alpha_\eta) - \text{Trace}\left[\sigma(x,t,\alpha_\eta)\sigma^T(x,t,\alpha_\eta)D^2\varphi(x,t)\right] \\ + f(x,t,u(x,t),s(x,t)D\varphi(x,t)) \geq -\eta.\end{aligned} \tag{8}$$

(iii) A locally bounded function $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ is a viscosity solution of (5) *if its USC envelope u^* is a subsolution and its LSC envelope u_* is a supersolution.*

Note that (7) and (8) is only a way to write the definition of sub- and supersolutions without writing a supremum which could not exist because of assumption (6).

We say that a function $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ is in the class $\mathcal{C}_p$ if

$$\frac{u(x,t)}{1+|x|^p} \underset{|x|\to+\infty}{\longrightarrow} 0, \quad \text{uniformly with respect to } t \in [0,T]. \tag{9}$$

Note that $u \in \mathcal{C}_p$ if and only if, for all $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$|u(x,t)| \leq M_\varepsilon + \varepsilon(1+|x|^p) \quad \text{for all } (x,t) \in \mathbb{R}^N \times [0,T].$$

In particular, for all $\lambda > 0$,

$$\sup_{x\in\mathbb{R}^N} \{u(x,t) - \lambda(1+|x|^p)\} = M_\lambda < +\infty. \tag{10}$$

The main result of this Section is the

Theorem 2.1 *Let $p > 1$, assume **(A)-(B)** and suppose that ψ is a continuous function which belongs to $\mathcal{C}_p$. Let $u \in USC(\mathbb{R}^N \times [0,T])$ be a viscosity subsolution of* (5) *and $v \in LSC(\mathbb{R}^N \times [0,T])$ be a viscosity supersolution of* (5)*. Suppose that u and v are in the class $\mathcal{C}_p$ defined by* (9) *and satisfy $u(x,0) \leq \psi(x) \leq v(x,0)$. Then $u \leq v$ in $\mathbb{R}^N \times [0,T]$.*

Before giving the proof of the theorem, let us state an existence result and some examples of applications. As it was already observed in [13], the question of the existence of a continuous solution to (5) is not completely obvious and in general the solutions may exist only for short time (see Example 3.3). One way to obtain the existence is to establish a link between the solution of the PDE and related control problems or BDSE systems which have a solution. By using PDE methods, in the framework of viscosity solutions, the existence is usually a consequence of the comparison principle by means of Perron's method, as soon as we can build a sub- and a super-solution to the problem. Here, the comparison principle is proved in the class of functions belonging to $\mathcal{C}_p$. Therefore, to prove the existence, it suffices to build sub- and super-solutions to (35) in $\mathcal{C}_p$. We need to strengthen **(A)**(iii) and **(B)**(i) by assuming that $\ell(\cdot,t,\alpha), f(\cdot,t,u,z) \in \mathcal{C}_p$ uniformly with respect to α, t, u, z, i.e., for all $(x,t,\alpha,u,z) \in \mathbb{R}^N \times [0,T] \times A \times \mathbb{R} \times \mathbb{R}^N$,

$$\begin{aligned}\chi(x) \geq \ell(x,t,\alpha) \geq \nu|\alpha|^p - \chi(x), \quad |f(x,t,u,z)| \leq C_f(1+\gamma(x)+|u|+|z|^{p'}), \\ \text{and} \quad \lim_{|x|\to+\infty} \frac{\chi(x)}{1+|x|^p}, \frac{\gamma(x)}{1+|x|^p} = 0.\end{aligned} \tag{11}$$

We have

Theorem 2.2 *Let $p > 1$, assume **(A)**–**(B)** and* (11). *For all $\psi \in \mathcal{C}_p$, there is $\tau > 0$ such that there exist a subsolution $\underline{u} \in \mathcal{C}_p$ and a supersolution $\overline{u} \in \mathcal{C}_p$ of* (5) *in $I\!R^N \times [0, \tau]$. In consequence, Equation* (5) *has a unique continuous viscosity solution in $I\!R^N \times [0, \tau]$ in the class $\mathcal{C}_p$.*

The proof of this theorem is postponed at the end of the section.

Example 2.1 (A Stochastic Linear Quadratic Control Problem) SLQ-type problems are extensively studied in the literature both from the control and PDE point of view, see for instance Bensoussan [8], Fleming and Rishel [15], Fleming and Soner [16], Øksendal [22], Yong and Zhou [25] and the references therein for an overview of this problem. Let us describe a simple case. Consider the stochastic differential equation (in dimension 1 for sake of simplicity)

$$\begin{cases} dX_s = X_s ds + \sqrt{2}\alpha_s dW_s, & t \le s \le T,\ t \in (0, T], \\ X_0 = x \in I\!R, \end{cases}$$

where W_s is a standard Brownian motion, $(\alpha_s)_s$ is a real valued progressively measurable process and the value function is given by

$$V(x, t) = \inf_{(\alpha_s)_s} E_{tx} \left\{ \int_t^T |\alpha_s|^2 \, ds + \psi(X_T) \right\}.$$

(Note that in this case, $p = p' = 2$.) The Hamilton-Jacobi equation formally associated to this problem is

$$\begin{cases} -u_t + \sup_{\alpha \in I\!R} \{-\alpha^2(u'' + 1)\} - xu' = 0 & \text{in } I\!R \times (0, T], \\ u(x, T) = \psi(x) & \text{in } I\!R. \end{cases} \tag{12}$$

We observe that in this case if $u'' + 1 < 0$ then the Hamiltonian becomes $+\infty$. Nevertheless, we are able to prove that (12) has a unique continuous viscosity solution u as soon as the terminal cost $\psi \in \mathcal{C}_2$ (i.e., has a strictly sub-p growth). This is not completely satisfactory since, in the classical Linear Quadratic Control Problem, one expects to have quadratic terminal costs like $\psi(x) = |x|^2$. Let us mention that Krylov [21] succeeded in treating this latter case. But his proof consists on some algebraic computations which rely heavily on the particular form of the data (the datas are supposed to be polynomials of degree 1 or degree 2 in (x, α)). He proves that V is the unique viscosity solution in $\tilde{\mathcal{C}}_2$ (see Definition in 36) if $\psi(x) = |x|^2$. In our case, up to restrict slightly the growth, we are able to deal with general datas. Moreover, with a straightforward adaptation of the results of [13, Section 3] we obtain that $u = V$, providing a characterization of the value function. Let us mention that the rigorous connection between control problems and Hamilton-Jacobi-Bellman equations in the framework of unbounded controls may be rather delicate. Besides the above results, some results in this direction were obtained for infinite horizon in the deterministic case by Barles [5] and in the stochastic case by Alvarez [1, 2].

Remark 2.1 With straightforward adaptations, Theorems 2.1 and 3.1 still hold for the Isaacs equation of [13],

$$\frac{\partial u}{\partial t} + H(x, t, Du, D^2u) + G(x, t, Du, D^2u) + f(x, t, u, s(x, t)Du) = 0 \tag{13}$$

where

$$G(x,t,q,X) = \inf_{\beta \in B} \left\{ -\langle g(x,t,\beta), q \rangle - l(x,t,\beta) - \text{Trace}\left[c(x,t,\beta) c^T(x,t,\beta) X \right] \right\},$$

is a concave Hamiltonian, B is bounded, g, l, c satisfy respectively **(A)**(ii),(iii),(iv) (with bounded controls β). The case where both the control sets A and B are unbounded is rather delicate. It is the aim of a future work.

Let us turn to the proof of the comparison theorem.

Proof of Theorem 2.1. Our result is stated for $p > 1$ but we do the proof for $p \geq 2$ which is the more difficult case. The proof of the case $p < 2$ follows the same line with simpler arguments.

We are going to show that for every $\mu \in (0,1)$, $\mu u - v \leq 0$, in $I\!R^N \times [0,T]$. To this end we argue by contradiction assuming that there exists $(\hat{x}, \hat{t}) \in I\!R^N \times [0,T]$ such that

$$u(\hat{x}, \hat{t}) - v(\hat{y}, \hat{t}) > \delta > 0. \tag{14}$$

We divide the proof in several steps.

1. The μ-equation for the subsolution. If u is a subsolution of (5), then $\tilde{u} = \mu u$ is a subsolution of

$$\begin{aligned}
&\tilde{u}_t + \sup_{\alpha \in A} \{ -\text{Trace}\big(\sigma(x,t,\alpha)\sigma(x,t,\alpha)^T D^2 \tilde{u}\big) + \langle b(x,t,\alpha), D\tilde{u} \rangle - \mu \ell(x,t,\alpha) \} \\
&+ \mu f\big(x, t, \frac{1}{\mu}\tilde{u}(x,t), \frac{1}{\mu} s(x,t) D\tilde{u}\big) \leq 0,
\end{aligned}$$

with the initial condition $\mu u(x,0) \leq \mu \psi(x)$.

2. Test-function and estimates on the penalization terms. For all $\varepsilon > 0$, $\eta > 0$ and $\theta, L > 0$ (to be chosen later) we consider the auxiliary function

$$\Phi(x,y,t) = \mu u(x,t) - v(y,t) - e^{Lt}\big(\frac{|x-y|^2}{\varepsilon^2} + \theta(1-\mu)(1 + |x|^2 + |y|^2)^{p/2} \big) - \rho t.$$

Since $u, v \in \mathcal{C}_p$, the supremum of Φ in $I\!R^N \times I\!R^N \times [0,T]$ is achieved at a point $(\bar{x}, \bar{y}, \bar{t})$. We will drop for simplicity of notation the dependence on the various parameters. If θ and ρ are small enough we have

$$\Phi(\bar{x}, \bar{y}, \bar{t}) \geq \mu u(\hat{x}, \hat{y}) - v(\hat{x}, \hat{t}) - \theta(1-\mu)(1 + 2|\hat{x}|^p) - \rho \hat{t} > \frac{\delta}{2},$$

which implies

$$\frac{|\bar{x} - \bar{y}|^2}{\varepsilon^2} + \theta(1-\mu)(1 + |\bar{x}|^2 + |\bar{y}|^2)^{p/2} \leq \mu u(\bar{x}, \bar{t}) - v(\bar{y}, \bar{t}).$$

Therefore, by (10), we get

$$
\begin{aligned}
& \frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} + \theta\frac{1-\mu}{2}(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2} \\
\leq & \sup_{(x,t)\in I\!R^N\times[0,T]} \{\mu u(x,t) - \theta\frac{1-\mu}{2}(1+|x|^p)\} + \sup_{(x,t)\in I\!R^N\times[0,T]} \{-v(x,t) - \theta\frac{1-\mu}{2}(1+|x|^p)\} \\
\leq & M
\end{aligned}
$$

for some $0 < M = M(\mu,\theta,u,v)$. Thus

$$|\bar{x}|, |\bar{y}| \leq R_{\mu,\theta} \tag{15}$$

with $R_{\mu,\theta}$ independent of ε and $|\bar{x}-\bar{y}| \to 0$ as $\varepsilon \to 0$. Up to extract a subsequence, we can assume that

$$\bar{x}, \bar{y} \to x_0 \in \overline{B}(0, R_{\mu,\theta}), \quad \bar{t} \to t_0 \text{ as } \varepsilon \to 0 \tag{16}$$

