Sub-criticality of non-local Schrödinger systems with antisymmetric potentials and applications to half-harmonic maps

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Abstract

We consider nonlocal linear Schrödinger-type critical systems of the type

$$\Delta^{1/4} v = \Omega\, v \quad \text{in } I\!R\,, \tag{1}$$

where Ω is antisymmetric potential in $L^2(I\!R, so(m))$, v is a $I\!R^m$ valued map and $\Omega\, v$ denotes the matrix multiplication. We show that every solution $v \in L^2(I\!R, I\!R^m)$ of (1) is in fact in $L^p_{loc}(I\!R, I\!R^m)$, for every $2 \le p < +\infty$, in other words, we prove that the system (1) which is a-priori only critical in L^2 happens to have a subcritical behavior for antisymmetric potentials. As an application we obtain the $C^{0,\alpha}_{loc}$ regularity of weak 1/2-harmonic maps into C^2 compact sub-manifolds without boundary.

Key words. Harmonic maps, nonlinear elliptic PDE's, regularity of solutions, commutator estimates.

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Contents

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1 Introduction

In this paper we consider maps $v = (v_1, \cdots, v_m) \in L^2(I\!R, I\!R^m)$ solving a system of the form

$$\forall i = 1 \cdots m \qquad \Delta^{1/4} v^i = \sum_{j=1}^{m} \Omega^i_j \, v^j \,, \tag{2}$$

where $\Omega = (\Omega^j_i)_{i,j=1\cdots m} \in L^2(I\!R, so(m))$ is an L^2 map from $I\!R$ into the space $so(m)$ of $m \times m$ antisymmetric matrices. The operator $\Delta^{1/4}$ on $I\!R$ is defined by means of the the Fourier transform as follows

$$\widehat{\Delta^{1/4} u} = |\xi|^{1/2} \hat{u} \,,$$

(given a function f, $\hat{f}$ or $\mathcal{F}[f]$ denotes the Fourier transform of f).

We will also simply denote such a system in the following way

$$\Delta^{1/4} v = \Omega \, v \,.$$

We remark that the system (5) is a-priori critical for $v \in L^2(I\!R)$. Indeed under the assumptions that $v, \Omega \in L^2$ we obtain that $\Delta^{1/4} v \in L^1$ and using classical theory on singular integrals we deduce that $v \in L^{2,\infty}_{loc}$, the weak-L^2 space, which has the same homogeneity of L^2. Thus we are more or less back to the initial assumption which is a property that characterizes critical equations.

In such a critical situation it is a-priori not clear whether solutions have some additional regularity or whether weakly converging sequences of solutions tends to another solution (stability of the equation under weak convergence)...etc.

In [10] and [11] the second author proved the sub-criticality of local *a-priori* critical Schödinger systems of the form

$$\forall i = 1 \cdots m \qquad - \Delta u^i = \sum_{j=1}^{m} \Omega^i_j \cdot \nabla u^j \,, \tag{3}$$

where $u = (u^1, \cdots, u^m) \in W^{1,2}(D^2, I\!R^m)$ and $\Omega \in L^2(D^2, I\!R^2 \otimes so(m))$, or of the form

$$\forall i = 1 \cdots m \qquad - \Delta v^i = \sum_{j=1}^{m} \Omega^i_j \, v^j \,, \tag{4}$$

where $v \in L^{n/(n-2)}(B^n, I\!R^m)$ and $\Omega \in L^{n/2}(B^n, so(m))$. In each of these two situations the antisymmetry of Ω was responsible for the regularity of the solutions or for the stability of the system under weak convergence.

Our first main result in this paper is to establish the sub-criticality of non-local Schrödinger systems of the form (2). Precisely we prove the following theorem which extends to a non-local setting the phenomena observed in [10] and [11] for the above local systems.

Theorem 1.1 *Let $\Omega \in L^2(I\!R, so(m))$ and $v \in L^2(I\!R)$ be a weak solution of*

$$\Delta^{1/4} v = \Omega\, v\,. \tag{5}$$

Then $v \in L^p_{loc}(I\!R)$ for every $1 \le p < +\infty$.

As in the previous works the main technique to prove Theorem 1.1 is to perform a *change of gauge* by rewriting the system after having multiplied v by a well chosen rotation valued map $P \in H^{1/2}(I\!R, SO(m))$. [(1)] In [10] the choice of P for systems of the form (3) was given by the geometrically relevant *Coulomb Gauge* satisfying

$$div\left[P^{-1}\nabla P + P^{-1}\Omega P\right] = 0\,. \tag{6}$$

In this context there is not hope to solve an equation of the form (6) with the operator ∇ replaced by $\Delta^{1/4}$, since for $P \in SO(m)$ the matrix $P^{-1}\Delta^{1/4}P$ is not in general antisymmetric. The novelty here, like in [11], is to choose the gauge P satisfying the following (maybe less geometrically relevant) equation which involves the antisymmetric part of $P^{-1}\Delta^{1/4}P$[(2)]:

$$Asymm\left(P^{-1}\Delta^{1/4}P\right) := 2^{-1}\left[P^{-1}\Delta^{1/4}P - \Delta^{1/4}P^{-1}P\right] = \Omega\,. \tag{7}$$

The local existence of such P is given by the following theorem.

Theorem 1.2 *There exists $\varepsilon > 0$ and $C > 0$ such that for every $\Omega \in L^2(I\!R; so(m))$ satisfying $\int_{I\!R} |\Omega|^2 dx \le \varepsilon$, there exists $P \in \dot{H}^{1/2}(I\!R, SO(m))$ such that*

$$\left\{\begin{array}{lll} (i) & & P^{-1}\Delta^{1/4}P - \Delta^{1/4}P^{-1}P = 2\,\Omega\,; \\[3mm] (ii) & & \displaystyle\int_{I\!R} |\Delta^{1/4}P|^2 dx \le C \int_{I\!R} |\Omega|^2 dx\,. \end{array}\right. \tag{8}$$

□

[(1)] $SO(m)$ is the space of $m \times m$ matrices R satisying $R^t R = R R^t = Id$ and $det(R) = +1$

[(2)] Given a $m \times m$ matrix M, we denote by $Asymm(M)$ and by $Symm(M)$ respectively the antisymmetric and the symmetric part of M, namely $Asymm(M) := \frac{M-M^t}{2}$ and $Symm(M) := \frac{M+M^t}{2}$, M^t is the transpose of M .

The proof of this theorem is established by following an approach introduced by K.Uhlenbeck in [18] to construct *Coulomb Gauges* for L^2 curvatures in 4 dimension. The construction does not provide the continuity of the map which to $\Omega \in L^2$ assigns $P \in \dot{H}^{1/2}$. This illustrates the difficulty of the proof of Theorem 1.2 which is not a direct consequence of an application of the local inversion theorem but requires more elaborated arguments.

Thus if the L^2 norm of Ω is small, Theorem 1.2 gives a P for which $w := Pv$ satisfies

$$\begin{aligned} \Delta^{1/4} w &= -\left[P\Omega P^{-1} - \Delta^{1/4} P\, P^{-1}\right]\, w + N(P,v) \\ &= -Symm\left((\Delta^{1/4}P)\, P^{-1}\right)\, w + N(P,v)\,. \end{aligned} \tag{9}$$

where N is the bilinear operator defined as follows. For an arbitrary integer n, for every $Q \in \dot{H}^{1/2}(I\!R^n, \mathcal{M}_{\ell\times m}(I\!R^n))$ $\ell \geq 0$(3) and $v \in L^2(I\!R^n, I\!R^m)$, N is given by

$$N(Q,v) := \Delta^{1/4}(Q\, v) - Q\Delta^{1/4} v + \Delta^{1/4} Q\; v\,. \tag{10}$$

One of the key result used in [4] establishes that, under the above assumptions on $Q \in H^{1/2}(I\!R^n, M_m(I\!R))$ and $v \in L^2(I\!R^n, I\!R^m)$, $N(Q,v)$ is more regular than each of its three generating terms respectively $\Delta^{1/4}(Q\; v)$, $Q\Delta^{1/4}v$ and $\Delta^{1/4}Q\, v$ (4). We proved that $N(Q,v)$ is in fact in $H^{-1/2}(I\!R, I\!R^m)$. Such a result in [4] was called a 3-commutator estimate (see Theorem 1.3).

In the paper [5] we improve the gain of regularity by compensation obtained in [4]. In order to make it more precise we recall the definition of the Hardy space $\mathcal{H}^1(I\!R^n)$ which is the space of L^1 functions f on $I\!R^n$satisfying

$$\int_{I\!R^n} \sup_{t\in I\!R} |\phi_t * f|(x)\; dx < +\infty \quad ,$$

where $\phi_t(x) := t^{-n}\; \phi(t^{-1}x)$ and where ϕ is some function in the Schwartz space $\mathcal{S}(I\!R^n)$ satisfying $\int_{I\!R^n} \phi(x)\; dx = 1$. (5)

Lemma 1.1 *There exists a constant $C > 0$ such that, for any $Q \in \dot{H}^{1/2}(I\!R^n, M_m(I\!R))$ and $v \in L^2(I\!R^n, I\!R^m)$, $N(Q,v) = \Delta^{1/4}(Q\, v) - Q\Delta^{1/4}v + \Delta^{1/4}Q\; v$ is in $\mathcal{H}^1(I\!R^n)$ and the following estimate holds*

$$\|N(Q,v)\|_{\mathcal{H}^1} \leq C\, \|Q\|_{\dot{H}^{1/2}}\; \|v\|_{L^2(I\!R)}\,. \tag{11}$$

Thus in equation (9) the last term in the r.h.s happens to be slightly more regular. It remains to deal with the first term in this r.h.s. : $-Symm\left(\Delta^{1/4}PP^{-1}\right)\; w$. A-priori $Symm\left((\Delta^{1/4}P)P^{-1}\right) = 2^{-1}[\Delta^{1/4}P\, P^{-1} + P\,\Delta^{1/4}P^{-1}]$ is only in L^2 but here again we are going to take advantage of a gain of regularity due to a compensation. Though, individually

(3) $\mathcal{M}_{\ell\times m}(I\!R)$ denotes, as usual, the space of $\ell \times m$ real matrices.
(4) The last one for example being only a-priori in L^1.
(5) For more properties on the Hardy space $\mathcal{H}^1$ we refer to [7] and [8].

each of the terms $\Delta^{1/4}P\,P^{-1}$ and its transposed $P\,\Delta^{1/4}P^{-1}$ are only in L^2, the sum happens to belong to the "slightly" smaller space $L^{2,1}$ defined as follows: $L^{2,1}(I\!R)$ is the Lorentz space of measurable functions satisfying

$$\int_{I\!R_+} t^{-1/2} f^*(t)dt < +\infty\,,$$

where f^* is the decreasing rearrangement of $|f|$.

The fact that $Symm\left((\Delta^{1/4}P)\,P^{-1}\right)$ belongs to $L^{2,1}(I\!R)$ comes from the combination of the following lemma according to which $\Delta^{1/4}(Symm\left((\Delta^{1/4}P)\,P^{-1}\right)) \in \mathcal{H}^1(I\!R)$ and the sharp Sobolev embedding (6) which says that $f \in \mathcal{H}^1(I\!R)$ implies that $\Delta^{-1/4}f \in L^{2,1}$. Precisely we have

Lemma 1.2 *Let $P \in H^{1/2}(I\!R, SO(m))$ then $\Delta^{1/4}(Symm\left(\Delta^{1/4}P\ P^{-1}\right))$ is in the Hardy space $\mathcal{H}^1(I\!R)$ and the following estimates hold*

$$\|\Delta^{1/4}[\Delta^{1/4}P\ P^{-1} + P\ \Delta^{1/4}P^{-1}]\|_{\mathcal{H}^1} \le C\|P\|^2_{H^{1/2}}\,,$$

where $C > 0$ is a constant independent of P. This implies in particular that

$$\|Symm\left((\Delta^{1/4}P)\ P^{-1}\right)\|_{L^{2,1}} \le C\|P\|^2_{H^{1/2}}\,. \tag{12}$$

The proof of Lemma 1.2 is a consequence of the *3-commutator estimates* in [4] (see Theorem 1.5 below).

Remark 1 The fact that, for rotation valued maps $P \in W^{2,n/2}(I\!R^n, SO(m))$ $(n > 2)$, $Symm\,(\Delta P\ P^{-1})$ happens to be more regular than $Asymm\,(\Delta P\ P^{-1})$ was also one of the key points in [11].

As we explain in Section 3, Theorem 1.1 is a consequence of this special choice of P for which the new r.h.s. in the gauge transformed equation (9) is slightly more regular due to Lemma 1.1 and Lemma 1.2. More precisely this *gain of regularity* in the right of equation (9) combined with suitable localization arguments permit to obtain the following local Morrey type estimate for Pv and thus for v, (since P is bounded in the L^∞ norm)

$$\sup_{\substack{x_0 \in B(0,\rho)\\ 0<r<\rho/8}} r^{-\beta} \int_{B(x_0,r)} |\Delta^{1/4}v|dx \le C\,, \tag{13}$$

for ρ small enough, $0 < \beta < 1/2$ independent on x_0 and $C > 0$ depending only on the dimension. Proposition 3.2 in [1] yields that $v \in L^q_{loc}(I\!R)$ for some $q > 2$. (7)

(6) The fact that $v \in \mathcal{H}^1$ implies $\Delta^{-1/4}v \in L^{2,1}$ is deduced by duality from the fact that $\Delta^{1/4}v \in L^{2,\infty}$ implies that $v \in BMO(I\!R)$ - This last embedding has been proved by Adams in [1]

(7) In a paper in preparation [5] we show that the solutions of (5) are actually in $L^\infty_{loc}(I\!R)$.

Our study of the linear systems has been originally motivated by the following non-linear problem.

In the joint paper [4] we proved the $C^{0,\alpha}_{loc}$ regularity of weak 1/2-harmonic maps into a sphere S^{m-1}. The second aim of the present paper is to extend this result to weak 1/2-harmonic maps with values in a k dimensional sub-manifold $\mathcal{N}$, which is supposed at least C^2, compact and without boundary. We recall that 1/2-harmonic maps are functions u in the space $\dot{H}^{1/2}(I\!R,\mathcal{N})=\{u\in\dot{H}^{1/2}(I\!R,I\!R^m):\ \ u(x)\in\mathcal{N},\text{a.e,}\}$, which are critical points for perturbation of the type $\Pi^N_{\mathcal{N}}(u+t\varphi)$, ($\varphi\in C^\infty$ and $\Pi^N_{\mathcal{N}}$ is the normal projection on $\mathcal{N}$) of the functional

$$\mathcal{L}(u)=\int_{I\!R}|\Delta^{1/4}u(x)|^2dx\,, \tag{14}$$

(see Definition 1.1 in [4]) . The Euler Lagrange equation associated to this non linear problem can be written as follows :

$$\Delta^{1/2}u\wedge\nu(u)=0\qquad\text{in }\mathcal{D}'(I\!R)\,, \tag{15}$$

where $\nu(z)$ is the Gauss Maps at $z\in\mathcal{N}$ taking values into the grassmannian $\tilde{G}r_{m-k}(I\!R^m)$ of oriented $m-k$ planes in $I\!R^m$ which is given by the oriented normal $m-k-$plane to $T_z\mathcal{N}$. [8]

The Euler Lagrange equation in the form (15) is hiding fundamental properties of this equation such as in particular its elliptic nature and is difficult to use directly to solve problems related to regularity and compactness. One of the first task is then to rewrite it in a form that will make some of its analysis features more apparent. This is the purpose of the next proposition. Before stating it, we need some additional notations

We denote by $P^T(z)$ and $P^N(z)$ the projections respectively to the tangent space $T_z\mathcal{N}$ and to the normal space $N_z\mathcal{N}$ to $\mathcal{N}$ at $z\in\mathcal{N}$. For $u\in\dot{H}^{1/2}(I\!R,\mathcal{N})$ we simply denote by P^T and P^N the compositions $P^T\circ u$ and $P^N\circ u$. In Section 5 we establish that , under the assumption $\mathcal{N}$ to be C^2, $P^T\circ u$ as well as $P^N\circ u$ are matrix valued maps in $\dot{H}^{1/2}(I\!R,M_m(I\!R))$.

A useful formulation of the 1/2-harmonic map equation is given by the following result.

Proposition 1.1 *Let $u\in\dot{H}^{1/2}(I\!R,\mathcal{N})$ be a weak 1/2-harmonic map. Then the following equation holds*

$$\Delta^{1/4}v=\tilde{\Omega}_1+\tilde{\Omega}_2\,v+\Omega\,v\,, \tag{16}$$

where $v\in L^2(I\!R,I\!R^{2m})$ is given by

$$v:=\begin{pmatrix}P^T\Delta^{1/4}u\\ \mathcal{R}P^N\Delta^{1/4}u\end{pmatrix}\,,$$

[8] We can identify the unit simple $m-k$ vector $\nu(z)$ with an oriented $m-k$ plane (see for instance [6]). Moreover since we are assuming that $\mathcal{N}$ is C^2, ν is a C^1 map on $\mathcal{N}$ and the paracomposition gives that $\nu(u)$ is in $\dot{H}^{1/2}(I\!R,\wedge^{m-k}I\!R^m)$ hence, since $\Delta^{1/2}u$ is *a-priori* in $\dot{H}^{-1/2}$ the product $\Delta^{1/2}u\wedge\nu(u)$ makes sense in $\mathcal{D}'$ using the duality $\dot{H}^{1/2}-\dot{H}^{-1/2}$

and where $\mathcal{R}$ *is the Fourier multiplier of symbol* $\sigma(\xi) = i\frac{\xi}{|\xi|}$.
$\Omega \in L^2(I\!R, so(2m))$ *is given by*

$$\Omega = 2 \begin{pmatrix} -\omega & \omega_{\mathcal{R}} \\ \\ \omega_{\mathcal{R}} & -\mathcal{R}\omega_{\mathcal{R}} \end{pmatrix}$$

the maps ω *and* $\omega_{\mathcal{R}}$ *are in* $L^2(I\!R, so(m))$ *and given respectively by*

$$\omega = \frac{\Delta^{1/4}P^TP^T - P^T\Delta^{1/4}P^T}{2}\,,$$

and

$$\omega_{\mathcal{R}} = \frac{(\mathcal{R}\Delta^{1/4}P^T)P^T - P^T(\mathcal{R}\Delta^{1/4}P^T)}{2}\,.$$

Finally the maps $\tilde{\Omega}_1 := \tilde{\Omega}_1(P^N, P^T) \in H^{-1/2}(I\!R, I\!R^{2m})$ *and* $\tilde{\Omega}_2 = \tilde{\Omega}_2(P^N, P^T, \Delta^{1/4}u) \in L^{2,1}(I\!R, M_{2m}(I\!R))$ *and satisfy*

$$\|\tilde{\Omega}_1\|_{H^{-1/2}(I\!R, I\!R^{2m})} \le C\left(\|P^N\|^2_{H^{-1/2}} + \|P^T\|^2_{H^{-1/2}}\right)\;; \tag{17}$$

and

$$\|\tilde{\Omega}_2\|_{L^{2,1}(I\!R, M_{2m}(I\!R))} \le C\left(\|P^N\|_{H^{-1/2}} + \|P^T\|_{H^{-1/2}}\right)\|\Delta^{1/4}u\|_{L^{(2,\infty)}}\,.\quad \Box \tag{18}$$

The explicit formulations of $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are given in Section 5. The control on $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ is a consequence of regularity by compensation results on some operators that we now introduce.

For every $Q, v \in L^2(I\!R^n)$ we define the operator F by

$$F(Q, v) := \mathcal{R}(Q)\mathcal{R}(v) - Qv\,. \tag{19}$$

From the commutator estimates obtained in [3], one can deduce that $F(Q, v) \in H^{-1/2}(I\!R)$ and

$$\|F(Q, v)\|_{H^{-1/2}(I\!R)} \le C\|Q\|_{L^2(I\!R)}\|v\|_{L^2(I\!R)}\,. \tag{20}$$

By a suitable estimate on the dual operator of F (Lemma B.5) we show the following sharper estimate

$$\|F(Q, v)\|_{H^{-1/2}(I\!R)} \le C\|Q\|_{L^2(I\!R)}\|v\|_{L^{2,\infty}(I\!R)}\,. \tag{21}$$

Next we recall some *commutator estimates* we obtained in [4].

Theorem 1.3 *Let* $n \in I\!N^*$ *and let* $u \in BMO(I\!R^n)$, $Q \in \dot{H}^{1/2}(I\!R^n, \mathcal{M}_{\ell\times m}(I\!R^n))$ *. Denote*

$$T(Q, u) := \Delta^{1/4}(Q\Delta^{1/4}u) - Q\Delta^{1/2}u + \Delta^{1/4}u\Delta^{1/4}Q \quad ,$$

then $T(Q, u) \in H^{-1/2}(I\!R^n)$ *and there exists* $C > 0$, *depending only on* n, *such that*

$$\|T(Q, u)\|_{H^{-1/2}(I\!R^n)} \le C\;\|Q\|_{\dot{H}^{1/2}(I\!R^n)}\;\|u\|_{BMO(I\!R^n)}\,.\Box \tag{22}$$

Theorem 1.4 *Let $n \in \mathbb{N}^*$ and let $u \in BMO(\mathbb{R}^n)$, $Q \in \dot{H}^{1/2}(\mathbb{R}^n, \mathcal{M}_{\ell\times m}(\mathbb{R}^n))$. Denote*

$$S(Q,u) := \Delta^{1/4}[Q\Delta^{1/4}u] - \mathcal{R}(Q\nabla u) + \mathcal{R}(\Delta^{1/4}Q\mathcal{R}\Delta^{1/4}u)\,.$$

Then $S(Q,u) \in H^{-1/2}(\mathbb{R}^n)$ and there exists C depending only on n such that

$$\|S(Q,u)\|_{H^{-1/2}(\mathbb{R}^n)} \le C\ \|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)}\|u\|_{BMO(\mathbb{R}^n)}\,.\Box \tag{23}$$

As it is observed in [4], Theorems 1.3 and 1.4 are consequences respectively of the following results which are their "dual versions" .

Theorem 1.5 *Let $u, Q \in \dot{H}^{1/2}(\mathbb{R}^n)$, denote*

$$T^*(Q,u) = \Delta^{1/4}(Q\Delta^{1/4}u) - \Delta^{1/2}(Qu) + \Delta^{1/4}((\Delta^{1/4}Q)u)\,.$$

then $T^*(Q,u) \in \mathcal{H}^1(\mathbb{R}^n)$ and

$$\|T^*(Q,u)\|_{\mathcal{H}^1(\mathbb{R}^n)} \le C\|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)}\|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}\,.\Box \tag{24}$$

Theorem 1.6 *Let $u, Q \in \dot{H}^{1/2}(\mathbb{R}^n)$, denote*

$$S^*(Q,u) = \Delta^{1/4}(Q\Delta^{1/4}u) - \nabla(Q\mathcal{R}u) + \mathcal{R}\Delta^{1/4}(\Delta^{1/4}Q\mathcal{R}u)\,.$$

Then $S^*(Q,u) \in \mathcal{H}^1(\mathbb{R}^n)$ and

$$\|S^*(Q,u)\|_{\mathcal{H}^1(\mathbb{R}^n)} \le C\|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)}\|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}\,.\Box \tag{25}$$

Since the operators T^* and S^* are the duals respectively of T and S, by combining Theorems 1.3 and 1.5 and Theorems 1.4 and 1.6 one gets the followings sharper estimates for T and S:

$$\|T(Q,u)\|_{H^{-1/2}(\mathbb{R}^n)} \le C\ \|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)}\|\Delta^{1/4}u\|_{L^{2,\infty}(\mathbb{R}^n)}\,; \tag{26}$$

$$\|S(Q,u)\|_{H^{-1/2}(\mathbb{R}^n)} \le C\ \|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)}\|\Delta^{1/4}u\|_{L^{2,\infty}(\mathbb{R}^n)}\,. \tag{27}$$

An adaptation of Theorem 1.1 to the Euler Lagrange equation of the 1/2-Energy written in the form (16) leads to the following theorem which is the second main result of the present paper.

Theorem 1.7 *Let $\mathcal{N}$ be a closed C^2 submanifold of $\mathbb{R}^m$ without boundary Let $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ be a weak $1/2-$harmonic map into $\mathcal{N}$, then $u \in C^{0,\alpha}_{loc}(\mathbb{R}, \mathcal{N})$, for all $0 < \alpha < 1$.* $\Box$

Finally a classical elliptic type bootstrap argument leads to the following result (see [5] for the details of this argument).

Theorem 1.8 *Let $\mathcal{N}$ be a smooth closed submanifold of $\mathbb{R}^m$. Let u be a weak $1/2$-harmonic map in $\dot{H}^{1/2}(\mathbb{R}, \mathcal{N}))$, then u is C^∞ .* $\Box$

The regularity of critical points of non-local functionals has been recently investigated by Moser [9]. In this work critical points to the functional that assigns to any $u \in \dot{H}^{1/2}(I\!R, \mathcal{N})$ the minimal Dirichlet energy among all possible extensions in $\mathcal{N}$ are considered, while in the present paper the classical $\dot{H}^{1/2}$ Lagrangian corresponds to the minimal Dirichlet energy among all possible extensions in $I\!R^m$. Hence the approach in [9] consists in working with an intrinsic version of $H^{1/2}-$energy while we are considering here an extrinsic one. The drawback of considering the intrinsic energy is that the Euler Lagrange equation is almost impossible to write explicitly and is then implicit while in the present case it has the explicit form (15). However the intrinsic version of the $1/2-$harmonic map is more closely related to the existing regularity theory of Dirichlet Energy minimizing maps into $\mathcal{N}$.

Finally the regularity of $n/2$ harmonic maps in odd dimension $n > 1$ with values into a sphere has been recently investigated by Schikorra [15] . In this work the author extends the results obtained in [4] by adapting some compensation arguments introduced by Tartar [16].

The paper is organized as follows.

- In Section 3 we prove Theorem 1.1 .
- In Section 4 we prove Theorem 1.2 .
- In Section 5 we derive the Euler-Lagrange equation (16) associated to the Lagrangian (14) and we prove Theorem 1.7 .
- In Appendix A we prove some localization estimates related to the solutions to the linear nonlocal Schröndiger systems (9) .
- In Appendix B we provide some commutator estimates that are crucial for the construction of the gauge P.

2 Preliminaries: function spaces and the fractional Laplacian

In this Section we introduce some notations and definitions we are going to use in the sequel.

For $n \geq 1$, we denote respectively by $\mathcal{S}(I\!R^n)$ and $\mathcal{S}'(I\!R^n)$ the spaces of Schwartz functions and tempered distributions. Moreover given a function v we will denote by $\hat{v}$ and $\mathcal{F}[v]$ the Fourier Transform of v :

$$\hat{v}(\xi) = \mathcal{F}[v](\xi) := \int_{I\!R^n} v(x) e^{-i\xi\cdot x}\, dx\,.$$

Throughout the paper we use the convention that x, y denote variables in the space and ξ, η the variables in the phase .

We recall the definition of fractional Sobolev space (see for instance [17]).

Definition 2.1 *For a real* $s \geq 0$*,*

$$H^s(I\!R^n) = \left\{ v \in L^2(I\!R^n) : \quad |\xi|^s \mathcal{F}[v] \in L^2(I\!R^n) \right\}$$

For a real $s < 0$*,*

$$H^s(I\!R^n) = \left\{ v \in \mathcal{S}'(I\!R^n) : \quad (1 + |\xi|^2)^{s/2} \mathcal{F}[v] \in L^2(I\!R^n) \right\} . \square$$

It is known that $H^{-s}(I\!R^n)$ is the dual of $H^s(I\!R^n)$.

For $0 < s < 1$, another classical characterization of $H^s(I\!R^n)$ which does not make use the Fourier transform is the following, (see for instance [17]).

Lemma 2.1 *For* $0 < s < 1$*,* $u \in H^s(I\!R^n)$ *is equivalent to* $u \in L^2(I\!R^n)$ *and*

$$\left(\int_{I\!R^n} \int_{I\!R^n} \left(\frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \right) dxdy \right)^{1/2} < +\infty .$$

$\square$

For $s > 0$ we set

$$\|u\|_{H^s(I\!R^n)} = \|u\|_{L^2(I\!R^n)} + \| |\xi|^s \mathcal{F}[v] \|_{L^2(I\!R^n)} ,$$

and

$$\|u\|_{\dot{H}^s(I\!R^n)} = \| |\xi|^s \mathcal{F}[v] \|_{L^2(I\!R^n)} .$$

For an open set $\Omega \subset I\!R^n$, $H^s(\Omega)$ is the space of the restrictions of functions from $H^s(I\!R^n)$ and

$$\|u\|_{\dot{H}^s(\Omega)} = \inf \left\{ \|U\|_{\dot{H}^s(I\!R^n)}, \quad U = u \text{ on } \Omega \right\} .$$

In the case $0 < s < 1$ then $u \in H^s(\Omega)$ if and only if $u \in L^2(\Omega)$ and

$$\left(\int_{\Omega} \int_{\Omega} \left(\frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \right) dxdy \right)^{1/2} < +\infty .$$

Moreover

$$\|u\|_{\dot{H}^s(\Omega)} \simeq \left(\int_{\Omega} \int_{\Omega} \left(\frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \right) dxdy \right)^{1/2} < +\infty ,$$

see for instance [17].