Actually we can obtain a more precise estimate: we have

$$\Phi(\bar{x},\bar{y},\bar{t}) \geq \max_{I\!R^N\times[0,T]} \{\mu u(x,t) - v(x,t) - e^{Lt}\theta(1-\mu)(1+2|x|^2)^{p/2} - \rho t\} := M_{\mu,\theta}.$$

Thus

$$\liminf_{\varepsilon\to 0} \Phi(\bar{x},\bar{y},\bar{t}) \geq M_{\mu,\theta}.$$

On the other hand

$$
\begin{aligned}
& \limsup_{\varepsilon\to 0} \Phi(\bar{x},\bar{y},\bar{t}) \\
& \leq \limsup_{\varepsilon\to 0} [\mu u(\bar{x},\bar{t}) - v(\bar{y},\bar{t}) - e^{L\bar{t}}\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2} - \rho\bar{t}] - \liminf_{\varepsilon\to 0} e^{L\bar{t}}\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} \\
& \leq M_{\mu,\theta} - \liminf_{\varepsilon\to 0} e^{L\bar{t}}\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2}.
\end{aligned}
$$

By combining the above inequalities we get, up to subsequences, that

$$\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} \to 0 \text{ as } \varepsilon \to 0. \tag{17}$$

Note that we have

$$|\bar{x}-\bar{y}|, \frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} = m(\varepsilon), \tag{18}$$

where m denotes a modulus of continuity independent of ε (but which depends on θ, μ).

3. Ishii matricial theorem and viscosity inequalities. We set

$$\Theta(x,y,t) = e^{Lt}\big(\frac{|x-y|^2}{\varepsilon^2} + \theta(1-\mu)(1+|x|^2+|y|^2)^{p/2}\big) + \rho t.$$

We claim that there is a subsequence ε_n such that $\bar{t} = 0$. Suppose by contradiction that for all $\varepsilon > 0$ we have $\bar{t} > 0$. Next Steps are devoted to prove some estimates in order to obtain the desired contradiction at the end of Step 8.

By Theorem 8.3 in the User's guide [11], for every $\varrho > 0$, there exist $a_1, a_2 \in I\!\!R$ and $X, Y \in \mathcal{S}_N$ such that

$$\begin{aligned}(a_1, D_x\Theta(\bar{x}, \bar{y}, \bar{t}), X) &\in \bar{\mathcal{P}}^{2,+}(\mu u)(\bar{x}, \bar{t}),\\ (a_2, -D_y\Theta(\bar{x}, \bar{y}, \bar{t}), Y) &\in \bar{\mathcal{P}}^{2,-}(v)(\bar{y}, \bar{t}),\\ a_1 - a_2 &= \Theta_t(\bar{x}, \bar{y}, \bar{t}),\end{aligned}$$

and

$$-(\frac{1}{\varrho} + |M|)I \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le M + \varrho M^2$$

where $M = D^2\Theta(\bar{x}, \bar{y}, \bar{t})$. Note that

$$a_1 - a_2 = Le^{L\bar{t}}(\frac{|\bar{x} - \bar{y}|^2}{\varepsilon^2} + \theta(1 - \mu)(1 + |\bar{x}|^2 + |\bar{y}|^2)^{p/2}) + \rho,$$

and, setting $p_\varepsilon = 2e^{L\bar{t}}\frac{\bar{x} - \bar{y}}{\varepsilon^2}$, $q_x = e^{L\bar{t}}p\theta(1 - \mu)\bar{x}(1 + |\bar{x}|^2 + |\bar{y}|^2)^{p/2-1}$, $q_y = -e^{L\bar{t}}p\theta(1 - \mu)\bar{y}(1 + |\bar{x}|^2 + |\bar{y}|^2)^{p/2-1}$ we have

$$D_x\Theta(\bar{x}, \bar{y}, \bar{t}) = p_\varepsilon + q_x \quad \text{and} \quad D_y\Theta(\bar{x}, \bar{y}, \bar{t}) = -p_\varepsilon - q_y,$$

and

$$M = A_1 + A_2 + A_3$$

where

$$\begin{aligned}A_1 &= \frac{2e^{L\bar{t}}}{\varepsilon^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix},\\ A_2 &= e^{L\bar{t}}p\theta(1 - \mu)(1 + |\bar{x}|^2 + |\bar{y}|^2)^{p/2-1}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},\\ A_3 &= e^{L\bar{t}}p(p - 2)\theta(1 - \mu)(1 + |\bar{x}|^2 + |\bar{y}|^2)^{p/2-2}\begin{pmatrix} x \otimes x & x \otimes y \\ x \otimes y & y \otimes y \end{pmatrix}.\end{aligned}$$

It follows

$$\begin{aligned}\langle X\xi, \xi\rangle - \langle Y\zeta, \zeta\rangle \ \le\ & \frac{2e^{L\bar{t}}}{\varepsilon^2}|\xi - \zeta|^2\\ & + e^{L\bar{t}}p\theta(1 - \mu)(1 + |\bar{x}|^2 + |\bar{y}|^2)^{p/2-1}(|\xi|^2 + |\zeta|^2)\\ & + 2e^{L\bar{t}}p(p - 2)\theta(1 - \mu)(1 + |\bar{x}|^2 + |\bar{y}|^2)^{p/2-2}\left(\langle\xi, x\rangle^2 + \langle\zeta, y\rangle^2\right)\\ & + m\left(\frac{\varrho}{\varepsilon^4}\right),\end{aligned} \qquad (19)$$

where m is a modulus of continuity which is independent of ρ and ε.

We now write the viscosity inequalities satisfied by the subsolution μu and the supersolution v (recall that we assume $\bar{t} > 0$).

For all $\alpha \in A$ we have

$$\begin{aligned} a_1 &- \mathrm{Trace}(\sigma(\bar{x}, \bar{t}, \alpha)\sigma(\bar{x}, \bar{t}, \alpha)^T X) + \langle b(\bar{x}, \bar{t}, \alpha), p_\varepsilon + q_x \rangle - \mu \ell(\bar{x}, \bar{t}, \alpha) \\ &+ \mu f(\bar{x}, \bar{t}, u(\bar{x}, \bar{t}), \frac{1}{\mu} s(\bar{x}, \bar{t})(p_\varepsilon + q_x)) \leq 0. \end{aligned} \tag{20}$$

On the other hand, for all $\eta > 0$, there exists $\alpha_\eta \in A$ such that

$$\begin{aligned} a_2 &- \mathrm{Trace}(\sigma(\bar{y}, \bar{t}, \alpha_\eta)\sigma^T(\bar{y}, \bar{t}, \alpha_\eta) Y) + \langle b(\bar{y}, \bar{t}, \alpha_\eta), p_\varepsilon + q_y \rangle - \ell(\bar{y}, \bar{t}, \alpha_\eta) \\ &+ f(\bar{y}, \bar{t}, v(\bar{y}, \bar{t}), \frac{1}{\mu} s(\bar{y}, \bar{t})(p_\varepsilon + q_y)) \geq -\eta. \end{aligned} \tag{21}$$

We set for simplicity

$$\begin{aligned} &\sigma_x := \sigma(\bar{x}, \bar{t}, \alpha_\eta), \ \ \sigma_y = \sigma(\bar{y}, \bar{t}, \alpha_\eta) \\ &b_x = b(\bar{x}, \bar{t}, \alpha_\eta), \ \ b_y = b(\bar{y}, \bar{t}, \alpha_\eta), \ \ s_x = s(\bar{x}, \bar{t}), \ \ s_y = s(\bar{y}, \bar{t}). \end{aligned}$$

By subtracting (20) and (21) we get

$$\begin{aligned} & Le^{L\bar{t}}\big(\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} + \theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2}\big) + \rho \\ \leq \ & \mathrm{Trace}(\sigma_x \sigma_x^T X - \sigma_y \sigma_y^T Y) + \langle b_y, p_\varepsilon + q_y \rangle - \langle b_x, p_\varepsilon + q_x \rangle \\ & -\ell(\bar{y}, \bar{t}, \alpha_\eta) + \mu \ell(\bar{x}, \bar{t}, \alpha_\eta) \\ & + f(\bar{y}, \bar{t}, v(\bar{y}, \bar{t}), s_y(p_\varepsilon + q_y)) - \mu f(\bar{x}, \bar{t}, u(\bar{x}, \bar{t}), \frac{1}{\mu} s_x(p_\varepsilon + q_x)) + \eta. \end{aligned} \tag{22}$$

4. Estimates of the second-order terms. From (19) and **(A)**(iv), it follows

$$\begin{aligned} & \mathrm{Trace}\left[\sigma_x {\sigma_x}^T X - \sigma_y {\sigma_y}^T Y\right] - m\left(\frac{\varrho}{\varepsilon^4}\right) \\ \leq \ & e^{L\bar{t}}\Bigg(\frac{2}{\varepsilon^2}|\sigma_x - \sigma_y|^2 + p\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2-1}(|\sigma_x|^2 + |\sigma_y|^2) \\ & \qquad + 2p(p-2)\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2-2}(|\sigma_x|^2|x|^2 + |\sigma_y|^2|y|^2)\Bigg) \\ \leq \ & 2C_\sigma^2 e^{L\bar{t}}\left(\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} + p(p-1)\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2-1}(1+|\bar{x}|^2+|\bar{y}|^2+|\alpha_\eta|^2)\right) \\ \leq \ & 2C_\sigma^2 e^{L\bar{t}}\Bigg(\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} + p(p-1)\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2} \\ & \qquad + p(p-1)\theta(1-\mu)|\alpha_\eta|^2(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2-1}\Bigg). \end{aligned}$$

By Young's inequality for $p > 2$ (for $p = 2$ the inequality is obvious),

$$|\alpha_\eta|^2(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2-1} \leq \frac{2}{p}|\alpha_\eta|^p + \frac{p-2}{p}(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2}.$$

It follows, using (18),

$$\begin{array}{rcl}\mathrm{Trace}\left[\sigma_x{\sigma_x}^T X - \sigma_y{\sigma_y}^T Y\right] & \leq & 4(p-1)^2 C_\sigma^2 e^{L\bar{t}}\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2} \\ & & +4(p-1)C_\sigma^2 e^{L\bar{t}}\theta(1-\mu)|\alpha_\eta|^p + m(\varepsilon) + m\left(\frac{\varrho}{\varepsilon^4}\right). \quad (23)\end{array}$$