Finally for a submanifold $\mathcal{N}$ of $I\!R^m$ we can define

$$H^s(I\!R^n, \mathcal{N}) = \{ u \in H^s(I\!R^n, I\!R^m) : \quad u(x) \in \mathcal{N}, \text{a.e.} \} .$$

Given $q > 1$ we also set

$$W^{s,q}(I\!R^n) := \{ v \in L^q(I\!R^n) : \quad \mathcal{F}^{-1}[|\xi|^s \mathcal{F}[v]] \in L^q(I\!R^n) \} .$$

We shall make use of the Littlewood-Paley dyadic decomposition of unity that we recall here. Such a decomposition can be obtained as follows . Let $\phi(\xi)$ be a radial Schwartz function supported in $\{\xi \in I\!R^n : \ |\xi| \le 2\}$, which is equal to 1 in $\{\xi \in I\!R^n : \ |\xi| \le 1\}$. Let $\psi(\xi)$ be the function given by

$$\psi(\xi) := \phi(\xi) - \phi(2\xi)\,.$$

ψ is then a "bump function" supported in the annulus $\{\xi \in I\!R^n : \ 1/2 \le |\xi| \le 2\}$.

Let $\psi_0 = \phi$, $\psi_j(\xi) = \psi(2^{-j}\xi)$ for $j \neq 0$. The functions ψ_j, for $j \in Z\!\!\!Z$, are supported in $\{\xi \in I\!R^n : \ 2^{j-1} \le |\xi| \le 2^{j+1}\}$ and they realize a dyadic decomposition of the unity :

$$\sum_{j \in Z\!\!\!Z} \psi_j(x) = 1\,.$$

We further denote

$$\phi_j(\xi) := \sum_{k=-\infty}^{j} \psi_k(\xi)\,.$$

The function ϕ_j is supported on $\{\xi, \ |\xi| \le 2^{j+1}\}$.

We recall the definition of the homogeneous Besov spaces $\dot{B}^s_{p,q}(I\!R^n)$ and homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{pq}(I\!R^n)$ in terms of the above dyadic decomposition.

Definition 2.2 *Let $s \in I\!R$, $0 < p, q \le \infty$. For $f \in \mathcal{S}'(I\!R^n)$ we set*

$$\begin{aligned} \|u\|_{\dot{B}^s_{p,q}(I\!R^n)} &= \left(\sum_{j=-\infty}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}[\psi_j \mathcal{F}[u]]\|^q_{L^p(I\!R^n)} \right)^{1/q} && \textit{if } q < \infty \\ \|u\|_{\dot{B}^s_{p,q}(I\!R^n)} &= \sup_{j \in Z\!\!\!Z} 2^{js} \|\mathcal{F}^{-1}[\psi_j \mathcal{F}[u]]\|_{L^p(I\!R^n)} && \textit{if } q = \infty \end{aligned} \tag{28}$$

When $p, q < \infty$ we also set

$$\|u\|_{\dot{F}^s_{p,q}(I\!R^n)} = \left\| \left(\sum_{j=-\infty}^{\infty} 2^{jsq} |\mathcal{F}^{-1}[\psi_j \mathcal{F}[u]]|^q \right)^{1/q} \right\|_{L^p}\,.$$

□

The space of all tempered distributions u for which the quantity $\|u\|_{\dot{B}^s_{p,q}(I\!R^n)}$ is finite is called the homogeneous Besov space with indices s, p, q and it is denoted by $\dot{B}^s_{p,q}(I\!R^n)$. The space of all tempered distributions f for which the quantity $\|f\|_{\dot{F}^s_{p,q}(I\!R^n)}$ is finite is called the homogeneous Triebel-Lizorkin space with indices s, p, q and it is denoted by $\dot{F}^s_{p,q}(I\!R^n)$. A classical result says [9] that $\dot{W}^{s,p}(I\!R^n) = \dot{B}^s_{p,2}(I\!R^n) = \dot{F}^s_{p,2}(I\!R^n)$.

[9] See for instance [7]

Finally we denote by $\mathcal{H}^1(I\!R^n)$ the homogeneous Hardy Space in $I\!R^n$. A less classical result [10] asserts that $\mathcal{H}^1(I\!R^n) \simeq \dot{F}^0_{2,1}$, thus we have

$$\|u\|_{\mathcal{H}^1(I\!R^n)} \simeq \int_{I\!R} \left(\sum_j |\mathcal{F}^{-1}[\psi_j \mathcal{F}[u]]|^2 \right)^{1/2} dx\,.$$

We recall that in dimension $n = 1$, the space $\dot{H}^{1/2}(I\!R)$ is continuously embedded in the Besov space $\dot{B}^0_{\infty,\infty}(I\!R)$. More precisely we have

$$\dot{H}^{1/2}(I\!R) \hookrightarrow BMO(I\!R) \hookrightarrow \dot{B}^0_{\infty,\infty}(I\!R)\,, \tag{29}$$

where $BMO(I\!R)$ is the space of bounded mean oscillation dual to $\mathcal{H}^1(I\!R^n)$ (see for instance [14], page 31).

The s-fractional Laplacian of a function $u \colon I\!R^n \to I\!R$ is defined as a pseudo differential operator of symbol $|\xi|^{2s}$:

$$\widehat{\Delta^s u}(\xi) = |\xi|^{2s}\hat{u}(\xi)\,. \tag{30}$$

In the case where $s = 1/2$, we can write $\Delta^{1/2}u = -\mathcal{R}(\nabla u)$ where $\mathcal{R}$ is Fourier multiplier of symbol $\frac{i\xi}{|\xi|}$.

To conclude we introduce some basic notations.

We denote by $B_r(\bar{x})$ the ball of radius r and centered at $\bar{x}$. If $\bar{x} = 0$ we simply write B_r. If $x, y \in I\!R^n$, $x \cdot y$ denote the scalar product between x, y.

Given a subset K of $I\!R^n$, $1\!\!1_K$ denotes the characteristic function of B.

For every function $u \colon I\!R^n \to I\!R$ we denote by $M(u)$ the maximal function of u, namely

$$M(u) = \sup_{r>0,\ x\in I\!R^n} |B(x,r)|^{-1} \int_{B(x,r)} |u(y)|dy\,. \tag{31}$$

Given $q > 1$ we denote by q' the coniugate of q: $q^{-1} + q'^{-1} = 1$.

In the sequel we will often use the symbols $\lesssim$ and $\simeq$ instead of $\le$ and $=$, if the constants appearing in the estimates are not relevant and therefore they are omitted.

3 Regularity of nonlocal Schrödinger type systems

In this Section we prove Theorem 1.1. The proof is based on " ad-hoc" *localization estimates* given in the Appendix A and on the *3 terms commutator estimates* (26) and (24).

Proof of theorem 1.1.

Let $\rho > 0$ be such that $\|1\!\!1_{B(0,\rho)}\Omega\|_{L^2} \le \varepsilon_0$, with ε_0 small enough. We decompose Ω as follows $\Omega_1 = 1\!\!1_{B(0,\rho)}\Omega$ and $\Omega_2 = (1 - 1\!\!1_{B(0,\rho)})\Omega$.

[10] See for instance [8].

Let $P \in \dot H^{1/2}(I\!R, SO(m))$ given by Theorem 1.2 (with Ω replaced by Ω_1). We have

$$\Delta^{1/4}(Pv) = \left[P\Omega P^{-1} - (\Delta^{1/4}P)P^{-1}\right] Pv + N(P,v) \tag{32}$$

where N is the operator defined in 10.

Since P satisfies (8)(i) we have

$$\begin{aligned} P\Omega P^{-1} - \Delta^{1/4}PP^{-1} &= -\frac{(\Delta^{1/4}P)P^{-1} + P\Delta^{1/4}P^{-1}}{2} \\ &= -Symm\left((\Delta^{1/4}P)P^{-1}\right). \end{aligned} \tag{33}$$

From Theorem 1.5 it follows that $Symm\left((\Delta^{1/4}P)P^{-1}\right) \in L^{2,1}(I\!R)$. We stress that the fact that $Symm\left((\Delta^{1/4}P)P^{-1}\right)$ is in $L^{2,1}(I\!R)$ (which is stricly contained in L^2) will play a crucial role.

Claim 1. From Theorems 1.3 and 1.5 we can deduce the estimate (26), which can be expressed in term of the operator N as follows:

$$\|N(Q,v)\|_{\dot H^{1/2}(I\!R^n)} \le C\|v\|_{L^{2,\infty}(I\!R^n)}\ \|Q\|_{H^{1/2}(I\!R^n)}.$$

for every $Q \in \dot H^{1/2}(I\!R^n)$ and $v \in L^2(I\!R^n)$.

Proof of Claim 1.

$$\begin{aligned} \|N(Q,v)\|_{\dot H^{-1/2}(I\!R^n)} &= \sup_{\|h\|_{\dot H^{1/2}}\le 1} \int_{I\!R^n} N(Q,v)h\,dx \\ &= \sup_{\|h\|_{\dot H^{1/2}}\le 1} \int_{I\!R^n} v[Q(\Delta^{1/4}h) - \Delta^{1/4}(Qh) + (\Delta^{1/4}Q)h]dx \\ &= \sup_{\|h\|_{\dot H^{1/2}}\le 1} \int_{I\!R^n} v\Delta^{-1/4}(T^*(Q,h))dx \\ &\text{by applying Theorem 1.5} \\ &\lesssim \sup_{\|h\|_{\dot H^{1/2}}\le 1} \|v\|_{L^{2,\infty}}\|\Delta^{-1/4}(T^*(Q,h))\|_{L^{2,1}} \\ &\lesssim \|v\|_{L^{2,\infty}}\|Q\|_{\dot H^{1/2}}. \end{aligned} \tag{34}$$

This concludes the proof of claim 1.

We set now $w = Pv$ and $\omega = -Symm\left((\Delta^{1/4}P)P^{-1}\right)$ and rewrite equation (32) as follows

$$\Delta^{1/4}w = \omega\, w + N(P, P^{-1}w) + \Omega_2 P^{-1}w. \tag{35}$$

where by construction $\|\omega\|_{L^{2,1}}, \|P\|_{\dot H^{1/2}} \le \varepsilon_0$.

Claim 2 : *There exists $q > 2$ such that $v \in L^q_{loc}(I\!R)$.*

Proof of Claim 2 . In order to establish the claim 2, we are going to establish the following bound

$$\sup_{x_0\in B(0,\rho/8)\,,0<r<\rho/16} r^{-\beta}\ \|w\|_{L^{2,\infty}(B(x_0,r))}<+\infty\,.$$

Let $x_0\in B(0,\rho/8)$ and $r\in(0,\rho/16)$. We argue by duality and multiply (35) by ϕ which is given as follows. Let $g\in L^{2,1}(I\!R)$, with $\|g\|_{L^{2,1}}\le 1$ and set $g_{r\alpha}=1\!\!1_{B(x_0,r\alpha)}g$, with $0<\alpha<1/4$ and $\phi=\Delta^{-1/4}(g_{r\alpha})\in L^\infty(I\!R)\cap\dot H^{1/2}(I\!R)$. We take the scalar product of both sides of equation (35) with ϕ and we integrate.

Left hand side of the equation (35):

$$\begin{aligned}\sup_{\|g\|_{L^{2,1}}\le 1}\int_{I\!R}\phi\ \Delta^{1/4}w dx &= \sup_{\|g\|_{L^{2,1}}\le 1}\int_{I\!R} g_{r\alpha} w dx\\ &= \|w\|_{L^{2,\infty}(B(x_0,r\alpha))}\,.\end{aligned}\tag{36}$$

Right hand side of the equation (35):

We apply Lemmas A.5, A.3, A.4 and A.6 and we respectively obtain

$$\begin{aligned}\int_{I\!R}\phi\ \omega\, w dx &\le \|\omega\|_{L^{2,1}}\ \|g\|_{L^{2,1}}\ \|w\|_{L^{2,\infty}(B(x_0,r))}\\ &\quad +\alpha^{1/2}\sum_{h=-1}^{+\infty}2^{-h/2}\|\omega\|_{L^{2,1}}\ \|g\|_{L^{2,1}}\ \|w\|_{L^{2,\infty}(B(x_0,2^{h+1}r)\setminus B(x_0,2^{h-1}r))}\\ &\lesssim \varepsilon_0\|w\|_{L^{2,\infty}(B(x_0,r))}+\alpha^{1/2}\sum_{h=-1}^{+\infty}2^{-h/2}\|w\|_{L^{2,\infty}(B(x_0,2^{h+1}r)\setminus B(x_0,2^{h-1}r))}\,;\end{aligned}\tag{37}$$

$$\begin{aligned}\int_{I\!R}\phi\ N(P,P^{-1}w)dx &\le \varepsilon_0\|w\|_{L^{2,\infty}(B(x_0,r)}\\ &+\ C\alpha^{1/2}\sum_{h=1}^{+\infty}2^{-h/2}\|w\|_{L^{2,\infty}(B(x_0,2^{h+1}r)\setminus B(x_0,2^{h-1}r))}\,,\end{aligned}\tag{38}$$

and finally

$$\int_{I\!R}\Omega_2P^{-1}\,w\ \phi dx\le C\alpha^{1/2}r^{1/2}\,.\tag{39}$$

Thus combining (36)...(38) we get

$$\begin{aligned}\|w\|_{L^{2,\infty}(B(x_0,r\alpha))} &\lesssim \varepsilon_0\|w\|_{L^{2,\infty}(B(x_0,r)}\\ &+\ \alpha^{1/2}\sum_{h=1}^{+\infty}2^{-h/2}\ \|w\|_{L^{2,\infty}(B_{2^{h+1}r(x_0)}\setminus B_{2^{h-1}r}(x_0))}+\left(\frac{r}{\rho}\right)^{1/2}\alpha^{1/2}\,.\end{aligned}\tag{40}$$

If α and ε are small enough the formula (40) implies that for all $x_0 \in B(0,\rho/8)$ and $0<r<\rho/16$ we have $\|w\|_{L^{2,\infty}(B(x_0,r))} \le Cr^{\beta}$, for some $\beta \in (0,1/2)$ and $C>0$ independent on r . Since $P^{-1} \in L^\infty$, this implies that

$$\sup_{\substack{x_0 \in B(0,\rho/8)\\ 0<r<\rho/16}} r^{-\beta} \int_{B(x_0,r)} |\Delta^{1/4} v| dx < +\infty\,. \tag{41}$$

Proposition 3.2 in [1] yields that $v \in L^q_{loc}(I\!R)$ for some $q>2$ which finishes the proof of claim 2.

Claim 3: *$v \in L^p_{loc}(I\!R)$ for every $p>2$.*

Proof of Claim 3 . We argue as in the proof of claim 2. We consider again $\rho>0$ such that $\|1\!\!1_{B(0,\rho)}\Omega\|_{L^2} \le \varepsilon_0$, with ε_0 small enough. We write $\Omega = \Omega_1 + \Omega_2$ with $\Omega_1 = 1\!\!1_{B(0,\rho)}\Omega$ and $\Omega_2 = (1 - 1\!\!1_{B(0,\rho)})\Omega$. We consider an arbitrary $q>2$ such that $v \in L^q_{loc}$.

Let $x_0 \in B(0,\rho/8)$, $r \in (0,\rho/16)$, $g \in L^{q'}(I\!R)$, with $\|g\|_{L^{q'}} \le 1$ and set $g_{r\alpha} = 1\!\!1_{B(x_0,r\alpha)} g$, with $0<\alpha<1/4$ and $\phi = \Delta^{-1/4}(g_{r\alpha})$. We observe that $\phi \in W^{1/2,q'}(I\!R)$. Moreover since $q'<2$ and $W^{1/2,q'}(I\!R) \hookrightarrow L^{\frac{2q}{q-2}}(I\!R)$, we also have $\phi \in L^{\frac{2q}{q-2}}(I\!R)$.

We write the equation (5) as follows

$$\begin{aligned} \Delta^{1/4} v &= \Omega_1 1\!\!1_{B(x_0,r/2)} v + \sum_{h=0}^{+\infty} \Omega_1 1\!\!1_{B(x_0,2^h r)\setminus B(x_0,2^{h-1}r)} v \\ &+ \Omega_2 v\,. \end{aligned} \tag{42}$$

We take the scalar product of the equation (42) with $\Delta^{-1/4}(g_{r\alpha})$ and integrate. By using Lemmas A.6-A.9 we get that

$$\begin{aligned} \|v\|_{L^q(B(x_0,r\alpha))} &\lesssim \varepsilon_0 \|v\|_{L^q(B(x_0,r/4)} \\ &+ \alpha^{1/q} \sum_{h=1}^{+\infty} 2^{-h/q}\, \|w\|_{L^q(B_{2^{h+1}r(x_0)} \setminus B_{2^{h-1}r}(x_0))} + \left(\frac{r}{\rho}\right)^{1/q} \alpha^{1/q}\,. \end{aligned} \tag{43}$$

If α and ε are small enough, the formula (43) implies

$$\sup_{\substack{x_0 \in B(0,\rho/8)\\ 0<r<\rho/16}} r^{-\gamma} \left[\int_{B(x_0,r)} |v|^q dx\right]^{1/q} < +\infty\,, \tag{44}$$

with $0<\gamma<1/4$ independent on q . Thus by plugging (44) in the equation (5) we obtain for the same $\gamma>0$ independent of q

$$\sup_{\substack{x_0 \in B(0,\rho/8)\\ 0<r<\rho/16}} r^{-\gamma}\, \|\Delta^{1/4} v\|_{L^{2q/(q+2)} B(x_0,r)}\; dx < +\infty\,. \tag{45}$$

Theorem 3.1 in [1] yields that $v \in L^{\tilde{q}}_{loc}$, with $\tilde{q} > q$ given by

$$\tilde{q}^{-1} = q^{-1} - 2^{-1}[1 - \gamma(q^{-1} + 2^{-1})^{-1}]^{-1} .$$

Since $q > 2$ we have

$$\tilde{q}^{-1} < q^{-1} - \frac{2}{(1 - 4\gamma)} .$$

By repeating the above arguments with q replaced by $\tilde{q}$, one finally gets that $v \in L^p_{loc}$ for every $p > 2$. This concludes the proof of theorem 1.1. □

4 Construction of an optimal gauge P : the proof of Theorem 1.2.

Proof of Theorem 1.2.

We follow the strategy of [11] to construct solutions to $Asymm(P^{-1}\,\Delta P) = \Omega$ which was itself inspired by Uhlenbeck's construction in [18] of Coulomb Gauges solving (6).

Let $2 < q < +\infty$ and consider

$$\mathcal{U}^q_\varepsilon = \left\{ \Omega \in L^q(I\!\!R, so(m)) \cap L^{q'}(I\!\!R, so(m)) : \int_{I\!\!R} |\Omega|^2 dx \le \varepsilon \right\} .$$

Claim: There exist $\varepsilon > 0$ small enough and $C > 0$ large enough such that

$$\mathcal{V}^q_{\varepsilon,C} := \left\{ \begin{array}{c} \Omega \in \mathcal{U}^q_\varepsilon \ \text{ : there exits } P \in \dot{W}^{1/2,q}(I\!\!R, SO(m)) \cap \dot{W}^{1/2,q'}(I\!\!R, SO(m)) \\ \\ \text{satisfying (8) (i)-(ii) \quad and} \\ \\ \displaystyle\int_{I\!\!R} |\Delta^{1/4} P|^q dx \le C \int_{I\!\!R} |\Omega|^q dx, \quad \int_{I\!\!R} |\Delta^{1/4} P|^{q'} dx \le C \int_{I\!\!R} |\Omega|^{q'} dx \end{array} \right\}$$

is open and closed in $\mathcal{U}^q_\varepsilon$ and thus $\mathcal{V}^q_\varepsilon \equiv \mathcal{U}^q_\varepsilon$. Actually the set $\mathcal{U}^q_\varepsilon$ is star-shaped with respect to the origin (if $\Omega \in \mathcal{U}^q_\varepsilon$, then $t\Omega \in \mathcal{U}^q_\varepsilon$ for every $0 \le t \le 1$) and therefore it is path connected.
Proof of the claim.

We first observe that $\mathcal{V}^q_{\varepsilon,C} \ne \emptyset$, $(0 \in \mathcal{V}^q_{\varepsilon,C})$.
Step 1: *For any $\varepsilon > 0$ and $C > 0$, $\mathcal{V}^q_{\varepsilon,C}$ is closed in $L^q \cap L^{q'}(I\!\!R, so(m))$.*

Let $\Omega_n \in \mathcal{V}^q_{\varepsilon,C}$ such that $\Omega_n \to \Omega_\infty$ in the norm $L^q \cap L^{q'}$, as $n \to +\infty$ and let P_n be a solution of

$$P_n^{-1}\Delta^{1/4} P_n - \Delta^{1/4} P_n^{-1} P_n = 2\Omega_n$$

with

$$\int_{I\!R} |\Delta^{1/4} P_n|^q dx \le C_0 \int_{I\!R} |\Omega_n|^q dx \,,$$

$$\int_{I\!R} |\Delta^{1/4} P_n|^{q'} dx \le C_0 \int_{I\!R} |\Omega_n|^{q'} dx \,.$$

Since $\Omega_n \to \Omega_\infty$ in the norm $L^q \cap L^{q'}$ and $\int_{I\!R} |\Omega_n|^2 dx \le \varepsilon$, we can pass to the limit in this inequality and we have

$$\int_{I\!R} |\Omega_\infty|^2 dx \le \varepsilon \,, \tag{46}$$

which implies that $\Omega_\infty \in \mathcal{U}_\varepsilon$.

One can extract a subsequence $P_{n'} \rightharpoonup P_\infty$ in $\dot{W}^{1/2,q} \cap \dot{W}^{1/2,q'}$. By the Rellich-Kondrachov Theorem we also have $P_{n'} \to P_\infty$ in L^2_{loc} and hence $P_\infty \in SO(m)$ a.e. Thus $P_\infty \in \dot{W}^{1/2,q}(I\!R, SO(m)) \cap \dot{W}^{1/2,q'}(I\!R, SO(m))$ and the lower semi-continuity of the $\dot{H}^{1/2}$, $\dot{W}^{1/2,q}$ and $\dot{W}^{1/2,q'}$ norms implies that

$$\begin{aligned} &\int_{I\!R} |\Delta^{1/4} P_\infty|^2 dx \le C_0 \int_{I\!R} |\Omega_\infty|^2 dx \,, \\ &\int_{I\!R} |\Delta^{1/4} P_\infty|^q dx \le C_0 \int_{I\!R} |\Omega_\infty|^q dx \\ \text{and} \quad &\int_{I\!R} |\Delta^{1/4} P_\infty|^{q'} dx \le C_0 \int_{I\!R} |\Omega_\infty|^{q'} dx \,. \end{aligned} \tag{47}$$

We have

$$P_n^{-1} \Delta^{1/4} P_n - \Delta^{1/4} P_n^{-1} P_n \to P_\infty^{-1} \Delta^{1/4} P_\infty - \Delta^{1/4} P_\infty^{-1} P_\infty \qquad \text{in } \mathcal{D}'(I\!R) \,.$$

Since $P_n^{-1} \Delta^{1/4} P_n - \Delta^{1/4} P_n^{-1} P_n = \Omega_n \to \Omega_\infty$ in $\mathcal{D}'$ as well, we deduce that

$$P_\infty^{-1} \Delta^{1/4} P_\infty - \Delta^{1/4} P_\infty^{-1} P_\infty = \Omega_\infty \qquad \text{a.e.} \tag{48}$$

and combining (46), (47) and (48) we deduce that $\Omega_\infty \in \mathcal{V}^q_{\varepsilon,C}$ which concludes the proof of Step 1.

Step 2: *For $\varepsilon > 0$ small enough and $C > 0$ large enough $\mathcal{V}^q_{\varepsilon,C}$ is open.*

For every $P_0 \in \dot{W}^{1/2,q}(I\!R, SO(m)) \cap \dot{W}^{1/2,q'}(I\!R, SO(m))$ we introduce the map

$$F^{P_0} \ : \ \dot{W}^{1/2,q} \cap \dot{W}^{1/2,q'}(I\!R, so(m)) \longrightarrow L^q \cap L^{q'}(I\!R, so(m))$$

$$U \longrightarrow (P_0 \exp U)^{-1} \Delta^{1/4} (P_0 \exp U) - \Delta^{1/4} (P_0 \exp U)^{-1} (P_0 \exp U) \,.$$

We claim first that F^{P_0} is a C^1 map between the two Banach spaces $\dot{W}^{1/2,q} \cap \dot{W}^{1/2,q'}(I\!R, so(m))$ and $L^q \cap L^{q'}(I\!R, so(m))$

i) Since $\dot W^{1/2,q}$ for $q>2$ embedds continuously in C^0, the map $V\to exp\,(V)$ is clearly smooth from $\dot W^{1/2,q}\cap\dot W^{1/2,q'}(I\!R,so(m))$ into $\dot W^{1/2,q}\cap\dot W^{1/2,q'}(I\!R,SO(m))$.

ii) The operator $\Delta^{1/4}$ is a smooth linear map from $\dot W^{1/2,q}\cap\dot W^{1/2,q'}(I\!R,M_m(I\!R))$ into $L^q\cap L^{q'}(I\!R,M_m(I\!R))$.

iii) Since again $\dot W^{1/2,q}$ embedds continuously in L^∞ - $\dot W^{1/2,q}\cap\dot W^{1/2,q'}$ is an algebra - the following map

$$\begin{array}{rcl}\Pi\ :\ \dot W^{1/2,q}\cap\dot W^{1/2,q'}(I\!R,M_n(I\!R))\times L^q\cap L^{q'}(I\!R,M_n(I\!R)) & \longrightarrow & L^q\cap L^{q'}(I\!R,M_n(I\!R))\\ (A,B) & \longrightarrow & A\,B\end{array}$$

is also smooth.

Now we show that $dF_0^{P_0}=L^{P_0}$ (11)

$$L^{P_0}(\eta):=-\eta\ P_0^{-1}\Delta^{1/4}P_0+\Delta^{1/4}(\eta\ P_0^{-1})P_0$$
$$+P_0^{-1}\Delta^{1/4}(P_0\eta)-\Delta^{1/4}P_0^{-1}P_0\eta\,.$$

• **Differentiability of F^{P_0} at $U=0$:**

$$\left\|F^{P_0}(\eta)-F^{P_0}(0)-L^{P_0}\cdot\eta\right\|_{L^q\cap L^{q'}}=\left\|F^{P_0}(\eta)-F^{P_0}(0)+\eta P_0^{-1}\Delta^{1/4}P_0\right.$$
$$\left.-\Delta^{1/4}(\eta P_0^{-1})P_0-P_0^{-1}\Delta^{1/4}(P_0\eta)+\Delta^{1/4}P_0^{-1}P_0\eta\right\|_{L^q\cap L^{q'}}$$

First of all we estimate

$$\begin{aligned}&\left\|(P_0\exp(\eta))^{-1}\Delta^{1/4}(P_0\exp\eta)-P_0^{-1}\Delta^{1/4}P_0+\eta P_0^{-1}\Delta^{1/4}P_0-P_0^{-1}\Delta^{1/4}(\eta P_0)\right\|_{L^q\cap L^{q'}}\\&\le\left\|\Delta^{1/4}(P_0)\right\|_{L^q\cap L^2}\left\|(P_0\exp(\eta))^{-1}-P_0^{-1}+\eta(P_0)^{-1}\right\|_{L^\infty}\\&\quad+\left\|(P_0\exp(\eta))^{-1}\right\|_{L^\infty}\ \left\|\Delta^{1/4}(P_0\exp(\eta))-\Delta^{1/4}(P_0)-\Delta^{1/4}(P_0\eta)\right\|_{L^q\cap L^{q'}}\\&\quad+\left\|\Delta^{1/4}(P_0\eta)\right\|_{L^q\cap L^{q'}}\ \|P_0\exp(\eta)-P_0\|_{L^\infty}\\&\le C\ o(\|\eta\|_{\dot W^{1/2,q}(I\!R)})\end{aligned}\tag{49}$$

The estimate of

$$\left\|(P_0\exp\eta)^{-1}\Delta^{1/4}(P_0\exp(\eta))-P_0^{-1}\Delta^{1/4}(P_0)-P_0^{-1}\Delta^{1/4}(P_0\eta)+\Delta^{1/4}P_0^{-1}P_0\eta\right\|_{L^q\cap L^{q'}}\ .$$

(11) In order to define L^{P_0} as a map from $\dot W^{1/2,q}\cap\dot W^{1/2,q'}$ into $L^q\cap L^{q'}$ we recall again that we make use of the embedding $\dot W^{1/2,q}(I\!R)\hookrightarrow L^\infty(I\!R)$ if $q>2$ (see for instance [14], pag 33).

is analogous. Hence we have proved that $dF_0^{P_0} = L^{P_0}$.

• $d_0F^{P_0}$ **is an isomorphism from** $\dot W^{1/2,q} \cap \dot W^{1/2,q'}(I\!R, so(m))$ **into** $L^q \cap L^{q'}(I\!R, so(m))$ **.**

Precisely we prove the following lemma.