5. Estimates of the drift terms. By using **(A)**(ii) and, from (18), by taking ε small enough in order that $|\bar{x}-\bar{y}| \leq 1$, we get

$$\begin{array}{rl} & \langle b_y, p_\varepsilon + q_y\rangle - \langle b_x, p_\varepsilon + q_x\rangle \\ \leq & \langle b_y - b_x, p_\varepsilon + q_y\rangle + \langle b_x, q_y - q_x\rangle \\ \leq & |b_y - b_x||p_\varepsilon| + |b_y - b_x||q_y| + |b_x||q_x - q_y| \\ \leq & C_b e^{L\bar{t}}\Bigg(2(1+|\alpha_\eta|)\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} + 2p\theta(1-\mu)(1+|\alpha_\eta|)|\bar{x}-\bar{y}|(1+|\bar{x}|^2+|\bar{y}|^2)^{(p-1)/2} \\ & \qquad +2p\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2}\Bigg) \\ \leq & C_b e^{L\bar{t}}\Bigg(2\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} + 4p\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2} \\ & \qquad +m(\varepsilon)|\alpha_\eta| + \theta(1-\mu)m(\varepsilon)|\alpha_\eta|(1+|\bar{x}|^2+|\bar{y}|^2)^{(p-1)/2}\Bigg). \end{array}$$

By Young's inequality, we get

$$\begin{array}{rl} & m(\varepsilon)|\alpha_\eta| + \theta(1-\mu)m(\varepsilon)|\alpha_\eta|(1+|\bar{x}|^2+|\bar{y}|^2)^{(p-1)/2} \\ \leq & \dfrac{m(\varepsilon)}{(\theta(1-\mu))^{1/(p-1)}} + \theta(1-\mu)|\alpha_\eta|^p + \theta(1-\mu)m(\varepsilon) + \theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2}. \end{array}$$

It follows

$$\begin{array}{rl} & \langle b_y, p_\varepsilon + q_y\rangle - \langle b_x, p_\varepsilon + q_x\rangle \\ \leq & (4p+1)C_b e^{L\bar{t}}\theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2} + C_b e^{L\bar{t}}\theta(1-\mu)|\alpha_\eta|^p + m(\varepsilon). \end{array} \tag{24}$$

6. Estimates of running cost terms. Recall that we chose ε small enough in order that $|\bar{x}-\bar{y}| \leq 1$. Setting $R=1$, from **(A)**(iii), we get

$$\begin{array}{rcl}\mu\ell(\bar{x},\bar{t},\alpha_\eta) - \ell(\bar{y},\bar{t},\alpha_\eta) & = & (\mu-1)\ell(\bar{x},\bar{t},\alpha_\eta) + \ell(\bar{x},\bar{t},\alpha_\eta) - \ell(\bar{y},\bar{t},\alpha_\eta) \\ & \leq & (1-\mu)|\alpha_\eta|^p\left(-\nu + \dfrac{m_1(|\bar{x}-\bar{y}|)}{1-\mu}\right) \\ & & +C_\ell(1-\mu)(1+|\bar{x}|^p) + m_1(|\bar{x}-\bar{y}|). \end{array}$$

Since $m_1(|\bar{x}-\bar{y}|) = m(\varepsilon)$ by (18), we obtain

$$\mu\ell(\bar{x},\bar{t},\alpha_\eta) - \ell(\bar{y},\bar{t},\alpha_\eta) \leq (1-\mu)|\alpha_\eta|^p\left(-\nu + m(\varepsilon)\right) + C_\ell(1-\mu)(1+|\bar{x}|^p) + m(\varepsilon). \tag{25}$$

Note that it is the term "$-(1-\mu)\nu|\alpha_\eta|^p$" which will allow to control all the unbounded control terms in the sequel.

7. Estimates of f-terms. To simplify, we replace **(B)**(v) by the assumption that f is nondecreasing with respect to the u variable. By some changes of functions as in Lemma 3.1, we can reduce to this case without loss of generality.

We write

$$f\big(\bar y,\bar t,v(\bar y,\bar t),s_y(p_\varepsilon+q_y)\big)-\mu\, f\big(\bar x,\bar t,u(\bar x,\bar t),\frac{1}{\mu}s_x(p_\varepsilon+q_x)\big)=\mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3$$

with

$$\begin{aligned}
\mathcal{T}_1&=f\big(\bar y,\bar t,v(\bar y,\bar t),s_y(p_\varepsilon+q_y)\big)-f\big(\bar x,\bar t,v(\bar y,\bar t),s_y(p_\varepsilon+q_y)\big),\\
\mathcal{T}_2&=f\big(\bar x,\bar t,v(\bar y,\bar t),s_y(p_\varepsilon+q_y)\big)-f\big(\bar x,\bar t,u(\bar x,\bar t),s_y(p_\varepsilon+q_y)\big),\\
\mathcal{T}_3&=f\big(\bar x,\bar t,u(\bar x,\bar t),s_y(p_\varepsilon+q_y)\big)-\mu\, f\big(\bar x,\bar t,u(\bar x,\bar t),\frac{1}{\mu}s_x(p_\varepsilon+q_x)\big),
\end{aligned}$$

and we estimate the three terms separately.

From **(B)**(ii), we have

$$\mathcal{T}_1\le m_{R_{\mu,\theta}}\left((1+|v(\bar y,\bar t)|+|s_y(p_\varepsilon+q_y)||\bar x-\bar y|\right),$$

where $R_{\mu,\theta}$ is given by (15). Using **(B)**(iv) and the fact that $v\in\mathcal{C}_p$, we get

$$|v(\bar y,\bar t)|,|s_yq_y|=O(R_{\mu,\theta})\quad\text{and}\quad |s_yp_\varepsilon||\bar x-\bar y|\le m(\varepsilon),\tag{26}$$

and therefore

$$\mathcal{T}_1\le m(\varepsilon).\tag{27}$$

To deal with $\mathcal{T}_2$, we first note that

$$\begin{aligned}
\mu u(\bar x,\bar t)-v(\bar y,\bar t)&\ge \Phi(\bar x,\bar y,\bar t)\\
&\ge \Phi(\hat x,\hat x,\hat t)\\
&\ge \mu u(\hat x,\hat t)-v(\hat y,\hat t)-e^{L\hat t}\theta(1-\mu)(1+2|\hat x|^2)^{p/2}-\rho\hat t.
\end{aligned}$$

Since $u(\hat x,\hat t)>v(\hat y,\hat t)$ by (14), if we take μ close enough to 1 and ρ,θ close enough to 0, we obtain that

$$\mu u(\bar x,\bar t)\ge v(\bar x,\bar t).$$

From **(B)**(v) (monotonicity of f in u), it follows that

$$\begin{aligned}
\mathcal{T}_2&\le f\big(\bar x,\bar t,v(\bar y,\bar t),s_y(p_\varepsilon+q_y)\big)-f\big(\bar x,\bar t,\mu u(\bar y,\bar t),s_y(p_\varepsilon+q_y)\big)\\
&\quad+f\big(\bar x,\bar t,\mu u(\bar y,\bar t),s_y(p_\varepsilon+q_y)\big)-f\big(\bar x,\bar t,u(\bar y,\bar t),s_y(p_\varepsilon+q_y)\big)\\
&\le (1-\mu)|u(\bar x,\bar t)|\\
&\le C_u(1-\mu)(1+|\bar x|^2)^{p/2},
\end{aligned}\tag{28}$$

since $u\in\mathcal{C}_p$.

To estimate $\mathcal{T}_3$, we first recall the following convex inequality. If $\Psi : I\!R^N \to I\!R$ is convex and $0 < \mu < 1$, then, for all $\xi, \zeta \in I\!R^N$, we have

$$-\mu\,\Psi(\xi) + \Psi(\zeta) \le (1-\mu)\Psi\left(\frac{\mu\xi - \zeta}{\mu - 1}\right). \tag{29}$$

By **(B)**(iii) (convexity of f with respect to the gradient variable), for all $z_1, z_2 \in I\!R^N$, we obtain

$$f(x,t,u,z_1) - \mu\, f\left(x,t,u,\frac{z_2}{\mu}\right) \le (1-\mu)\, f\left(x,t,u,\frac{z_1 - z_2}{1-\mu}\right).$$

Therefore

$$\begin{aligned} \mathcal{T}_3 &\le (1-\mu)\, f\left(\bar{x}, \bar{t}, u(\bar{x},\bar{t}), \frac{1}{1-\mu}(s_y(p_\varepsilon + q_y) - s_x(p_\varepsilon + q_x))\right) \\ &\le C_f(1-\mu)\big(1 + |\bar{x}|^p + |u(\bar{x},\bar{t})| + \left|\frac{s_y(p_\varepsilon + q_y) - s_x(p_\varepsilon + q_x)}{1-\mu}\right|^{p'}\big) \end{aligned} \tag{30}$$

by **(B)**(i). But

$$s_y(p_\varepsilon + q_y) - s_x(p_\varepsilon + q_x) = (s_y - s_x)p_\varepsilon + (s_y - s_x)q_y + s_x(q_y - q_x).$$

Hence for some $C > 0$ depending only on p (which may change during the computation), we have

$$\begin{aligned} \left|\frac{s_y(p_\varepsilon + q_y) - s_x(p_\varepsilon + q_x)}{1-\mu}\right|^{p'} &\le \frac{CC_s^{p'}}{(1-\mu)^{p'}}\left((|\bar{x}-\bar{y}||p_\varepsilon|)^{p'} + (|\bar{x}-\bar{y}||q_y|)^{p'} + |q_x - q_y|^{p'}\right) \\ &\le e^{p'L\bar{t}}m(\varepsilon) + CC_s^{p'}e^{p'L\bar{t}}\theta^{p'}(1 + |\bar{x}|^2 + |\bar{y}|^2)^{p/2}, \end{aligned}$$

by using (26). Finally, since $u \in \mathcal{C}_p$, we get from (30)

$$\mathcal{T}_3 \le (1-\mu)C_f(1 + C_u + CC_s^{p'}e^{p'L\bar{t}}\theta^{p'})(1 + |\bar{x}|^2 + |\bar{y}|^2)^{p/2} + e^{p'L\bar{t}}m(\varepsilon). \tag{31}$$

8. End of the case $\bar{t} > 0$, choice of the various parameters. By plugging estimates (23), (24), (25), (27), (28) and (31) in (22), we get

$$\begin{aligned} Le^{L\bar{t}}\theta(1-\mu)(1 + |\bar{x}|^2 + |\bar{y}|^2)^{p/2} + \rho &\le \left(C_1e^{L\bar{t}}\theta + C_2 + C_3e^{p'L\bar{t}}\theta^{p'}\right)(1-\mu)(1 + |\bar{x}|^2 + |\bar{y}|^2)^{p/2} \\ &\quad + \left(-\nu + \theta e^{L\bar{t}}(C_4 + m(\varepsilon))\right)(1-\mu)|\alpha_\eta|^p \\ &\quad + (1 + e^{p'L\bar{t}})m(\varepsilon) + m(\varrho/\varepsilon^4) + \eta, \end{aligned} \tag{32}$$

where

$$\begin{aligned} C_1 &= 4(p-1)^2C_\sigma^2 + 4(p+1)C_b, \qquad C_2 = C_\ell + C_u + C_f(1 + C_u), \\ C_3 &= C_fCC_s^{p'}, \qquad C_4 = 4(p-1)C_\sigma^2 + C_b, \end{aligned}$$

are positive constants which depend only on the given datas of the problem.