Lemma 4.1 *There exists $\varepsilon > 0$ such that if $\Omega_0 \in \mathcal{U}^q_{\varepsilon,C}$ and if $P_0 \in \dot W^{1/2,q} \cap \dot W^{1/2,q'}(I\!R, SO(m))$ is a solution of (8) (i)-(ii), satisfying*

$$\left\{\begin{array}{l} \int_{I\!R} |\Delta^{1/4} P_0|^q dx \le C \int_{I\!R} |\Omega_0|^q dx \\ \\ \int_{I\!R} |\Delta^{1/4} P_0|^{q'} dx \le C \int_{I\!R} |\Omega_0|^{q'} dx\,, \end{array}\right. \tag{50}$$

then for every $\omega \in L^q \cap L^{q'}(I\!R, so(m))$ there exists a unique $\eta \in \dot W^{1/2,q} \cap \dot W^{1/2,q'}(I\!R, so(m))$ such that

$$\omega = -\eta P_0^{-1} \Delta^{1/4} P_0 + \Delta^{1/4}(\eta P_0^{-1}) P_0 + P_0^{-1} \Delta^{1/4}(P_0 \eta) - (\Delta^{1/4} P_0^{-1}) P_0 \eta \tag{51}$$

and

$$\|\eta\|_{\dot W^{1/2,q} \cap \dot W^{1/2,q'}} \le C\ \|\omega\|_{L^q \cap L^{q'}}\,.$$

Proof of Lemma 4.1. Let $\Omega_0 \in \mathcal{U}^q_{\varepsilon,C}$. Suppose that $P_0 \in \dot W^{1/2,q} \cap \dot W^{1/2,q'} I\!R, SO(m))$ is a solution of (8) (i)-(ii), satisfying (50).

Claim 1. *Let $1 < r < 2$. L^{P_0} is an isomorphism between $\dot W^{1/2,r}(I\!R, so(m))$ and $L^r(I\!R, so(m))$, namely for any $\omega \in L^r(I\!R, so(m))$ there exists a unique $\eta \in \dot W^{1/2,r}(I\!R, so(m))$ solution to $L^{P_0}(\eta) = \omega$ and*

$$\|\eta\|_{\dot W^{1/2,r}} \le C\ \|\omega\|_{L^r}$$

for $C > 0$.

We rewrite the equation (51) in the following way

$$\begin{aligned} \omega &= 2\Delta^{1/4}\eta - 2\eta P_0^{-1} \Delta^{1/4} P_0 - 2\Delta^{1/4} P_0^{-1} P_0 \eta \\ &+ Q(\eta, P_0) - Q^t(\eta, P_0)\,, \end{aligned} \tag{52}$$

where

$$Q(\eta, P_0) = \Delta^{1/4}(\eta P_0^{-1}) P_0 + \eta\ P_0^{-1} \Delta^{1/4} P_0 - \Delta^{1/4}\eta\,. \tag{53}$$

From Lemma B.2 and Lemma B.3 it follows that

$$\begin{aligned} \|Q(\eta, P_0)\|_{L^r} &\le C \|\eta P_0^{-1}\|_{\dot W^{1/2,r}} \|P_0\|_{\dot H^{1/2}} \\ &\le C\, \|\eta\|_{\dot W^{1/2,r}}\ \left(\|P_0^{-1}\|_{L^\infty} + \|P_0^{-1}\|_{\dot H^{1/2}}\right) \|P_0\|_{\dot H^{1/2}} \\ &\le C\, \|\eta\|_{\dot W^{1/2,r}}\ \left(\|P_0\|_{L^\infty} \|P_0\|_{\dot H^{1/2}} + \|P_0\|^2_{\dot H^{1/2}}\right)\,. \end{aligned} \tag{54}$$

Since $2^{-1} + (2-r)\,(2r)^{-1} = r^{-1}$, by applying Hölder Inequality we get

$$\|\eta\ P_0^{-1}\Delta^{1/4}P_0\|_{L^r} \le \|\eta\|_{L^{2r/(2-r)}}\|P_0^{-1}\Delta^{1/4}P_0\|_{L^2} \tag{55}$$

$$\text{since } \dot{W}^{1/2,r}(I\!\!R) \hookrightarrow L^{\frac{2r}{2-r}}(I\!\!R)$$

$$\le C|\eta\|_{\dot{W}^{1/2,r}}\|P_0^{-1}\Delta^{1/4}P_0\|_{L^2}\,.$$

We consider the following map $H^{P_0}\colon \dot{W}^{1/2,r}(I\!\!R, so(m)) \to L^r(I\!\!R, so(m))$,

$$H^{P_0}(\eta) \;\;=\;\; -2\,\eta P_0^{-1}\Delta^{1/4}P_0 - 2\Delta^{1/4}P_0^{-1}P_0\,\eta + Q(\eta, P_0) - Q^t(\eta, P_0)\,.$$

From (54) and (55), it follows that there exists a constant $C > 0$ (independent of P_0) such that

$$\|H^{P_0}(\eta)\|_{L^r} \le C\,\|\eta\|_{\dot{W}^{1/2,r}}\ \left(\|P_0\|_{L^\infty}\|P_0\|_{\dot{H}^{1/2}} + \|P_0\|^2_{\dot{H}^{1/2}}\right)\,.$$

Because of (50), $\|P_0\|_{\dot{H}^{1/2}} \le (C\varepsilon)^{1/2}$ and hence, if $\varepsilon > 0$ is small enough, $L^{P_0} = 2\Delta^{1/4} + H_{P_0}\colon \dot{W}^{1/2,r}(I\!\!R, so(m)) \to L^r(I\!\!R, so(m))$ is invertible which proves the first claim.

Claim 2. *Let $q' < r < 2$. Let $\omega \in L^q \cap L^r$ and $\eta \in \dot{W}^{1/2,r}$ be the solution of $L^{P_0}(\eta) = \omega$, then η is in $\dot{W}^{1/2,q} \cap \dot{W}^{1/2,r}$.*

We apply Lemma B.3 to

$$\Delta^{1/4}\eta - P_0^{-1}\Delta^{1/4}(P_0\eta) = \Delta^{1/4}(P_0^{-1}P_0\eta) - P_0^{-1}\Delta^{1/4}(P_0\eta)$$

and we obtain

$$\|\Delta^{1/4}\eta - P_0^{-1}\Delta^{1/4}(P_0\eta)\|_{L^t} \le C\|P_0\eta\|_{\dot{W}^{1/2,r}}\ \|P_0^{-1}\|_{\dot{W}^{1/2,q}(I\!\!R, SO(m))}$$

$$\text{by Lemma B.2} \tag{56}$$

$$\le C\|\eta\|_{\dot{W}^{1/2,r}}\,[\|P_0\|_{L^\infty} + \|P_0\|_{\dot{H}^{1/2}}]\ \|P_0\|_{\dot{W}^{1/2,q}(I\!\!R, SO(m))}\,,$$

where t is given by $\frac{1}{t} = \frac{1}{q} + \frac{2-r}{2r}$.

In a similar way we have

$$\|\Delta^{1/4}\eta - \Delta^{1/4}(\eta\, P_0^{-1})P_0\|_{L^t} \le C\|\eta\|_{\dot{W}^{1/2,r}}\,[\|P_0\|_{L^\infty} + \|P_0\|_{\dot{H}^{1/2}}]\ \|P_0\|_{\dot{W}^{1/2,q}(I\!\!R, SO(m)}\,.$$

On the other hand we also have

$$\|(\eta\, P_0^{-1})\Delta^{1/4}P_0\|_{L^t} \le \|\eta\|_{L^{\frac{2r}{2-r}}}\ \|\Delta^{1/4}P_0\|_{L^q}\,. \tag{57}$$

Thus $Q(\eta, P_0), Q^t(\eta, P_0)$ and $H_{P_0}(\eta)$ are in L^t. Since $\omega \in L^q \cap L^r$, we have $\Delta^{1/4}\eta \in L^t$ as well. Since $q' < r < 2$ and $\frac{1}{t} = \frac{1}{q} + \frac{1}{r} - \frac{1}{2}$, we have that $t > 2$.

The fact that $\Delta^{1/4}\eta \in L^r \cap L^t$ for some $r < 2$ and $t > 2$ implies that $\eta \in L^\infty$ (see for instance [2], pag 25).

From the fact that $\eta \in L^\infty$ we deduce that $\eta P_0^{-1}\Delta^{1/4}P_0 \in L^q$ and $(\Delta^{1/4}P_0^{-1})P_0\,\eta \in L^q$. Now we apply Lemma B.4 respectively to $a = P_0\,\eta \in \dot{H}^{1/2} \cap L^\infty$, $b = P_0^{-1} \in \dot{W}^{1/2,q}$ and $a = \eta P_0^{-1}$, $b = P_0$ and we get that $H_{P_0}(\eta) \in L^q$. Since $\omega \in L^q \cap L^r$ we have $\Delta^{1/4}\eta \in L^q$ as well. Moreover the following estimate holds

$$\|\Delta^{1/4}\eta\|_{L^q} \leq C\;\|\omega\|_{L^q\cap L^r}\,,$$

which proves the claim 2.

Claim 3. *Let $w \in L^q \cap L^{q'}$ and $\eta \in \cap_{q'<r\leq q}\dot{W}^{1/2,r}$ be the solution of $L^{P_0}(\eta) = \omega$. Then η is in $\dot{W}^{1/2,q} \cap \dot{W}^{1/2,q'}$.*

It is enough to apply apply Lemma B.4 respectively to $a = P_0\,\eta \in L^\infty$, $b = P_0^{-1} \in \dot{W}^{1/2,q'}$ and $a = \eta P_0^{-1}$, $b = P_0$ in order to get that $H_{P_0}(\eta) \in L^{q'}$. Since $\omega \in L^q \cap L^{q'}$ we have $\Delta^{1/4}\eta \in L^{q'}$ as well.

Combining claim 1, claim 2 and claim 3 we obtain that for any $\omega \in L^q \cap L^{q'}(I\!R, so(m))$ there exists a unique $\eta \in \dot{W}^{1/2,q} \cap \dot{W}^{1/2,q'}(I\!R, so(m))$ such that

$$L^{P_0}\eta = \omega\,,$$

and

$$\|\eta\|_{\dot{W}^{1/2,q}\cap\dot{W}^{1/2,q'}} \leq C\;\|\omega\|_{L^q\cap L^{q'}}\,.$$

This finishes the proof of lemma 4.1. □

Proof of step 2 continued. We take $\Omega_0 \in \mathcal{V}^q_{C,\varepsilon}$. By definition of $\mathcal{V}^q_{C,\varepsilon}$ there exists $P_0 \in \dot{W}^{1/2,q'} \cap \dot{W}^{1/2,q}(I\!R, SO(m)$ that solves (8) (i)-(ii) and satisfies (50). Now we apply the Implicit Function Theorem to F^{P_0} and we deduce that for every P in some neighborhood of P_0 and Ω in a neighborhood of Ω_0 (both neighborhoods having a size depending on P_0 and Ω_0 of course) the equation (8)(i) is satisfied and for some constant $C > 0$ independent on q one has

$$\|\Delta^{1/4}P\|_{L^q} \leq C\|\Omega\|_{L^q}\,,\quad \text{and}\quad \|\Delta^{1/4}P\|_{L^{q'}} \leq C\|\Omega\|_{L^{q'}}\,. \tag{58}$$

The inequality (58) is satisfied by Ω_0 and P_0 by definition of $\mathcal{V}^q_{C,\varepsilon}$.

By possibly taking a smaller neighborhood of P_0 we may always assume that

$$\int_{I\!R} |\Delta^{1/4}P|^2dx \leq \varepsilon' < 1\,.$$

Step 3: *The fact that $\int_{I\!R} |\Delta^{1/4}P|^2dx \leq \varepsilon' < 1$ implies that $\int_{I\!R} |\Delta^{1/4}P|^2dx \leq C\int_{I\!R} |\Omega|^2dx$.*

We write

$$\begin{aligned} P^{-1}\Delta^{1/4}P &= \frac{1}{2}(P^{-1}\Delta^{1/4}P - (P^{-1}\Delta^{1/4}P)^t) + \frac{1}{2}(P^{-1}\Delta^{1/4}P + (P^{-1}\Delta^{1/4}P)^t)\\ &= Asymm(P^{-1}\Delta^{1/4}P) + Symm(P^{-1}\Delta^{1/4}P)\,. \end{aligned}$$

We apply the estimate (12) and we get

$$\|P^{-1}\Delta^{1/4}P+\Delta^{1/4}P^{-1}P\|_{L^2(I\!R)}\le C\|\Delta^{1/4}P\|^2_{L^2}$$

$$\le C\|P^{-1}\Delta^{1/4}P\|^2_{L^2}$$

$$\le C\|\Delta^{1/4}P\|_{L^2}\left(\|Symm(P^{-1}\Delta^{1/4}P)\|_{L^2}+\|Asymm(P^{-1}\Delta^{1/4}P)\|_{L^2}\right)\,.$$

Thus we get

$$\|Symm(P^{-1}\Delta^{1/4}P)\|_{L^2}\le C\varepsilon'\left(\|sym(P^{-1}\Delta^{1/4}P)\|_{L^2}+\|Asymm(P^{-1}\Delta^{1/4}P)\|_{L^2}\right)\,.$$

If $C\varepsilon'<1/2$ then

$$\|Symm(P^{-1}\Delta^{1/4}P)\|_{L^2}\le C\|Asymm(P^{-1}\Delta^{1/4}P)\|_{L^2}=C\|\Omega\|_{L^2}$$

which ends the proof of Step 3.

Step 4. Take now $\Omega\in L^2$ and $\int_{I\!R}|\Omega|^2dx\le\varepsilon$. Let $\Omega_k\in\mathcal{U}^q_\varepsilon$ be such that $\Omega_k\to\Omega$ as $k\to+\infty$ in L^2. By arguing as in the proof of that $\mathcal{V}^q_\varepsilon$ is closed one gets that there exists $P\in\dot{H}^{1/2}$ satisfying (8)(i)-(ii). □

5 Euler Equation for Half-Harmonic Maps into Manifolds

We consider a compact k dimensional C^2 manifold without boundary $\mathcal{N}\subset I\!R^m$. Let $\Pi_{\mathcal{N}}$ be the orthogonal projection on $\mathcal{N}$. We also consider the Dirichlet energy (14).

The weak 1/2-harmonic maps are defined as critical points of the functional (14) with respect to perturbation of the form $\Pi_{\mathcal{N}}(u+t\phi)$, where ϕ is an arbitrary compacted supported smooth map from $I\!R$ into $I\!R^m$.

Definition 5.1 *We say that $u\in H^{1/2}(I\!R,\mathcal{N})$ is a weak 1/2-harmonic map if and only if, for every maps $\phi\in H^{1/2}(I\!R,I\!R^m)\cap L^\infty(I\!R,I\!R^m)$ we have*

$$\frac{d}{dt}\mathcal{L}(\Pi_{\mathcal{N}}(u+t\phi))_{|t=0}=0\,. \tag{59}$$

□

We introduce some notations. We denote by $\bigwedge(I\!R^m)$ the exterior algebra (or Grassmann Algebra) of $I\!R^m$ and by the symbol $\wedge$ the *exterior or wedge product.* For every $p=1,\ldots,m$, $\bigwedge_p(I\!R^m)$ is the vector space of p-vectors.