Now we choose the different parameters in order to have a contradiction in the above inequality. We first assume that the final time T such that

$$T = 1/L > 0$$

(we will recover the result on any interval $[0,T]$ by a step-by-step argument). The main difficulty in the above estimate is to deal with the term in $|\alpha_\eta|^p$ since the control α_η is unbounded. Taking $\theta > 0$ such that

$$\theta e^1(C_4+1) \leq \frac{\nu}{2},$$

we obtain that the coefficient in front of $|\alpha_\eta|^p$ is negative (we can assume that ε is small enough in order to have $m(\varepsilon) \leq 1$). Then we fix

$$L > C_1 + \frac{C_2}{\theta} + C_3 e^{p'-1}\theta^{p'-1} \quad \text{and} \quad \eta < \frac{\rho}{2}. \tag{33}$$

Therefore (32) implies

$$\frac{\rho}{2} \leq (1+e^{p'})m(\varepsilon) + m(\varrho/\varepsilon^4).$$

Sending first $\varrho \to 0$, we obtain a contradiction for small ε. In conclusion, up to a suitable choice of the parameters θ, L, η, the claim of the Step 3 is proved if $T \leq 1/L$.

9. Case when $\bar{t} = 0$. We have just proved that there is a subsequence ε_n such that $\bar{t} = 0$. Therefore for n large enough, for all $(x,t) \in I\!\!R^N \times [0,T]$, $T \leq 1/L$, we have

$$\begin{aligned}
& \mu u(x,t) - v(x,t) - \theta(1-\mu)e^{Lt}(1+2|x|^2)^{p/2} - \rho t \\
\leq\ & \mu u(\bar{x},0) - v(\bar{y},0) - \theta(1-\mu)(1+|\bar{x}|^2+|\bar{y}|^2)^{p/2} - \frac{|\bar{x}-\bar{y}|^2}{\varepsilon_n^2} \\
\leq\ & (1-\mu)(|u(\bar{x},0)| - \theta(1+|\bar{x}|^2)^{p/2}) + u(\bar{x},0) - v(\bar{y},0) \\
\leq\ & (1-\mu)M_\theta + u(\bar{x},0) - v(\bar{y},0)
\end{aligned}$$

where M_θ is given by (10) since $u \in \mathcal{C}_p$. Since $u - v$ is upper-semicontinuous, from (16), we get

$$\limsup_{\varepsilon_n \to 0} u(\bar{x},0) - v(\bar{y},0) \leq u(x_0,0) - v(x_0,0) \leq 0,$$

using that $u(x_0,0) \leq \psi(x_0) \leq v(x_0,0)$. It follows

$$\mu u(x,t) - v(x,t) - \theta(1-\mu)e^{Lt}(1+2|x|^2)^{p/2} - \rho t \leq (1-\mu)M_\theta.$$

Sending $\mu \to 1$ and $\rho \to 0$, we get $u \leq v$ in $I\!\!R^N \times [0,T]$, $T \leq 1/L$. Noticing that L given by (33) depends only on the given constants of the problem, we recover the comparison on $[0,T]$ for any $T > 0$ by a classical step-by-step argument. It completes the proof of the theorem. □

We end with the proof of the existence result.

Proof of Theorem 2.2. The point is to build a sub- and a supersolution. We treat the case of the subsolution (the case of the supersolution being simpler). It suffices to prove that, there exists $\tau > 0$ such that, for all $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$u_\varepsilon(x,t) = -e^{\rho t}(M_\varepsilon + \varepsilon(1+|x|^p)) \tag{34}$$

is a subsolution of (5) in $I\!R^N \times [0,\tau]$ with initial data ψ. Indeed, u_ε does not belong to $\mathcal{C}_p$ but $\underline{u} := \sup_{\varepsilon>0} u_\varepsilon \in \mathcal{C}_p$ and $\underline{u}$ is still a subsolution.

Let $\varepsilon > 0$. Since $\psi, \ell(\cdot,t,\alpha), f(\cdot,t,u,z) \in \mathcal{C}_p$, there exists $M_\varepsilon = M_\varepsilon(\psi,\ell,f)$ such that $|\psi|, |\chi|, |\gamma| \leq M_\varepsilon + \varepsilon(1+|x|^p)$. Let u_ε defined by (34) with this choice of M_ε. Let $\alpha \in A$. In the following computation, $C > 0$ is a constant which depends only on the given datas of the problem and may change line to line. We have, for all $(x,t) \in I\!R^N \times [0,T]$,

$$\begin{aligned}
\mathcal{L}(u_\varepsilon) &:= \frac{\partial u_\varepsilon}{\partial t} - \langle b, Du_\varepsilon\rangle - \ell - \mathrm{Trace}\left[\sigma\sigma^T D^2 u_\varepsilon\right] + f(x,t,u_\varepsilon, sDu_\varepsilon) \\
&\leq -\rho|u_\varepsilon| + C\varepsilon e^{\rho t}(1+|x|+|\alpha|)|x|^{p-1} - \nu|\alpha|^p + |\chi| + C\varepsilon e^{\rho t}(1+|x|^2+|\alpha|^2)|x|^{p-2} \\
&\quad +|\gamma| + C|u_\varepsilon| + C\varepsilon^{p'} e^{p'\rho t}|x|^{p'(p-1)} \\
&\leq -\rho|u_\varepsilon| + C|u_\varepsilon| + C\varepsilon^{p'-1}e^{(p'-1)\rho t}|u_\varepsilon| - \frac{\nu}{2}|\alpha|^p,
\end{aligned}$$

since $p'(p-1) = p$,

$$|\alpha||x|^{p-1} + |\alpha|^2|x|^{p-2} \leq \frac{\nu}{2}|\alpha|^p + C|x|^p \quad \text{and} \quad |\chi| + |\gamma| \leq 2(M_\varepsilon + \varepsilon(1+|x|^p)) = 2|u_\varepsilon|.$$

By choosing ρ large enough such that $\rho = C + Ce^1$ and $\tau > 0$ such that $(p'-1)\rho\tau \leq 1$, we obtain $\mathcal{L}(u_\varepsilon) \leq 0$ in $I\!R^N \times [0,\tau]$. Since $u_\varepsilon(\cdot,0) \leq \psi$ by the choice of M_ε, we obtain that u_ε is a subsolution, which ends the proof. $\square$

3 Equations with superlinear growth on the datas and the solutions

In this Section we extend the comparison result of [13] for equations with an additional nonlinearity and with $p > 1$ growth conditions on the datas and on the solutions. More precisely, we consider

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{Trace}(\sigma\sigma^T D^2 u) + \langle b, Du\rangle + f(x,t,u,sDu) = 0 & \text{in } I\!R^N \times [0,T], \\ u(x,0) = \psi(x) & \text{in } I\!R^N. \end{cases} \tag{35}$$

For simplicity, we choose to consider an equation with no control. With straightforward adaptations (see [13]), we may replace the linear part of the operator with (2) under assumption **(A)** but with σ bounded with respect to the control, i.e., (6) reads

$$|\sigma(x,t,\alpha)| \leq C_\sigma(1+|x|).$$

The hypothesis on the data are the following:

(C) (Asssumptions on the diffusion and the drift)

(i) $b \in C(I\!R^N \times [0,T]; I\!R^N)$ and there exists $C_b > 0$ such that, for all $x, y \in I\!R^N$, $t \in [0,T]$,

$$\begin{aligned} |b(x,t) - b(y,t)| &\leq C_b|x-y|, \\ |b(x,t)| &\leq C_b(1+|x|) \ ; \end{aligned}$$

(ii) $\sigma \in C(I\!R^N \times [0,T]; \mathcal{M}_{N,M}(I\!R))$ is Lipschitz continuous with respect to x (uniformly in t), namely, there exists $C_\sigma > 0$ such that, for all $x, y \in I\!R^N$ and $t \in [0,T]$,

$$|\sigma(x,t) - \sigma(y,t)| \leq C_\sigma|x-y|.$$

Note that σ satisfies, for every $x \in I\!R^N$, $t \in [0,T]$,

$$|\sigma(x,t)| \leq C_\sigma(1+|x|).$$

We are able to consider functions which are in a larger class than in Section 2. We say that a function $u\colon I\!R^N \times [0,T] \to I\!R$ is in the class $\tilde{\mathcal{C}}_p$ if u is locally bounded and for some $C > 0$ we have

$$|u(x,t)| \leq C(1+|x|^p), \quad \text{for all } (x,t) \in I\!R^N \times [0,T]. \tag{36}$$

The main result of this Section if the following

Theorem 3.1 *Let $p > 1$. Assume that σ and b satisfy* **(C)**, *that f satisfies* **(B)** *and that $\psi \in \tilde{\mathcal{C}}_p$. Let $u \in USC(I\!R^N \times [0,T])$ be a viscosity subsolution of* (35) *and $v \in LSC(I\!R^N \times [0,T])$ be a viscosity supersolution of* (35). *Suppose that u and v are in the class $\tilde{\mathcal{C}}_p$ and satisfy $u(x,0) \leq \psi(x) \leq v(x,0)$. Then $u \leq v$ in $I\!R^N \times [0,T]$.*

Before giving the proof of the theorem, we state an existence result and provide some examples. As observed in Section 2, we can prove the existence of solutions of (35) (at least for small time) as soon as we are able to build sub- and supersolutions in the class $\tilde{\mathcal{C}}_p$. In Example 3.3, we see that solutions may not exist for all time.

Theorem 3.2 *Let $p > 1$ and assume (B)–(C). If $K, \rho > 0$ are large enough, then $\overline{u}(x,t) = Ke^{\rho t}(1+|x|^2)^{p/2}$ is a viscosity supersolution of* (35) *in $I\!R^N \times [0,T]$ and there exists $0 < \tau \leq T$ such that $\underline{u}(x,t) = -Ke^{\rho t}(1+|x|^2)^{p/2}$ is a viscosity subsolution of* (35) *in $I\!R^N \times [0,\tau]$. In consequence, for all $\psi \in \tilde{\mathcal{C}}_p$, there exists a unique continuous viscosity solution of* (35) *in $I\!R^N \times [0,\tau]$ in $\tilde{\mathcal{C}}_p$.*

The proof is very close to the one of [13, Lemma 2.1], thus we omit it. Let us give some examples of Equations for which Theorem 3.1 applies and some examples.

Example 3.1 A typical (simple) case we have in mind is

$$u_t - \Delta u + |Du|^{p'} = -f(x,t) \quad \text{in } I\!R^N \times [0,T], \tag{37}$$

where f satisfies **(B)**. Note that (36) can be written

$$u_t - \Delta u + p \sup_{\alpha \in I\!R^N} \{\langle \alpha, Du \rangle - \frac{|\alpha|^{p'}}{p'}\} + f(x,t) = 0$$

and therefore is on the form (4) or (5). The stationary version of this equation was studied in Alvarez [1] under more restrictive assumptions on the datas and the growth of the solution. More precisely, he assumed conditions like (9) and (11).

Example 3.2 As far as the coefficient f is concerned, a typical case we have in mind is

$$f(x,t,u,z) = g(x,t,u) + |z|^{p'},$$

with continuous g satisfying **(B)**(i),(ii) and (v). It leads to nonlinearities like "$g(x,t,u) + |s(x,t)Du|^{p'}$" in the equation. Note that the power-p' term depends on x via $s(x,t)$. This dependence brings an additional difficulty, especially when doubling the variables in viscosity type's proofs, see the proof of Lemma 3.2 and compare with [13, Lemma 2.2].

Example 3.3 (Deterministic Control Problem) Consider the control problem (in dimension 1 for sake of simplicity)

$$\begin{cases} dX_s = \alpha_s \, ds, \quad s \in [t,T], \; 0 \le t \le T, \\ X_t = x \in I\!R, \end{cases}$$

where the control $\alpha \in \mathcal{A}_t := L^p([t,T]; I\!R)$ and the value function is given by

$$V(x,t) = \inf_{\alpha \in \mathcal{A}_t} \{ \int_t^T (\frac{|\alpha_s|^p}{p} + \rho |X_s|^p) \, ds - |X_T|^p \} \quad \text{for some } \rho > 0.$$

The Hamilton-Jacobi equation associated to this problem is

$$\begin{cases} -w_t + \frac{1}{p'}|w_x|^{p'} = \rho |x|^p & \text{in } I\!R \times (0,T), \\ w(x,T) = -|x|^p & \text{in } I\!R. \end{cases} \tag{38}$$

By easy adaptation of some results of [13], V is the unique viscosity solution of (37). Looking for a solution V under the form $V(x,t) = \varphi(t)|x|^p$, we obtain that φ is a solution of the differential equation

$$-\varphi' + \frac{|\varphi|^{p'}}{p'} = \rho \;\text{ in } (0,T), \qquad \varphi(T) = -1.$$

We get

$$\int_{-1}^{\varphi(t)} \frac{p'}{|y|^{p'} - \rho p'} dy = t - T.$$

One can check that if $0 < \rho p' < 1$ and $T > \int_{-\infty}^{-1} \frac{p'}{|y|^{p'} - \rho p'} dy$, then there is $\tau \in (0, T)$ such that the solution blows up at $t = \tau$. A sufficient condition to ensure that V is defined on $[0, T]$ is to take $\rho p' \geq 1$. Note that if we choose $\rho p' > 1$ and $T > -\int_{-1}^{0} \frac{p'}{|y|^{p'} - \rho p'} dy$ then the solution of (37) is neither bounded from below nor from above and therefore the results of [2] and [18] do not apply.

Example 3.4 In the framework of BSDEs, one generally considers forward-backward systems of the form

$$\begin{cases} dX_s^{x,t} = b(X_s^{x,t}, s)ds + \sigma(X_s^{t,x}, s)dW_s, \quad t \leq s \leq T, \\ X_t^{t,x} = x, \end{cases} \tag{39}$$

$$\begin{cases} -dY_s^{x,t} = f(X_s^{x,t}, s, Y_s^{x,t}, Z_s^{x,t})ds - Z_s^{x,t} dW_s, \quad t \leq s \leq T, \\ Y_T^{x,t} = \psi(x), \end{cases} \tag{40}$$

where $(W_s)_{s \in [0,T]}$ is standard Brownian motion on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$, with $(\mathcal{F}_t)_{t \in [0,T]}$ the standard Brownian filtration. (Note that b and σ do not depend on the control). The diffusion (38) is associated with the second-order elliptic operator L defined by

$$Lu = -\frac{1}{2}\mathrm{Trace}(\sigma\sigma^T D^2 u) - \langle b(x,t), Du \rangle.$$

The forward-backward system (38)-(39) is formally connected to the PDE of type (4),

$$\begin{cases} -\dfrac{\partial u}{\partial t} + Lu - f(x, t, u, \sigma(x,t)Du) = 0 & \text{in } I\!R^N \times (0, T) \\ u(x, T) = \psi(x) & \text{in } I\!R^N. \end{cases} \tag{41}$$

by the nonlinear Feynman-Kac formula

$$u(x,t) = Y_t^{x,t} \quad \text{for all } (x,t) \in I\!R^N \times [0, T]. \tag{42}$$

We recall that nonlinear BSDEs with Lipschitz continuous coefficients were first introduced by Pardoux and Peng [23], who proved existence and uniqueness. Their results were extended by Kobylanski [20] for bounded solutions in the case of coefficients f having a quadratic growth in the gradient. Briand and Hu [9] generalized this latter result to the case of solutions which are $O(1 + |x|^p)$, as $|x| \to \infty$, with $1 \leq p < 2$. In all these works, the connection with viscosity solutions to the related PDE (40) is established: u defined by (41) is a viscosity solution of (40) for $1 \leq p < 2$. Theorem 3.1 proves this solution is unique. We are able to deal with any $p > 1$ but we had to impose the regularity condition **(B)**(ii) on x for f, which is not needed for the BDSEs.

Let us turn to the proof of the comparison theorem.

Proof of Theorem 3.1. To avoid a lot of technicality, we start the proof with several lemmas collecting the main intermediate results. The proofs of the lemmas are postponed at the end of the section and can be skipped at first reading.

Lemma 3.1 *(Change of functions)*
Let $\tilde{u} = \mathrm{e}^{-Lt}u + h(x)$ *where* $h(x) = \overline{C}(1+|x|^p)$ *for some constants* $\overline{C}, L > 0$. *Then* $\tilde{u}$ *is a viscosity solution of*

$$\begin{cases} \tilde{u}_t - \mathrm{Trace}(\sigma(x,t)\sigma(x,t)^T D^2\tilde{u}) + \langle b(x,t), D\tilde{u}\rangle & \\ \qquad\qquad + \tilde{f}(x,t,\tilde{u}-h, s(x,t)(D\tilde{u}-Dh)) = 0 & \textit{in } I\!\!R^N \times (0,T], \\ \tilde{u}(x,0) = \psi(x) + h(x) & \textit{for all } x \in I\!\!R^N, \end{cases} \tag{43}$$

with, for all $(x,t,v,z) \in I\!\!R^N \times [0,T] \times I\!\!R^N$,

$$\tilde{f}(x,t,v,z) = Lv + \tilde{g}(x,t) + \mathrm{e}^{-Lt} f\left(x,t,\mathrm{e}^{Lt}v, \mathrm{e}^{Lt}z\right), \tag{44}$$

where

$$\tilde{g}(x,t) = \mathrm{Trace}(\sigma(x,t)\sigma(x,t)^T D^2 h(x)) - \langle b(x,t), Dh(x)\rangle. \tag{45}$$

Moreover,

$$\tilde{f}(x,t,v,z) - \tilde{f}(x,t,v',z) \le (\hat{C} - L)(v'-v) \quad \textit{if } v \le v'. \tag{46}$$

In the sequel, since $u, v, \psi \in \tilde{\mathcal{C}}_p$, we can choose $\overline{C} > 0$ such that

$$|u|, |v|, |\psi| \le \frac{\overline{C}}{2}(1+|x|^p). \tag{47}$$

In this case, note that

$$\psi(x) + h(x) = \psi(x) + \overline{C}(1+|x|^p) \ge 0$$

and the initial data is nonnegative in (42).

Moreover, we take

$$L > \hat{C} \quad \text{and} \quad L > 4p(p-1)NC_\sigma^2 + 4pC_b + 10\hat{C} \tag{48}$$

(the constants C_σ, C_b and $\hat{C}$ appear in **(B)**). The first condition ensures that the right-hand side of (45) is nonpositive (i.e. $v \mapsto \tilde{f}(x,t,v,z)$ is nondecreasing). The second condition appears naturally in the proof of the following lemma.

Lemma 3.2 *(A kind of linearization procedure)*
Let $\overline{C}, L > 0$ *be such that* (46) *and* (47) *hold. Let* $0 < \mu < 1$ *and set* $\tilde{w} = \mu\tilde{u} - \tilde{v}$. *Then* $\tilde{w}$ *is a viscosity subsolution of the variational inequality*

$$\begin{cases} \min\{w, \mathcal{L}[w]\} \le 0 & \text{in } I\!\!R^n \times (0,T), \\ w(\cdot,0) \le 0 & \text{in } I\!\!R^n, \end{cases} \tag{49}$$

where

$$\begin{aligned} \mathcal{L}[w] \;\; := \;\; & \frac{\partial w}{\partial t} - \mathrm{Trace}[\sigma(x,t)\sigma^T(x,t)D^2 w] - C_b(1+|x|)|Dw| + \frac{L}{4}(1-\mu)h(x,t) \\ & -(1-\mu)\mathrm{e}^{-Lt} f\left(x,t,0,\mathrm{e}^{Lt}s(x,t)(\frac{Dw}{\mu-1} - Dh(x))\right) \end{aligned} \tag{50}$$

and h *is defined in Lemma 3.1.*

Lemma 3.3 *(An auxiliary parabolic problem)*
Consider, for any $R > 0$, the parabolic problem

$$\begin{cases} \varphi_t - r^2\varphi_{rr} - r\varphi_r = 0 & \text{in } [0,+\infty)\times(0,T], \\ \varphi(r,0) = \max\{0, r-R\} & \text{in } [0,+\infty). \end{cases} \tag{51}$$

Then (50) has a unique solution $\varphi_R \in C([0,+\infty)\times[0,T]) \cap C^\infty([0,+\infty)\times(0,T])$ such that, for all $t \in (0,T]$, $\varphi_R(\cdot,t)$ is positive, nondecreasing and convex in $[0,+\infty)$. Moreover, for every $(r,t) \in [0,+\infty)\times(0,T]$,

$$\varphi_R(r,t) \geq \max\{0, r-R\}, \quad 0 \leq \frac{\partial \varphi_R}{\partial r}(r,t) \leq \mathrm{e}^T \quad \text{and} \quad \varphi_R(r,t) \underset{R\to+\infty}{\longrightarrow} 0. \tag{52}$$

For the proof of Lemma 3.3 we refer the reader to [13].