If $(\epsilon_i)_{i=1,\ldots,m}$ is the canonical orthonormal basis of $I\!R^m$, then every element $v \in \bigwedge_p(I\!R^m)$ is written as $v = \sum_I v_I \epsilon_I$ where $I = \{i_1, \ldots, i_p\}$ with $1 \le i_1 \le \ldots \le i_p \le m$, $v_I := v_{i_1,\ldots,i_p}$ and $\epsilon_I =:= \epsilon_{i_1} \wedge \ldots \wedge \epsilon_{i_p}$.

By the symbol $\llcorner$ we denote the interior multiplication $\llcorner\colon \bigwedge_p(I\!R^m) \times \bigwedge_q(I\!R^m) \to \bigwedge_{q-p}(I\!R^m)$ defined as follows.

Let $\epsilon_I = \epsilon_{i_1} \wedge \ldots \wedge \epsilon_{i_p}$, $\epsilon_J = \epsilon_{j_1} \wedge \ldots \wedge \epsilon_{j_q}$, with $q \ge p$. Then $\epsilon_J \llcorner \epsilon_I = 0$ if $I \not\subset J$, otherwise $\epsilon_J \llcorner \epsilon_I = (-1)^M \epsilon_K$ where ϵ_K is a $q-p$ vector (with $K \cup I = J$) and M is the number of pairs $(i,j) \in I \times J$ with $j > i$.

Finally by the symbol $*$ we denote the Hodge-star operator, $*\colon \bigwedge_p(I\!R^m) \to \bigwedge_{m-p}(I\!R^m)$, defined by $*\beta = (\epsilon_1 \wedge \ldots \wedge \epsilon_n) \llcorner \beta$. For an introduction of the Grassmann Algebra we refer the reader to the first Chapter of the book by Federer [6] .

In the sequel we denote by P^T and P^N respectively the tangent and the normal projection to the manifold $\mathcal{N}$.

They verify the following properties: $(P^T)^t = P^T, (P^N)^t = P^N$ (namely they are symmetric operators), $(P^T)^2 = P^T$, $(P^N)^2 = P^N$, $P^T + P^N = Id$, $P^N P^T = P^T P^N = 0$.

We set $e = \epsilon_1 \wedge \ldots \wedge \epsilon_k$ and $\nu = \epsilon_{k+1} \wedge \ldots \wedge \epsilon_m$. For avery $z \in \mathcal{N}$, $e(z)$ and $\nu(z)$ give the orientation respectively of the tangent k-plane and the normal $m-k$-plane to $T_z\mathcal{N}$.

We observe that for every $v \in I\!R^m$ we have

$$P^T v \;=\; (-1)^{k-1} * ((e \llcorner v) \wedge \nu)\,. \tag{60}$$

$$P^N v \;=\; (-1)^{m-1} * (e \wedge (\nu \llcorner e))\,. \tag{61}$$

We observe that $P^N$and P^T can be seen as matrices in $\dot H^{1/2}(I\!R) \cap L^\infty(I\!R)$.

Next we write the Euler equation associated to the functional (14) .

Proposition 5.1 *All weak* $1/2$*-harmonic maps* $u \in H^{1/2}(I\!R, \mathcal{N})$ *satisfy in a weak sense the following three equivalent equations:*

i) the equation

$$\int_{I\!R} (\Delta^{1/2} u) \cdot v \, dx = 0, \tag{62}$$

for every $v \in H^{1/2}(I\!R, I\!R^m) \cap L^\infty(I\!R, I\!R^m)$ *with* $v \in T_{u(x)}\mathcal{N}$ *almost everywhere ;*

ii) the equation

$$P^T \Delta^{1/2} u = 0 \quad in\ \mathcal{D}'\,, \tag{63}$$

and

iii) the equation

$$\Delta^{1/4}(P^T \Delta^{1/4} u) = T(P^T, u) - (\Delta^{1/4} P^T)\Delta^{1/4} u\,. \tag{64}$$

□

The Euler Lagrange equation (64) can be considered together with by the following "structure equation" involving the normal projection of $\Delta^{1/4}u$.

Proposition 5.2 *All maps in $\dot H^{1/2}(I\!R,\mathcal{N})$ satisfy the following identity*

$$\Delta^{1/4}(\mathcal{R}(P^N\Delta^{1/4}u)) = \mathcal{R}(S(P^N,u)) - (\Delta^{1/4}P^N)(\mathcal{R}\Delta^{1/4}u)\,. \tag{65}$$

□

For the proofs of Proposition 5.1 and 5.2 we refer the reader to [4] .

Next we see that by combining (64) and (65) we can obtain the new equation (16) for the vector field $v = (P^T\Delta^{1/4}u, \mathcal{R}(P^N\Delta^{1/4}u))$ whose right hand side contains an antisymmetric potential.

We introduce the following matrices

$$\omega_1 = \frac{(\Delta^{1/4}P^T)P^T + P^T\Delta^{1/4}P^T - \Delta^{1/4}(P^TP^T)}{2}\,, \tag{66}$$

$$\omega_2 = (\Delta^{1/4}P^T)P^N + P^T\Delta^{1/4}P^N - \Delta^{1/4}(P^TP^N)\,, \tag{67}$$

$$\omega = \frac{(\Delta^{1/4}P^T)P^T - P^T\Delta^{1/4}P^T}{2}\,; \tag{68}$$

and

$$\omega_3 = \frac{(\mathcal{R}\Delta^{1/4}P^T)P^T + P^T\Delta^{1/4}(\mathcal{R}\Delta^{1/4}P^T) - \mathcal{R}\Delta^{1/4}(P^TP^T)}{2}, \tag{69}$$

$$\omega_4 = (\mathcal{R}\Delta^{1/4}P^T)P^N + P^N(\mathcal{R}\Delta^{1/4}P^T) - \mathcal{R}\Delta^{1/4}(P^NP^T), \tag{70}$$

$$\omega_{\mathcal{R}} = \frac{(\mathcal{R}\Delta^{1/4}P^T)P^T - P^T(\mathcal{R}\Delta^{1/4}P^T)}{2}\,. \tag{71}$$

We observe that Theorem 1.3 and Theorem 1.4 imply respectively that $\Delta^{1/4}(\omega_1)$, $\Delta^{1/4}(\omega_2)$ and $\Delta^{1/4}(\omega_3)$, $\Delta^{1/4}(\omega_4)$ are in the homogeneous Hardy Space $\mathcal{H}^1(I\!R)$. Therefore $\omega_1,\omega_2,\omega_3,\omega_4 \in L^{2,1}(I\!R)$. The matrices ω and $\omega_{\mathcal{R}}$ are **antisymmetric**.

Proof of Proposition 1.1. From Propositions 5.1 and 5.2 it follows that u satisfies in a weak sense the equations (64) and (65).

The key point is to rewrite the terms $(\Delta^{1/4}P^T)(\Delta^{1/4}u)$ and $(\Delta^{1/4}P^N)\mathcal{R}(\Delta^{1/4}u)$.

- **Re-writing of** $(\Delta^{1/4}P^T)\Delta^{1/4}u$ **.**

$$\begin{aligned}(\Delta^{1/4}P^T)\Delta^{1/4}u &= (\Delta^{1/4}P^T)(P^T(\Delta^{1/4}u) + P^N(\Delta^{1/4}u))\\ &= ((\Delta^{1/4}P^T)P^T)(P^T(\Delta^{1/4}u)) + ((\Delta^{1/4}P^T)P^N)(P^N(\Delta^{1/4}u))\,.\end{aligned}$$

Now we have

$$(\Delta^{1/4}P^T)P^T = \omega_1 + \omega + \frac{\Delta^{1/4}P^T}{2}\,; \tag{72}$$

and

$$\begin{aligned}(\Delta^{1/4}P^T)P^N &= (\Delta^{1/4}P^T)P^N + P^T\Delta^{1/4}P^N - \Delta^{1/4}(P^TP^N) - P^T\Delta^{1/4}P^N \\ &= \omega_2 + P^T\Delta^{1/4}P^T \\ &= \omega_2 + \omega_1 - \omega + \frac{\Delta^{1/4}P^T}{2}\,.\end{aligned} \tag{73}$$

where in (73) we use that $\Delta^{1/4}P^N = -\Delta^{1/4}P^T$. Thus

$$\frac{(\Delta^{1/4}P^T)(P^T\Delta^{1/4}u)}{2} = \omega_1(P^T\Delta^{1/4}u) + \omega(P^T\Delta^{1/4}u) \tag{74}$$

$$\frac{(\Delta^{1/4}P^T)(P^N\Delta^{1/4}u)}{2} = (\omega_1+\omega_2)(P^N\Delta^{1/4}u) - \omega(P^N\Delta^{1/4}u) \tag{75}$$

$$\begin{aligned}&= \mathcal{R}(\omega_1+\omega_2)\mathcal{R}(P^N\Delta^{1/4}u) - \mathcal{R}(\omega)\mathcal{R}(P^N\Delta^{1/4}u) \\ &+ F(-\omega+\omega_1+\omega_2, (P^N\Delta^{1/4}u))\,.\end{aligned}$$

- **Re-writing of** $(\Delta^{1/4}P^N)(\mathcal{R}\Delta^{1/4}u)$ **.**

We have

$$\begin{aligned}(\Delta^{1/4}P^N)(\mathcal{R}\Delta^{1/4}u) &= (\mathcal{R}(\Delta^{1/4}P^N))(P^T(\Delta^{1/4}u) + P^N(\Delta^{1/4}u)) \\ &+ F((\mathcal{R}(\Delta^{1/4}P^N)), \Delta^{1/4}u)\,.\end{aligned}$$

We rewrite the terms $(\mathcal{R}\Delta^{1/4}P^N)P^T(\Delta^{1/4}u)$ and $(\mathcal{R}\Delta^{1/4}P^N)P^N(\Delta^{1/4}u)$. We have

$$\begin{aligned}(\mathcal{R}\Delta^{1/4}P^N)P^T &= -(\mathcal{R}\Delta^{1/4}P^T)P^T \\ &= -\omega_3 - \omega_{\mathcal{R}} - \frac{(\mathcal{R}\Delta^{1/4}P^T)}{2} \\ &= -\omega_3 - \omega_{\mathcal{R}} + \frac{(\mathcal{R}\Delta^{1/4}P^N)}{2}\,,\end{aligned}$$

and

$$\begin{aligned}(\mathcal{R}\Delta^{1/4}P^N)P^N &= -(\mathcal{R}\Delta^{1/4}P^T)P^N \pm P^T(\mathcal{R}\Delta^{1/4}P^N) \\ &= -[(\mathcal{R}\Delta^{1/4}P^T)P^N + P^T(\mathcal{R}\Delta^{1/4}P^N) - \mathcal{R}\Delta^{1/4}(P^NP^T)] \\ &+ P^T(\mathcal{R}\Delta^{1/4}P^N) \\ &= -\omega_4 - \omega_3 + \omega_{\mathcal{R}} + \frac{(\mathcal{R}\Delta^{1/4}P^N)}{2}\,.\end{aligned}$$

Thus

$$\frac{(\mathcal{R}\Delta^{1/4}P^N)P^T\Delta^{1/4}u}{2} = -\omega_3(P^T\Delta^{1/4}u) - \omega_{\mathcal{R}}(P^T\Delta^{1/4}u) \tag{76}$$

$$\begin{aligned}\frac{(\mathcal{R}\Delta^{1/4}P^N)P^N\Delta^{1/4}u}{2} &= -\omega_4(P^N\Delta^{1/4}u) - \omega_3(P^N\Delta^{1/4}u) + \omega_{\mathcal{R}}(P^N\Delta^{1/4}u) \\ &= \mathcal{R}(-\omega_3-\omega_4)\mathcal{R}(P^N\Delta^{1/4}u) \\ &+ \mathcal{R}(\omega_{\mathcal{R}})\mathcal{R}(P^N\Delta^{1/4}u) \\ &+ F(\omega_{\mathcal{R}}-\omega_3-\omega_4, P^N\Delta^{1/4}u)\,.\end{aligned} \tag{77}$$

By combining (74), (75), (76) and (77) we obtain

$$\begin{aligned}\Delta^{1/4}\begin{pmatrix} P^T\Delta^{1/4}u \\ \mathcal{R}P^N\Delta^{1/4}u\end{pmatrix} &= \tilde{\Omega}_1 + \tilde{\Omega}_2\begin{pmatrix} P^T\Delta^{1/4}u \\ \mathcal{R}P^N\Delta^{1/4}u\end{pmatrix} \\ &+ 2\begin{pmatrix} -\omega & \omega_{\mathcal{R}} \\ \omega_{\mathcal{R}} & -\mathcal{R}\omega_{\mathcal{R}}\end{pmatrix}\begin{pmatrix} P^T\Delta^{1/4}u \\ \mathcal{R}P^N\Delta^{1/4}u\end{pmatrix},\end{aligned} \tag{78}$$

where $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are given by

$$\tilde{\Omega}_1 = \begin{pmatrix} -2F(-\omega+\omega_1+\omega_2,(P^N\Delta^{1/4}u)) + T(P^T,u) \\ -2F(\mathcal{R}(\Delta^{1/4}P^N),\mathcal{R}(\Delta^{1/4}u)) - 2F(\omega_{\mathcal{R}}-\omega_3-\omega_4, P^N(\Delta^{1/4}u) + \mathcal{R}(S(P^N,u))\end{pmatrix}.$$

$$\tilde{\Omega}_2 = 2\begin{pmatrix} -\omega_1 & -[\mathcal{R}(\omega_1+\omega_2) + (\mathcal{R}(\omega) - \omega_{\mathcal{R}})] \\ \omega_3 & -\mathcal{R}(\omega_3-\omega_4)\end{pmatrix}.$$

The matrix

$$\Omega = 2\begin{pmatrix} -\omega & \omega_{\mathcal{R}} \\ \omega_{\mathcal{R}} & -\mathcal{R}\omega_{\mathcal{R}}\end{pmatrix}$$

is antisymmetric.

We observe that from the estimates on the operators F, T and S it follows that $\tilde{\Omega}_1 \in H^{-1/2}(I\!R, I\!R^{2m})$ and

$$\|\tilde{\Omega}_1\|_{H^{-1/2}(I\!R)} \le C(\|P^N\|_{\dot{H}^{1/2}(I\!R)} + \|P^T\|_{\dot{H}^{1/2}(I\!R)})\|\Delta^{1/4}u\|_{L^{2,\infty}}\,. \tag{79}$$

On the other hand $\tilde{\Omega}_2 \in L^{2,1}(I\!R, \mathcal{M}_{2m})$ and

$$\|\tilde{\Omega}_2\|_{L^{2,1}(I\!R)} \le C(\|P^N\|^2_{\dot{H}^{1/2}(I\!R)} + \|P^T\|^2_{\dot{H}^{1/2}(I\!R)})\,. \tag{80}$$

This concludes the proof of Proposition 1.1. □

Proof of Theorem 1.7.

From Proposition 1.1 it follows that $v = (P^T(\Delta^{1/4}u), \mathcal{R}(P^N(\Delta^{1/4}u)))$ solves equation (78) which is of the type (5) up to the terms $\tilde{\Omega}_1$ and $\tilde{\Omega}_2\, v$. The important point here is that the terms $\tilde{\Omega}_1$ and $\tilde{\Omega}_2\, v$ are not "dangerous" because of the key estimates (79) and (80).

Therefore the arguments are very similar to those of Theorem 1.1 and we give only a sketch of proof.

We aim at obtaining that $\Delta^{1/4}u \in L^p_{loc}(I\!R)$, for all $p \geq 1$. To this purpose we take $\rho > 0$ such that

$$\|\Omega\|_{L^2(B(0,\rho)}, \|P^T\|_{\dot{H}^{1/2}(B(0,\rho)}, \|P^N\|_{\dot{H}^{1/2}(B(0,\rho)} \leq \varepsilon_0,$$

with $\varepsilon_0 > 0$ small enough. Let $x_0 \in B(0,\rho/8)$ and $r \in (0,\rho/16)$. As in the case of equation (5) we argue by duality and multiply both sides of equation (78) by $\phi = \Delta^{-1/4}(g_{r\alpha})$, with $g \in L^{2,1}(I\!R)$, $\|g\|_{L^{2,1}} \leq 1$ and $g_{r\alpha} = 1\!\!1_{B(x_0,r\alpha)}g$, with $0 < \alpha < 1/4$.

It is enough to estimate the integral

$$\int_{I\!R} \tilde{\Omega}_1 \left(\begin{array}{c} \Delta^{-1/4}P^T(g_{r\alpha})) \\ \Delta^{-1/4}P^N(g_{r\alpha})) \end{array} \right) dx\,, \tag{81}$$

(the other terms have already estimated in the proof of Theorem 1.1).

We observe that

$$\|\Delta^{1/4}u\|_{L^{2,\infty}} \lesssim \left\| \sqrt{(P^T(\Delta^{1/4}u))^2 + (\mathcal{R}(P^N(\Delta^{1/4}u)))^2} \right\|_{L^{2,\infty}} = \|v\|_{L^{2,\infty}}\,. \tag{82}$$

By combining Lemma A.5-A.10 and the estimate (82) we obtain

$$\begin{aligned} (81) \;&\lesssim\; \varepsilon_0\|\Delta^{1/4}u\|_{L^{2,\infty}} + \alpha^{1/2}\sum_{h=1}^{+\infty} 2^{-h/2}\|\Delta^{1/4}u\|_{L^{2,\infty}(B(x_0,2^{h+1}r)\setminus B(x_0,2^{h-1}r))} \\ &\lesssim\; \varepsilon_0\|v\|_{L^{2,\infty}} + \alpha^{1/2}\sum_{h=1}^{+\infty} 2^{-h/2}\|v\|_{L^{2,\infty}(B(x_0,2^{h+1}r)\setminus B(x_0,2^{h-1}r))}\,. \end{aligned}$$

Since v satisfies an estimate of the type (40), for α and ε_0 small enough, we have

$$\sup_{x_0\in B(0,\rho/8), 0<r<\rho/16} \|v\|_{L^{2,\infty}(B(x_0,r))} \leq Cr^{\beta}\,,$$

for some $C > 0$ and $\beta \in (0,1/2)$.

By arguing as in Theorem 1.1 we deduce that $v \in L^p_{loc}(I\!R)$, for all $p \geq 1$. Therefore $\Delta^{1/4}u \in L^p_{loc}(I\!R)$, for all $p \geq 1$ as well.

This implies that $u \in C^{0,\alpha}_{loc}$ for all $0 < \alpha < 1$, since $W^{1/2,p}_{loc}(I\!R) \hookrightarrow C^{0,\alpha}_{loc}(I\!R)$ if $p > 2$ (see for instance [2]). This concludes the proof of Theorem 1.7. □

A Localization Estimates

The aim of this Appendix is to provide *localization estimates* for weak solutions to the equations (32) and (78) .

For $r > 0$, $h \in \mathbb{Z}$ and $x_0 \in \mathbb{R}$ we set

$$A_{h,x_0}(r) = B(x_0, 2^{h+1}r) \setminus B(x_0, 2^{h-1}r) \quad \text{and} \quad A'_{h,x_0}(r) = B(x_0, 2^{h}r) \setminus B(x_0, 2^{h-1}r)\,.$$

In the following two Lemmae we prove some estimates that will be often used in the sequel.

In the Lemma A.1 we estimate the L^1 and $\dot{H}^{1/2}$ norms of $\Delta^{-1/4}g$ respectively in a ball and in an annulus, where $g \in L^q(\mathbb{R})$, $q > 1$ has compact support .

Lemma A.1 *Let $g \in L^q(\mathbb{R})$, $q > 1$ supp $g \subset B(x_0, r\alpha)$, with $x_0 \in \mathbb{R}$, $\alpha > 0$. Then*

$$\|\Delta^{-1/4}g\|_{L^1(B(x_0,\gamma r)} \lesssim \gamma^{1/2}\alpha^{1/2}r^{\frac{q'+2}{2q'}}\|g\|_{L^q(\mathbb{R})}\,, \tag{83}$$

for all $\gamma > 0$ and

$$\|\Delta^{-1/4}g\|_{\dot{H}^{1/2}(A'_{h,x_0}(r))} \lesssim 2^{-h/2}\alpha^{1/q'}r^{(\frac{1}{q'}-\frac{1}{2})}\|g\|_{L^q(\mathbb{R})}\,. \tag{84}$$

Proof of Lemma A.1 .

First of all we may assume without restriction that $x_0 = 0$.

1. Estimate of (83).

We have

$$\begin{aligned}
\|\Delta^{-1/4}g\|_{L^1(B(0,\gamma r))} &\lesssim \||\cdot|^{-1/2} * g\|_{L^1(B(0,\gamma r))} \\
&\lesssim \||x|^{-1/2}\|_{L^1(B(0,\gamma r))}\|g\|_{L^1(B(0,r\alpha))} \\
&\lesssim (\gamma r)^{1/2}(r\alpha)^{1/q'}\|g\|_{L^q}\,.
\end{aligned}$$

2. Estimate of (84).

We have

$$
\begin{aligned}
&\|1\!\!1_{A'_{h,0}(r)}\Delta^{-1/4}g\|^2_{\dot H^{1/2}(A'_{h,0}(r))}\\
&=\int_{A'_{h,0}(r)}\int_{A'_{h,0}(r)}\frac{1}{|t-s|^2}\left(\int_{|x|<r\alpha}g(x)(\frac{1}{|x-s|^{1/2}}-\frac{1}{|x-t|^{1/2}})dx\right)^2 dtds
\end{aligned}
$$

by Mean Value Theorem

$$
\lesssim \int_{A'_{h,0}(r)}\int_{A'_{h,0}(r)}\left(\int_{|x|<r\alpha}g(x)\max(\frac{1}{|x-t|^{3/2}},\frac{1}{|x-s|^{3/2}})dx\right)^2 dtds
$$

$$
\lesssim \int_{A'_{h,0}(r)}\int_{A'_{h,0}(r)}2^{-3h}r^{-3}(r\alpha)^{2/q'}\left(\int_{|x|<r\alpha}|g(x)|^q dx\right)^{2/q} dtds
$$

$$
\lesssim 2^{-h}\alpha^{2/q'}r^{-1+2/q'}\|g\|^2_{L^q}\,.
$$

This concludes the proof of Lemma A.1 . □

In the next Lemma we estimate the integral over a ball of the product of $\Delta^{1/4}v$ with $v\in L^{2,\infty}$, supp $v\subset A_{h,x_0}(r)$ and $\Delta^{-1/4}g$ with $g\in L^q(I\!R)$, $q>1$, supp $g\subset B(x_0,r\alpha)$.

Lemma A.2 *Let $g\in L^q(I\!R)$, $q>1$, supp $g\subset B(x_0,r\alpha)$, with $x_0\in I\!R$, $0<\alpha<\frac{1}{4}$ and let $v\in L^{2,\infty}$, supp $v\subset A_{h,x_0}(r)$ with $h>-1$. Then for all $\delta>0$ we have*

$$
\int_{I\!R}\Delta^{1/4}v\,1\!\!1_{B(x_0,\delta r)}\Delta^{-1/4}g dx\lesssim 2^{-h}\delta^{1/2}\alpha^{1/q'}r^{\frac{1}{q'}-\frac{1}{2}}\|g\|_{L^q(I\!R)}\|v\|_{L^{2,\infty}(A_{h,0}(r))}\,. \tag{85}
$$

Proof of Lemma A.2 . We assume without restriction that $x_0=0$. We have

$$
\int_{I\!R}(\Delta^{1/4}v)(x)\,(1\!\!1_{B(0,\delta r)}\,\Delta^{-1/4}g)(x)\,dx
$$

by the Plancherel Theorem (86)

$$
=\int_{I\!R}\mathcal{F}\left[(\Delta^{1/4}v)\right](\xi)\,\mathcal{F}[(1\!\!1_{B(0,\delta r)}\,\Delta^{-1/4}g)](\xi)d\xi
$$

$$
\simeq\int_{I\!R}\mathcal{F}^{-1}(|\cdot|^{1/2})(x)[v*(1\!\!1_{B(0,\delta r)}\Delta^{-1/4}g)]\,d\xi\,.
$$

Now we observe that supp$[v*(1\!\!1_{B(0,\delta r)}\Delta^{-1/4}g)]\subset B(0,2^{h+2}r)\setminus B(0,2^{h-2}r)$.

Thus we have

$$
\begin{aligned}
(85) \;\lesssim\;& \||\xi|^{-3/2}\|_{L^\infty(B^c(0,2^{h-2}r))}\|v * (1\!\!1_{B(0,\delta r)}\Delta^{-1/4}g)\|_{L^1(I\!R)} \\
\lesssim\;& 2^{-3/2h}r^{-3/2}\|v\|_{L^1(A_{h,x_0}(r))}\|1\!\!1_{B(x_0,\delta r)}\Delta^{-1/4}g)\|_{L^1(I\!R)} \\
\lesssim\;& 2^{-3/2h}r^{-3/2}\|v\|_{L^1(A_{h,x_0}(r))}\|\Delta^{-1/4}g)\|_{L^1(B(0,\delta r))} \\
& \text{by (83)} \\
\lesssim\;& 2^{-h}\delta^{1/2}\alpha^{1/q'}r^{\frac{1}{q'}-\frac{1}{2}}\|g\|_{L^q(I\!R)}\|v\|_{L^{2,\infty}(A_{h,0}(r))}\,.
\end{aligned}
$$

This conclude the proof of Lemma A.2. □

1. Localization of the term $N(Q,v)=\Delta^{1/4}(Q\,v)-Q\Delta^{1/4}v+\Delta^{1/4}Q\;v$ **.**

Lemma A.3 *Let* $Q\in \dot H^{1/2}(I\!R)\cap L^\infty(I\!R)$, $\|Q\|_{\dot H^{1/2}(I\!R)}\le\varepsilon_0$, $v\in L^2(I\!R)$, $g\in L^{2,1}(I\!R)$, *supp* $g\subset B(x_0,r\alpha)$, *with* $x_0\in I\!R$, $0<\alpha<\frac{1}{4}$, $r>0$. *Then we have*

$$
\begin{aligned}
\int_{I\!R} N(Q,v)\Delta^{-1/4}g dx \;\lesssim\;& \varepsilon_0\|g\|_{L^{2,1}}\|v\|_{L^{2,\infty}(B(x_0,r))} \qquad (87)\\
+\;& \alpha^{1/2}(\|Q\|_{\dot H^{1/2}(I\!R)}+\|Q\|_{L^\infty})\|g\|_{L^{2,1}}\sum_{h=1}^{+\infty}2^{-h}\|v\|_{L^{2,\infty}(A_{h,x_0}(r))}\,.
\end{aligned}
$$

Proof of Lemma A.3. We suppose without restriction that $x_0=0$.

We consider a dyadic decomposition of the unity $\varphi_j\in C_0^\infty(I\!R)$ such that

$$
\text{supp}(\varphi_j)\subset B_{2^{j+1}r}(0)\setminus B_{2^{j-1}r}(0),\quad \sum_{-\infty}^{+\infty}\varphi_j=1\,. \qquad (88)
$$

We set $\chi_r:=\sum_{-\infty}^0\varphi_j$.

We observe that the function $\Delta^{-1/4}g\in L^\infty(I\!R)\cap \dot H^{1/2}(I\!R)$.

We take the scalar product of $N(Q,v)$ with $\Delta^{-1/4}g$ and we integrate. We write

$$
\begin{aligned}
\int_{I\!R} N(Q,v)\Delta^{-1/4}g dx \;=\;& \underbrace{\int_{I\!R} N(Q,\chi_r v)\Delta^{-1/4}g dx}_{(1)} \\
+\;& \underbrace{\int_{I\!R}\sum_{k=1}^{+\infty} N(Q,\varphi_k v)\Delta^{-1/4}g dx}_{(2)}\,.
\end{aligned}
$$

To estimate (1) we use the fact that $N(Q,v)\in \dot H^{-1/2}(I\!R)$ and (34) holds.

$$
\begin{aligned}
(1) &\lesssim \|\Delta^{-1/4}g\|_{\dot H^{-1/2}(I\!R)}\|Q\|_{\dot H^{1/2}(I\!R)}\|v\|_{L^{2,\infty}}\\
&\lesssim \varepsilon_0\|g\|_{L^{2,1}}\|v\|_{L^{2,\infty}(B(0,r)}\,.
\end{aligned}
$$

Next we spilt (2) in two parts:

$$
\begin{aligned}
(2) &= \underbrace{\sum_{k=1}^{\infty}\int_{I\!R} N(Q,\varphi_k v)\mathbb{1}_{B(0,r/4)}\Delta^{-1/4}g dx}_{(3)}\\
&+ \underbrace{\sum_{k=1}^{\infty}\sum_{h=-1}^{\infty}\int_{I\!R} N(Q,\varphi_k v)\mathbb{1}_{A'_{h,0}(r)}\Delta^{-1/4}g dx}_{(4)}\,.
\end{aligned}
$$

We observe that in (3) and (4) we can exchange the integral with the infinite sum "$\sum_{k=1}^{+\infty}$". Indeed one can easily check that

$$
\lim_{n\to+\infty}\int_{I\!R}\sum_{k=n}^{\infty} N(Q,\varphi_k v)\,\Delta^{-1/4}g dx = 0\,.
$$

(see also the arguments of Lemma A.3, Lemma A.4 and Corollary A.1 in [4]).