Lemma 3.4 *(Construction of a smooth strict supersolution)*
Let $\Phi(x,t) = \varphi_R(h(x), Ct)$ where φ_R is given by Lemma 3.3, $h(x) = \overline{C}(1+|x|^p)$, $\overline{C}$ satisfies (46) and $C > 0$. Then, for C and $L = L(\mu)$ large enough, we have

$$\mathcal{L}[\Phi(x,t)] > 0 \quad \text{for all } (x,t) \in I\!R^N \times (0, 1/L], \tag{53}$$

where $\mathcal{L}$ is defined by (49).

Now, we continue the **proof of Theorem 3.1**. Consider

$$\max_{I\!R^N\times[0,1/L]} \{\tilde{w} - \Phi\}, \tag{54}$$

where $\tilde{w}$ is given by Lemma 3.2 and Φ is the function built in Lemma 3.4. From (51), for $|x|$ large enough, we have $\Phi(x,t) \geq \overline{C}(1+|x|^p) - R$. Since $\tilde{w} \leq (\mu+1)\overline{C}(1+|x|^p)/2$, it follows that the maximum (53) is achieved at a point $(\bar{x},\bar{t}) \in I\!R^N \times [0,1/L]$. We can assume that $\tilde{w}(\bar{x},\bar{t}) > 0$ otherwise, arguing as in (54)-(55), we prove $\tilde{w} \leq 0$ in $I\!R^N \times [0,1/L]$ and the conclusion follows.

We claim that $\bar{t} = 0$. Indeed suppose by contradiction that $\bar{t} > 0$. Then since $\tilde{w}$ is a viscosity subsolution of (48) with $\tilde{w}(\bar{x},\bar{t}) > 0$, by taking Φ as a test-function, we would have $\mathcal{L}[\Phi](\bar{x},\bar{t}) \leq 0$ which contradicts the fact that Φ satisfies (52) . Thus, for all $(x,t) \in I\!R^N \times [0,1/L]$,

$$\tilde{w}(x,t) - \Phi(x,t) \leq \tilde{w}(\bar{x},0) - \Phi(\bar{x},0) \leq 0, \tag{55}$$

where the last inequality follows from (48) and the fact that $\Phi \geq 0$. Therefore, for every $(x,t) \in I\!R^N \times [0,1/L]$, we have

$$(\mu\tilde{u} - \tilde{v})(x,t) = \tilde{w}(x,t) \leq \Phi(x,t) = \varphi_R(h(x), Ct). \tag{56}$$

Letting R to $+\infty$, we get by (51), $\mu\tilde{u} - \tilde{v} \leq 0$ in $I\!R^N \times [0,1/L]$.

We can repeat the above arguments on $I\!R^N \times [1/L, 2/L]$ with the same constants. By a step-by-step argument, we then prove that $\mu\tilde{u} - \tilde{v} \leq 0$ in $I\!R^N \times [0,T]$. Letting μ go to 1, we obtain $u \leq v$ as well which completes the proof of the theorem. □

We turn to the proof of the Lemmas 3.1, 3.2 and 3.4.

Proof of Lemma 3.1. Since

$$\begin{aligned} u &= \mathrm{e}^{Lt}(\tilde{u} - h), & u_t &= \mathrm{e}^{Lt}(\tilde{u}_t + L(\tilde{u} - h)), \\ Du &= \mathrm{e}^{Lt}(D\tilde{u} - Dh), & D^2 u &= \mathrm{e}^{Lt}(D^2\tilde{u} - D^2 h), \end{aligned}$$

we obtain easily that $\tilde{u}$ is a viscosity solution of (42) with $\tilde{f}$ and $\tilde{g}$ given by (43) and (44). It remains to check (45)). Take $v, v' \in I\!R$ such that $v \leq v'$. From **(B)**(v), we obtain

$$\begin{aligned} & \tilde{f}(x,t,v,z) - \tilde{f}(x,t,v',z) \\ \leq \; & L(v - v') + \mathrm{e}^{-Lt}\left(f(x,t,\mathrm{e}^{Lt}v, \mathrm{e}^{Lt}z) - f(x,t,\mathrm{e}^{Lt}v', \mathrm{e}^{Lt}z)\right) \\ \leq \; & -L(v' - v) + \mathrm{e}^{-Lt}\hat{C}|\mathrm{e}^{Lt}(v - v')| \leq (\hat{C} - L)(v' - v). \end{aligned}$$

It ends the proof of the lemma. □

Proof of Lemma 3.2. For $0 < \mu < 1$, let $\tilde{u}^\mu = \mu\tilde{u}$ and $\tilde{w} = \tilde{u}^\mu - \tilde{v}$. We divide the proof in different steps.

Step 1. A new equation for $\tilde{u}^\mu$. It is not difficult to see that, if $\tilde{u}$ is a subsolution of (42), then $\tilde{u}^\mu$ is a subsolution of

$$\begin{cases} \tilde{u}^\mu_t - \mathrm{Trace}(\sigma(x,t)\sigma(x,t)^T D^2\tilde{u}^\mu) + \langle b(x,t), D\tilde{u}^\mu\rangle & \\ \qquad +\mu\, \tilde{f}(x,t,\dfrac{\tilde{u}^\mu}{\mu} - h, s(x,t)(\dfrac{D\tilde{u}^\mu}{\mu} - Dh)) = 0 & \text{in } I\!R^N \times (0,T], \\ \tilde{u}^\mu(x,0) = \mu\psi(x) + \mu h(x) & \text{for all } x \in I\!R^N. \end{cases} \tag{57}$$

Step 2. The equation for $\tilde{w}$. Let $\varphi \in C^2(I\!R^N \times [0,T])$ and suppose that we have

$$\max_{I\!R^N \times [0,T]} \tilde{w} - \varphi = (\tilde{w} - \varphi)(\bar{x}, \bar{t}). \tag{58}$$

We distinguish 3 cases.

At first, if the maximum is achieved for $\bar{t} = 0$, then, writing that $\tilde{u}^\mu$ is a subsolution of (56) and $\tilde{v}$ a supersolution of (42) at $t = 0$ we obtain $\tilde{u}^\mu(\bar{x},0) \leq \mu\psi(\bar{x}) + \mu h(\bar{x})$ and $\tilde{v}(\bar{x},0) \geq \psi(\bar{x}) + h(\bar{x})$. It follows that

$$\tilde{w}(\bar{x},0) \leq (\mu - 1)(\psi(\bar{x}) + h(\bar{x})) = (\mu - 1)(\psi(\bar{x}) + \overline{C}(1 + |\bar{x}|^p)) \leq 0$$

by (46). Therefore $\tilde{w}$ satisfies (48) at $(\bar{x},0)$.

Secondly, we suppose that $\bar{t} > 0$ and $\tilde{w}(\bar{x},\bar{t}) \leq 0$. Again, $\tilde{w}$ satisfies (48) at $(\bar{x},\bar{t})$.

From now on, we consider the last and most difficult case when

$$\bar{t} > 0 \quad \text{and} \quad \tilde{w}(\bar{x},\bar{t}) > 0. \tag{59}$$

Step 3. Viscosity inequalities for $\tilde{u}^\mu$ and $\tilde{v}$. This step is classical in viscosity theory. We can assume that the maximum in (57) at $(\bar{x},\bar{t})$ is strict in the some ball $\overline{B}(\bar{x},r) \times [\bar{t} - r, \bar{t} + r]$ (see [6] or [3]). Let

$$\Theta(x,y,t) = \varphi(x,t) + \frac{|x-y|^2}{\varepsilon^2}$$

and consider

$$M_\varepsilon := \max_{x,y\in\overline{B}(\bar{x},r),\, t\in[\bar{t}-r,\bar{t}+r]} \{\tilde{u}^\mu(x,t) - \tilde{v}(y,t) - \Theta(x,y,t)\}.$$

This maximum is achieved at a point $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$ and, since the maximum is strict, we know that

$$x_\varepsilon, y_\varepsilon \to \bar{x}, \qquad \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} \to 0, \tag{60}$$

and

$$M_\varepsilon = \tilde{u}^\mu(x_\varepsilon, t_\varepsilon) - \tilde{v}(y_\varepsilon, t_\varepsilon) - \Theta(x_\varepsilon, y_\varepsilon, t_\varepsilon) \longrightarrow (\tilde{w} - \varphi)(\bar{x}, \bar{t}) \quad \text{as } \varepsilon \to 0.$$

It means that, at the limit $\varepsilon \to 0$, we obtain some information on $\tilde{w} - \varphi$ at $(\bar{x}, \bar{t})$ which will provide the new equation for $\tilde{w}$. From (58), for ε small enough, we have

$$\tilde{u}^\mu(x_\varepsilon, t_\varepsilon) - \tilde{v}(y_\varepsilon, t_\varepsilon) > 0. \tag{61}$$

We can take Θ as a test-function to use the fact that $\tilde{u}^\mu$ is a subsolution of (56) and $\tilde{v}$ a supersolution of (42). Indeed $(x,t) \in \overline{B}(\bar{x}, r) \times [\bar{t} - r, \bar{t} + r] \mapsto \tilde{u}^\mu(x,t) - \tilde{v}(y_\varepsilon, t) - \Theta(x, y_\varepsilon, t)$ achieves its maximum at $(x_\varepsilon, t_\varepsilon)$ and $(y,t) \in \overline{B}(\bar{x}, r) \times [\bar{t} - r, \bar{t} + r] \mapsto -\tilde{u}^\mu(x_\varepsilon, t) + \tilde{v}(y,t) + \Theta(x_\varepsilon, y, t)$ achieves its minimum at $(y_\varepsilon, t_\varepsilon)$. Thus, by Theorem 8.3 in the User's guide [11], for every $\rho > 0$, there exist $a_1, a_2 \in I\!R$ and $X, Y \in \mathcal{S}_N$ such that

$$(a_1, D_x\Theta(x_\varepsilon, y_\varepsilon, t_\varepsilon), X) \in \bar{\mathcal{P}}^{2,+}(\tilde{u}^\mu)(x_\varepsilon, t_\varepsilon), \quad (a_2, -D_y\Theta(x_\varepsilon, y_\varepsilon, t_\varepsilon), Y) \in \bar{\mathcal{P}}^{2,-}(\tilde{v})(y_\varepsilon, t_\varepsilon),$$

$a_1 - a_2 = \Theta_t(x_\varepsilon, y_\varepsilon, t_\varepsilon) = \varphi_t(x_\varepsilon, t_\varepsilon)$ and

$$-(\frac{1}{\rho} + |M|)I \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le M + \rho M^2 \quad \text{where } M = D^2\Theta(x_\varepsilon, y_\varepsilon, t_\varepsilon). \tag{62}$$

Setting $p_\varepsilon = 2\dfrac{x_\varepsilon - y_\varepsilon}{\varepsilon^2}$, we have

$$D_x\Theta(x_\varepsilon, y_\varepsilon, t_\varepsilon) = p_\varepsilon + D\varphi(x_\varepsilon, t_\varepsilon) \quad \text{and} \quad D_y\Theta(x_\varepsilon, y_\varepsilon, t_\varepsilon) = -p_\varepsilon,$$

and

$$M = \begin{pmatrix} D^2\varphi(x_\varepsilon, t_\varepsilon) + 2I/\varepsilon^2 & -2I/\varepsilon^2 \\ -2I/\varepsilon^2 & 2I/\varepsilon^2 \end{pmatrix}.$$

Thus, from (61), it follows

$$\langle Xp, p\rangle - \langle Yq, q\rangle \le \langle D^2\varphi(x_\varepsilon, t_\varepsilon)p, p\rangle + \frac{2}{\varepsilon^2}|p - q|^2 + m\left(\frac{\rho}{\varepsilon^4}\right), \tag{63}$$

where m is a modulus of continuity which is independent of ρ and ε. In the sequel, m will always denote a generic modulus of continuity independent of ρ and ε.

Writing the subsolution viscosity inequality for $\tilde{u}^\mu$ and the supersolution inequality for $\tilde{v}$ by means of the semi-jets and subtracting the inequalities, we obtain

$$\begin{aligned}
& \varphi_t(x_\varepsilon, t_\varepsilon) \\
& -\text{Trace}\left[\sigma(x_\varepsilon,t_\varepsilon)\sigma^T(x_\varepsilon,t_\varepsilon)X\right] + \text{Trace}\left[\sigma(y_\varepsilon,t_\varepsilon)\sigma^T(y_\varepsilon,t_\varepsilon)Y\right] \\
& -\langle b(x_\varepsilon,t_\varepsilon), p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)\rangle + \langle b(y_\varepsilon,t_\varepsilon), p_\varepsilon\rangle \\
& +\mu\tilde{f}\left(x_\varepsilon, t_\varepsilon, \frac{\tilde{u}^\mu(x_\varepsilon,t_\varepsilon)}{\mu} - h(x_\varepsilon), s(x_\varepsilon,t_\varepsilon)(\frac{p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)}{\mu} - Dh(x_\varepsilon))\right) \\
& -\tilde{f}\left(y_\varepsilon, t_\varepsilon, \tilde{v}(y_\varepsilon,t_\varepsilon) - h(y_\varepsilon), s(y_\varepsilon,t_\varepsilon)(p_\varepsilon - Dh(y_\varepsilon))\right) \\
\leq\ & 0 \qquad (64)
\end{aligned}$$

Now, we derive some estimates for the various terms appearing in (63) in order to be able to send $\varepsilon \to 0$. The estimates for the σ and b terms are classical wheras those for the f terms are more involved.

For the sake of simplicity, for any function $g : I\!R^N \times [0,T] \to I\!R$, we set

$$g(x_\varepsilon, t_\varepsilon) = g_x \quad \text{and} \quad g(y_\varepsilon, t_\varepsilon) = g_y.$$

Step 4. Estimate of σ-terms. Let us denote by $(e_i)_{1\leq i\leq N}$ the canonical basis of $I\!R^N$. By using (62), we obtain

$$\begin{aligned}
\text{Trace}\left[\sigma_x\sigma_x^T X - \sigma_y\sigma_y^T Y\right] &= \sum_{i=1}^N \langle X\sigma_x e_i, \sigma_x e_i\rangle - \langle Y\sigma_y e_i, \sigma_y e_i\rangle \\
&\leq \text{Trace}\left[\sigma_x\sigma_x^T D^2\varphi(x_\varepsilon,t_\varepsilon)\right] + \frac{2}{\varepsilon^2}|\sigma_x - \sigma_y|^2 + m\left(\frac{\rho}{\varepsilon^4}\right) \\
&\leq \text{Trace}\left[\sigma_x\sigma_x^T D^2\varphi(x_\varepsilon,t_\varepsilon)\right] + 2C_{\sigma,r}^2\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} + m\left(\frac{\rho}{\varepsilon^4}\right) \\
&\leq \text{Trace}\left[\sigma\sigma^T(\bar{x},\bar{t})D^2\varphi(\bar{x},\bar{t})\right] + m(\varepsilon) + m\left(\frac{\rho}{\varepsilon^4}\right), \qquad (65)
\end{aligned}$$

where $C_{\sigma,r}$ is a Lipschitz constant for σ in $\bar{B}(x,r)$ and we used that σ is continuous, φ is C^2 and (59).

Step 5. Estimate of b-terms. From **(C)**, if $C_{b,r}$ is the Lipschitz constant of b in $\overline{B}(\bar{x},r)\times[\bar{t}-r,\bar{t}+r]$, then we have

$$\langle b(x_\varepsilon,t_\varepsilon) - b(y_\varepsilon,t_\varepsilon), p_\varepsilon\rangle \leq C_{b,r}|x_\varepsilon - y_\varepsilon||p_\varepsilon| \leq 2C_{b,r}\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} = m(\varepsilon)$$

and

$$\langle b(x_\varepsilon,t_\varepsilon), D\varphi(x_\varepsilon,t_\varepsilon)\rangle \leq C_b(1 + |x_\varepsilon|)|D\varphi(x_\varepsilon,t_\varepsilon)|.$$

It follows

$$\langle b(x_\varepsilon,t_\varepsilon), p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)\rangle - \langle b(y_\varepsilon,t_\varepsilon), p_\varepsilon\rangle \leq C_b(1 + |\bar{x}|)|D\varphi(\bar{x},\bar{t})| + m(\varepsilon) \qquad (66)$$

Step 6. Estimate of $\tilde{f}$-terms. We write

$$\begin{aligned}
& -\mu\tilde{f}\left(x_\varepsilon, t_\varepsilon, \frac{\tilde{u}^\mu(x_\varepsilon,t_\varepsilon)}{\mu} - h_x, s_x(\frac{p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)}{\mu} - Dh_x)\right) \\
& +\tilde{f}\left(y_\varepsilon, t_\varepsilon, \tilde{v}(y_\varepsilon,t_\varepsilon) - h_y, s_y(p_\varepsilon - Dh_y)\right) \\
= \;& \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{T}_1 &= -\mu\tilde{f}\left(x_\varepsilon, t_\varepsilon, \frac{\tilde{u}^\mu(x_\varepsilon,t_\varepsilon)}{\mu} - h_x, s_x(\frac{p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)}{\mu} - Dh_x)\right) \\
&\quad +\mu\tilde{f}\left(x_\varepsilon, t_\varepsilon, \tilde{v}(y_\varepsilon,t_\varepsilon) - h_y, s_x(\frac{p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)}{\mu} - Dh_x)\right), \\
\mathcal{T}_2 &= -\mu\tilde{f}\left(x_\varepsilon, t_\varepsilon, \tilde{v}(y_\varepsilon,t_\varepsilon) - h_y, s_x(\frac{p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)}{\mu} - Dh_x)\right) \\
&\quad +\mu\tilde{f}\left(y_\varepsilon, t_\varepsilon, \tilde{v}(y_\varepsilon,t_\varepsilon) - h_y, s_x(\frac{p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)}{\mu} - Dh_x)\right), \\
\mathcal{T}_3 &= -\mu\tilde{f}\left(y_\varepsilon, t_\varepsilon, \tilde{v}(y_\varepsilon,t_\varepsilon) - h_y, s_x(\frac{p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)}{\mu} - Dh_x)\right) \\
&\quad +\tilde{f}\left(y_\varepsilon, t_\varepsilon, \tilde{v}(y_\varepsilon,t_\varepsilon) - h_y, s_y(p_\varepsilon - Dh_y)\right).
\end{aligned}$$

We estimate $\mathcal{T}_1$. From (60), we have

$$\tilde{u}(x_\varepsilon,t_\varepsilon) = \frac{\tilde{u}^\mu(x_\varepsilon,t_\varepsilon)}{\mu} > \tilde{v}(y_\varepsilon,t_\varepsilon) + (1-\mu)\tilde{u}(x_\varepsilon,t_\varepsilon).$$

Using (45) (the monotonicity in u of $\tilde{f}$) and then **(B)**(v) (Lipschitz continuity in u of f), we get

$$\begin{aligned}
& \tilde{f}\left(x_\varepsilon, t_\varepsilon, \frac{\tilde{u}^\mu(x_\varepsilon,t_\varepsilon)}{\mu} - h_x, s_x(\frac{p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)}{\mu} - Dh_x)\right) \\
\geq \;& \tilde{f}\left(x_\varepsilon, t_\varepsilon, \tilde{v}(y_\varepsilon,t_\varepsilon) + (1-\mu)\tilde{u}(x_\varepsilon,t_\varepsilon) - h_x, s_x(\frac{p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)}{\mu} - Dh_x)\right) \\
\geq \;& \tilde{f}\left(x_\varepsilon, t_\varepsilon, \tilde{v}(y_\varepsilon,t_\varepsilon) + h_y, s_x(\frac{p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)}{\mu} - Dh_x)\right) \\
& -\hat{C}(1-\mu)|\tilde{u}(x_\varepsilon,t_\varepsilon)| - \hat{C}|h_x - h_y|.
\end{aligned}$$

Since h is continuous, we have $\hat{C}|h_x - h_y| = m(\varepsilon)$. By (46) and since $x_\varepsilon \to \bar{x}$, we obtain

$$\hat{C}(1-\mu)|\tilde{u}(x_\varepsilon,t_\varepsilon)| \leq \hat{C}\overline{C}(1-\mu)(1+|x_\varepsilon|^p) = \hat{C}(1-\mu)h(\bar{x}) + m(\varepsilon).$$

Therefore

$$\mathcal{T}_1 \leq \hat{C}(1-\mu)h(\bar{x}) + m(\varepsilon). \tag{67}$$

The estimate of $\mathcal{T}_2$ relies on **(B)**(ii). Setting $Q_\varepsilon = \mathrm{e}^{Lt_\varepsilon} s_x(\frac{p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)}{\mu} - Dh_x)$ and recalling that r is defined at the beginning of Step 3, we have

$$\begin{aligned}
|\mathcal{T}_2| &\leq \mu|g(x_\varepsilon,t_\varepsilon) - g(y_\varepsilon,t_\varepsilon)| \\
&\quad +\mu\mathrm{e}^{-Lt_\varepsilon}|f(x_\varepsilon,t_\varepsilon,\mathrm{e}^{Lt_\varepsilon}(\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y),Q_\varepsilon) - f(y_\varepsilon,t_\varepsilon,\mathrm{e}^{-Lt_\varepsilon}(\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y),Q_\varepsilon)| \\
&\leq \mu|g(x_\varepsilon,t_\varepsilon) - g(y_\varepsilon,t_\varepsilon)| \\
&\quad +\mu\mathrm{e}^{-Lt_\varepsilon} m_{2r}\left((1+|\mathrm{e}^{Lt_\varepsilon}(\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y)| + |Q_\varepsilon|)|x_\varepsilon - y_\varepsilon|\right) \\
&\leq m(\varepsilon),
\end{aligned} \tag{68}$$

since g is continuous, $|x_\varepsilon - y_\varepsilon| = m(\varepsilon)$ and $p_\varepsilon|x_\varepsilon - y_\varepsilon| = |x_\varepsilon - y_\varepsilon|^2/\varepsilon^2 = m(\varepsilon)$ by (59).

Let us turn to the estimate of $\mathcal{T}_3$. We have

$$\begin{aligned}
\mathcal{T}_3 &= L(1-\mu)(\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y) + (1-\mu)\tilde{g}(y_\varepsilon,t_\varepsilon) \\
&\quad -\mu\mathrm{e}^{-Lt_\varepsilon} f\left(y_\varepsilon,t_\varepsilon,\mathrm{e}^{Lt_\varepsilon}(\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y),\mathrm{e}^{Lt_\varepsilon}s_x(\frac{p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)}{\mu} - Dh_x)\right) \\
&\quad +\mathrm{e}^{-Lt_\varepsilon} f\left(y_\varepsilon,t_\varepsilon,\mathrm{e}^{Lt_\varepsilon}(\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y),\mathrm{e}^{Lt_\varepsilon}s_y(p_\varepsilon - Dh_y)\right).
\end{aligned}$$

At first, from (46), we have

$$L(1-\mu)(\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y) \leq -\frac{L(1-\mu)}{2}h(\bar{x}) + m(\varepsilon).$$

Using **(C)** (see (70) for the details), a straightforward computation gives an estimate for the continuous function $\tilde{g}$:

$$|\tilde{g}(y_\varepsilon,t_\varepsilon)| \leq (p(p-1)NC_\sigma^2 + pC_b)h(\bar{x}) + m(\varepsilon).$$

Now we estimate the f-terms. From **(B)**(iii) (convexity of f with respect to the gradient variable), we can apply (29) to obtain

$$\begin{aligned}
&-\mu\mathrm{e}^{-Lt_\varepsilon} f\left(y_\varepsilon,t_\varepsilon,\mathrm{e}^{Lt_\varepsilon}(\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y),\mathrm{e}^{Lt_\varepsilon}s_x(\frac{p_\varepsilon + D\varphi(x_\varepsilon,t_\varepsilon)}{\mu} - Dh_x)\right) \\
&+\mathrm{e}^{-Lt_\varepsilon} f\left(y_\varepsilon,t_\varepsilon,\mathrm{e}^{Lt_\varepsilon}(\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y),\mathrm{e}^{Lt_\varepsilon}s_y(p_\varepsilon - Dh_y)\right) \\
\leq\ & (1-\mu)\mathrm{e}^{-Lt_\varepsilon} f\left(y_\varepsilon,t_\varepsilon,\mathrm{e}^{Lt_\varepsilon}(\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y),Q_\varepsilon\right),
\end{aligned}$$

where

$$\begin{aligned}
Q_\varepsilon &= \frac{\mathrm{e}^{Lt_\varepsilon}}{\mu - 1}\left((s_x - s_y)p_\varepsilon + s_x D\varphi(x_\varepsilon,t_\varepsilon) + s_y Dh_y - \mu s_x Dh_x\right) \\
&= \mathrm{e}^{L\bar{t}}s(\bar{x},\bar{t})\left(\frac{D\varphi(\bar{x},\bar{t})}{\mu-1} - Dh(\bar{x})\right) + m(\varepsilon).
\end{aligned}$$

from **(B)**(iv) and (59). From **(B)**(v), (46) and the continuity of f, it follows

$$\begin{aligned}
&\mathrm{e}^{-Lt_\varepsilon} f\left(y_\varepsilon,t_\varepsilon,\mathrm{e}^{Lt_\varepsilon}(\tilde{v}(y_\varepsilon,t_\varepsilon)-h_y),Q_\varepsilon\right) \\
\leq\ & \mathrm{e}^{-L\bar{t}} f\left(\bar{x},\bar{t},0,\mathrm{e}^{L\bar{t}}s(\bar{x},\bar{t})\left(\frac{D\varphi(\bar{x},\bar{t})}{\mu-1} - Dh(\bar{x})\right)\right) + \frac{3\hat{C}}{2}h(\bar{x}) + m(\varepsilon).
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\mathcal{T}_3 &\leq (1-\mu)\left(-\frac{L}{2}+p(p-1)NC_\sigma^2+pC_b+\frac{3\hat{C}}{2}\right)h(\bar{x}) \\ &\quad +\mathrm{e}^{-L\bar{t}}f\left(\bar{x},\bar{t},0,\mathrm{e}^{L\bar{t}}s(\bar{x},\bar{t})\left(\frac{D\varphi(\bar{x},\bar{t})}{\mu-1}-Dh(\bar{x})\right)\right)+m(\varepsilon). \qquad (69)\end{aligned}$$

Step 7. End of the proof. Combining (63), (64), (65), (66), (67) and (68), setting $L > 4p(p-1)NC_\sigma^2+4pC_b+10\hat{C}$ and sending $\rho\to 0$ and then $\varepsilon\to 0$, we get

$$\begin{aligned}& \varphi_t(\bar{x},\bar{t})-\mathrm{Trace}\left[\sigma\sigma^T(\bar{x},\bar{t})D^2\varphi(\bar{x},\bar{t})\right]-C_b(1+|\bar{x}|)|D\varphi(\bar{x},\bar{t})|+\frac{L}{4}(1-\mu)h(\bar{x}) \\ & -(1-\mu)\mathrm{e}^{-L\bar{t}}f\left(\bar{x},\bar{t},0,\mathrm{e}^{L\bar{t}}s(\bar{x},\bar{t})\left(\frac{D\varphi(\bar{x},\bar{t})}{\mu-1}-Dh(\bar{x})\right)\right) \\ \leq\ & 0,\end{aligned}$$

which is exactly the new equation for $\tilde{w}$ in the case (58). It completes the proof of the lemma. □

Proof of Lemma 3.4. For simplicity, we fix R and set $\varphi=\varphi_R$ for simplicity. Therefore φ_r denotes the derivative of φ wrt the space variable. We compute

$$\begin{aligned}&\Phi_t = C\varphi_t, \quad D\Phi=\varphi_r Dh, \\ &D^2\Phi=\varphi_r D^2h+\varphi_{rr}Dh\otimes Dh,\end{aligned}$$

with

$$h=\overline{C}(1+|x|^p),\quad Dh=p\overline{C}|x|^{p-2}x,\quad D^2h=p\overline{C}(|x|^{p-2}Id+(p-2)|x|^{p-4}x\otimes x).$$

For all $(x,t)\in I\!R^N\times(0,T]$,

$$\begin{aligned}&\mathcal{L}(\Phi(x,t)) \\ =\ & C\varphi_t-\left(\mathrm{Trace}(\sigma\sigma^TD^2h)+C_b(1+|x|)|Dh|\right)\varphi_r-\mathrm{Trace}(\sigma\sigma^TDh\otimes Dh)\varphi_{rr} \\ & +(1-\mu)\frac{L}{4}h-(1-\mu)\mathrm{e}^{-Lt}f\left(x,t,0,\mathrm{e}^{Lt}s\left(\frac{\varphi_r}{\mu-1}+1\right)Dh\right). \qquad (70)\end{aligned}$$

Using **(C)**(ii) and the fact that $p'(p-1)=p$, we have the following estimates:

$$\begin{aligned}&|Dh|\leq p\overline{C}|x|^{p-1},\quad |D^2h|\leq p(p-1)\overline{C}|x|^{p-2}, \\ &C_b(1+|x|)|Dh|\leq pC_bh, \\ &|\mathrm{Trace}(\sigma\sigma^TD^2h)|\leq p(p-1)NC_\sigma^2h, \\ &0\leq \mathrm{Trace}(\sigma\sigma^TDh\otimes Dh)\leq C_\sigma^2(1+|x|^2)p^2\overline{C}^2|x|^{2(p-1)}\leq p^2C_\sigma^2h^2.\end{aligned} \qquad (71)$$

Now, the assumption **(B)**(i) on the growth of f plays a crucial role:

$$\begin{aligned} & f\left(x,t,0,\mathrm{e}^{Lt}s\left(\frac{\varphi_r}{\mu-1}+1\right)Dh\right) \\ \leq\ & C_f\left(1+|x|^p+\left|\mathrm{e}^{Lt}s\left(\frac{\varphi_r}{\mu-1}+1\right)Dh\right|^{p'}\right) \\ \leq\ & C_f\left(\frac{1}{\overline{C}}+p^{p'}C_s^{p'}\overline{C}^{p'-1}\mathrm{e}^{Lp't}\left(\frac{\mathrm{e}^T}{1-\mu}+1\right)^{p'}\right)h(x) \end{aligned}$$

since $\varphi_r \leq \mathrm{e}^T$ (Lemma 3.3) and $|Dh|^{p'} \leq p^{p'}\overline{C}^{p'-1}h(x)$ (because $p'(p-1)=p$). It follows from (69),

$$\begin{aligned} & \mathcal{L}(\Phi(x,t)) \\ \geq\ & C\varphi_t-(p(p-1)NC_\sigma^2+pC_b)\varphi_r-p^2C_\sigma^2h^2\varphi_{rr} \\ & +(1-\mu)\left(\frac{L}{4}-\frac{C_f\mathrm{e}^{-Lt}}{\overline{C}}-p^{p'}C_s^{p'}\overline{C}^{p'-1}\mathrm{e}^{Lp't}\left(\frac{\mathrm{e}^T}{1-\mu}+1\right)^{p'}\right)h(x). \end{aligned}$$

We take

$$\begin{aligned} C &> \max\left\{p(p-1)NC_\sigma^2+pC_b,p^2C_\sigma^2\right\}, \\ L &> \frac{4C_f}{\overline{C}}+4p^{p'}C_s^{p'}\overline{C}^{p'-1}\mathrm{e}^{p'}\left(\frac{\mathrm{e}^T}{1-\mu}+1\right)^{p'}+1 \quad \text{and (47) holds,} \\ \tau &= \frac{1}{L}. \end{aligned}$$

For this choice of parameters, for all $(x,t)\in I\!R^N\times(0,\tau]$, we have

$$\mathcal{L}(\Phi(x,t)) \geq C\left(\varphi_t(h,Ct)-h\varphi_r(h,Ct)-h^2\varphi_{rr}(h,Ct)\right)+(1-\mu)h>0$$

since φ is a solution of (50) and $h>0$. This proves the lemma. □

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