We estimate (3). We first observe that since $\mathbb{1}_{B(0,r/4)}$ and φ_k have disjoint supports, we have

$$
N(Q,\varphi_k v)\mathbb{1}_{B(0,r/4)}\,\Delta^{-1/4}g = \left[\Delta^{1/4}(Q\varphi_k v) - Q\Delta^{1/4}(\varphi_k v)\right]\mathbb{1}_{B(0,r/4)}\,\Delta^{-1/4}g\,.
$$

We have

$$(3) = \sum_{k=1}^{\infty} \int_{I\!R} \left[\Delta^{1/4}(Q\varphi_k v) - Q\Delta^{1/4}(\varphi_k v)\right] 1\!\!1_{B(0,r/4)}\, \Delta^{-1/4} g\, dx$$

by the Plancherel Theorem

$$= \sum_{k=1}^{\infty} \int_{I\!R} \mathcal{F}\left[\Delta^{1/4}(Q\varphi_k v) - Q\Delta^{1/4}(\varphi_k v)\right](\xi)\, \mathcal{F}[(1\!\!1_{B(0,r/4)}\, \Delta^{-1/4} g)](\xi) d\xi$$

$$\simeq \sum_{k=1}^{\infty} \int_{I\!R} \mathcal{F}^{-1}(|\cdot|^{1/2})(x)$$
$$\left[Q(\varphi_k v) * (1\!\!1_{B(0,r/4)}\Delta^{-1/4} g) - (\varphi_k v) * (Q 1\!\!1_{B(0,r/4)}\Delta^{-1/4} g)\right]\, d\xi$$

by applying Lemma A.2

$$\lesssim \sum_{k=1}^{\infty} \||\xi|^{-3/2}|\|_{L^\infty(B^c(0,2^{k-2}r))}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{-k}\alpha^{1/2}\left[\|Q\|_{L^\infty}\|v\|_{L^{2,\infty}(A_{k,0}(r))}\|g\|_{L^{2,1}(I\!R)}\right]$$

$$\lesssim \alpha^{1/2}\|Q\|_{L^\infty(I\!R)}\|g\|_{L^{2,1}(I\!R)} \sum_{k=1}^{\infty} 2^{-k}\|v\|_{L^{2,\infty}(A_{k,0}(r))}\,.$$

Next we split (4) as follows.

$$(4) = \underbrace{\sum_{k=1}^{\infty}\sum_{|k-h|\le 5}\int_{I\!R} N(Q,\varphi_k v) 1\!\!1_{A'_{h,0}(r)}\Delta^{-1/4} g dx}_{(5)}$$
$$= \underbrace{\sum_{k=1}^{\infty}\sum_{|k-h|\ge 5}\int_{I\!R} N(Q,\varphi_k v) 1\!\!1_{A'_{h,0}(r)}\Delta^{-1/4} g dx}_{(6)}\,.$$

We estimate (5).

$$
\begin{aligned}
(5) \quad &\lesssim \sum_{k=1}^{\infty} \sum_{|k-h|\le 5} \|N(Q,\varphi_k v)\|_{\dot H^{-1/2}(I\!R)} \|1\!\!1_{A'_{h,0}(r)} \Delta^{-1/4} g\|_{\dot H^{1/2}(I\!R)} \\
&\quad \text{by applying (34)} \\
&\lesssim \sum_{k=1}^{\infty} \sum_{|k-h|\le 5} \|Q\|_{\dot H^{1/2}(I\!R)} \|\varphi_k v\|_{L^{2,\infty}(I\!R)}\, \|\Delta^{-1/4} g\|_{\dot H^{1/2}(A'_{h,0}(r))} \\
&\quad \text{by applying (84)} \\
&\lesssim \alpha^{1/2} \|Q\|_{\dot H^{1/2}(I\!R)} \|g\|_{L^{2,1}(I\!R)} \left(\sum_{k=1}^{\infty} 2^{-k/2} \|v\|_{L^{2,\infty}(A_{k,0}(r))} \right).
\end{aligned}
$$

In order to estimate (6) we observe if $|k-h| \geq 6$ then $\varphi_k v$ and $1\!\!1_{A'_{h,0}(r)} \Delta^{-1/4} g$ have disjoint supports . Thus by arguing as in (3) we get

$$
\begin{aligned}
(6) \quad &\lesssim \alpha^{1/2} \|Q\|_{L^\infty} \|g\|_{L^{2,1}(I\!R)} \sum_{k=1}^{+\infty} 2^{-k} \|v\|_{L^{2,\infty}(A_{k,0}(r))} \\
&\lesssim \alpha^{1/2} \|Q\|_{L^\infty} \|g\|_{L^{2,1}(I\!R)} \sum_{k=1}^{+\infty} 2^{-k} \|v\|_{L^{2,\infty}(A_{k,0}(r))}\,.
\end{aligned}
$$

This concludes the proof of Lemma A.3 . □

Lemma A.4 *Let $Q \in \dot H^{1/2}(I\!R) \cap L^\infty(I\!R)$, supp $Q \subset B^c(0,\rho)$ for some $\rho > 0$, $v \in L^2(I\!R)$, $x_0 \in B(0,\rho/8)$, $g \in L^{2,1}(I\!R)$, supp $g \subset B(x_0, r\alpha)$, with , $0 < \alpha < 1$, $0 < r < \rho/16$.*

Then we have

$$
\begin{aligned}
\int_{I\!R} N(Q,v)\Delta^{-1/4} g dx &\lesssim \left(\frac{r}{\rho}\right)^{1/2} \|g\|_{L^{2,1}(I\!R)} \|Q\|_{\dot H^{1/2}(I\!R)} \|v\|_{L^{2,\infty}(B(x_0,r))} \\
&+ \alpha^{1/2} \|Q\|_{L^\infty} \|g\|_{L^{2,1}(I\!R)} \left(\sum_{h=1}^{+\infty} 2^{-h/2} \|v\|_{L^{2,\infty}(A_{h,x_0}(r))} \right).
\end{aligned} \tag{89}
$$

Proof of Lemma A.4. We write

$$\int_{\mathbb{R}} N(Q,v)\Delta^{-1/4} g dx = \underbrace{\int_{\mathbb{R}} N(Q,\chi_r v)\Delta^{-1/4} g dx}_{(7)} + \underbrace{\int_{\mathbb{R}} N(Q,(1-\chi_r)v)\Delta^{-1/4} g dx}_{(8)},$$

where χ_r is defined as in Lemma A.3.

We denote by $Q_\rho = |B_\rho(0)|^{-1}\int_{B_\rho(0)} Q(y)dy = 0$ and write $Q = \sum_{h=-1}^{+\infty} \tilde{\varphi}_h (Q - Q_\rho)$, with $\mathrm{supp}(\tilde{\varphi}_h) \subset B(0, 2^{h+1}\rho) \setminus B(0, 2^{h-1}\rho)$, $\tilde{\varphi}_h$ partition of unity.

We recall two key results obtained in [4]. The first one is a sort of Poincaré Inequality for functions in $\dot{H}^{1/2}(\mathbb{R})$ having compact support and the second one concerns with a geometric localization property of the $\dot{H}^{1/2}-$ norm on the real line .

Precisely from Lemma A.2 in [4] it follows that

$$\|\tilde{\varphi}_h(Q-Q_\rho)\|_{L^1} \le C 2^h \rho \|Q\|_{\dot{H}^{1/2}(\mathbb{R})}, \tag{90}$$

and from Lemma 4.1 in [4] one can deduce that

$$\sum_{h=0}^{+\infty} 2^{-h/2} \|\tilde{\varphi}_h(Q-Q_\rho)\|_{\dot{H}^{1/2}(\mathbb{R})} \lesssim \|Q\|_{\dot{H}^{1/2}(\mathbb{R})}. \tag{91}$$

We estimate (7).

$$(7) = \int_{I\!R} N(\sum_{h=-1}^{+\infty} \tilde{\varphi}_h(Q-Q_\rho), \chi_r v)\,(\Delta^{-1/4} g) dx$$

$$= \sum_{h=-1}^{+\infty} \int_{I\!R} [-\tilde{\varphi}_h(Q-Q_\rho)\Delta^{1/4}(\chi_r v)\Delta^{-1/4} g + \Delta^{1/4}(\tilde{\varphi}_h(Q-Q_\rho))(\chi_r v)\Delta^{-1/4} g] dx$$

by applying the Plancherel Theorem

$$= \sum_{h=-1}^{+\infty} \int_{I\!R} \mathcal{F}[\Delta^{1/4}(\chi_r v)]\,\mathcal{F}[-\tilde{\varphi}_h(Q-Q_\rho)\,\Delta^{-1/4} g]$$

$$+ \mathcal{F}[\Delta^{1/4}(\varphi_h(Q-Q_\rho)]\,\mathcal{F}[(\chi_r v)\Delta^{-1/4} g] d\xi$$

$$= \sum_{h=-1}^{+\infty} \int_{I\!R} \mathcal{F}^{-1}[|\cdot|^{1/2}](x)$$

$$\left[-(\chi_r v) * (\tilde{\varphi}_h(Q-Q_\rho)\Delta^{-1/4} g) + \tilde{\varphi}_h(Q-Q_\rho) * (\chi_r v \Delta^{-1/4} g)\right] dx$$

$$\lesssim \sum_{h=-1}^{+\infty} \||\xi|^{-3/2}\|_{L^\infty(B^c(0,2^{h-2}\rho))}$$

$$\|\left[-(\chi_r v) * (\tilde{\varphi}_h(Q-Q_\rho)\Delta^{-1/4} g) + \varphi_h(Q-Q_\rho) * (\chi_r v \Delta^{-1/4} g)\right]\|_{L^1(I\!R)}$$

$$\lesssim \sum_{h=-1}^{+\infty} \||\xi|^{-3/2}\|_{L^\infty(B^c(0,2^{h-2}\rho))} \left[\|\chi_r v\|_{L^1} \|\varphi_h(Q-Q_\rho)\|_{L^1} \|\Delta^{-1/4} g\|_{L^\infty}\right] .$$

Now we apply (90) and (91) and we get:

$$
\begin{aligned}
(7) \quad &\lesssim \sum_{h=-1}^{+\infty} 2^{-3/2h}\rho^{-3/2}r^{1/2}\|v\|_{L^{2,\infty}(B(x_0,r))}2^h\rho\|\varphi_h(Q-Q_\rho)\|_{H^{1/2}(I\!R)} \\
&\lesssim \|g\|_{L^{2,1}(I\!R)}\sum_{h=-1}^{+\infty} 2^{-h/2}\left(\frac{r}{\rho}\right)^{1/2}\|v\|_{L^{2,\infty}(B(x_0,r)}\|\varphi_h(Q-Q_\rho)\|_{\dot H^{1/2}(I\!R)} \\
&\lesssim \left(\frac{r}{\rho}\right)^{1/2}\|g\|_{L^{2,1}(I\!R)}\|Q\|_{\dot H^{1/2}(I\!R)}\|v\|_{L^{2,\infty}(B(x_0,r))}\,.
\end{aligned}
$$

By arguing as in (3) and (4) we get

$$
(8) \lesssim \|Q\|_{L^\infty}\|g\|_{L^{2,1}(I\!R)}\alpha^{1/2}\left(\sum_{h=1}^{+\infty} 2^{-h/2}\|v\|_{L^{2,\infty}(A_{h,x_0}(r))}\right)\,. \tag{92}
$$

This concludes the proof of Lemma A.4. □

The localization of the operator $S(Q,\Delta^{-1/4}v)$, with $v\in L^2(I\!R)$ is similar to that of $N(Q,v)$ and we omit it.

2. Localization of a term of the type Av with $A\in L^{2,1}$ and $v\in L^2$.

Lemma A.5 *Let $A\in L^{2,1}(I\!R)$, $x_0\in I\!R$, $r>0$, $0<\alpha<1/4$ and $g\in L^{2,1}(I\!R)$, supp $g\subset B(x_0,r\alpha)$. Then*

$$
\begin{aligned}
\int_{I\!R} Av\Delta^{-1/4}g dx \quad &\lesssim \quad \|A\|_{L^{2,1}}\|g\|_{L^{2,1}}\|v\|_{L^{2,\infty}(B(x_0,r))} \\
&+ \quad \alpha^{1/2}\|A\|_{L^{2,1}}\|g\|_{L^{2,1}}\sum_{h=-1}^{+\infty} 2^{-h/2}\|v\|_{L^{2,\infty}(A_{h,x_0}(r))}\,.
\end{aligned} \tag{93}
$$

Proof of Lemma A.5. We suppose again for simplicity that $x_0=0$. We write

$$
\int_{I\!R} Av\,\Delta^{-1/4}g dx \quad = \quad \underbrace{\int_{I\!R} Av\,1\!\!1_{B(0,r)}\Delta^{-1/4}g dx}_{(9)}+\underbrace{\sum_{h=0}^{+\infty}\int_{I\!R} Av1\!\!1_{A'_{h,0}}\Delta^{-1/4}g dx}_{(10)}\,.
$$

Now we observe that $\Delta^{-1/4}g=|x|^{-1/2}*g\in L^\infty(I\!R)$ since $|x|^{-1/2}\in L^{2,\infty}(I\!R)$ and $g\in L^{2,1}(I\!R)$ (see for instance [7]) .Thus we have

$$
\begin{aligned}
(9) \quad &\le \quad \|A\Delta^{-1/4}g\|_{L^{2,1}}\|v\|_{L^{2,\infty}(B(0,r))} \\
&\le \quad \|A\|_{L^{2,1}}\|\Delta^{-1/4}g\|_{L^\infty}\|v\|_{L^{2,\infty}(B(0,r))} \\
&\lesssim \quad \|A\|_{L^{2,1}}\|g\|_{L^{2,1}}\|v\|_{L^{2,\infty}(B(0,r))}\,.
\end{aligned}
$$

$$
\begin{aligned}
(10) \quad &\simeq \sum_{h=0}^{+\infty} \int_{I\!R} \mathcal{F}^{-1}[|\cdot|^{-1/2}](\xi) g * (1\!\!1_{A'_{h,0}} Av) d\xi \\
&\lesssim \sum_{h=0}^{+\infty} \||\xi|^{-1/2}\|_{L^\infty(B^c(0,2^{h-1}r))} \|g * (1\!\!1_{A'_{h,0}} Av)\|_{L^1} \\
&\lesssim \sum_{h=0}^{+\infty} 2^{-h/2} r^{-1/2} \|g\|_{L^1} \|1\!\!1_{A'_{h,0}} Av\|_{L^1} \\
&\lesssim \sum_{h=0}^{+\infty} 2^{-h/2} r^{-1/2} (r\alpha)^{1/2} \|g\|_{L^{2,1}} \|A\|_{L^{2,1}} \|v\|_{L^{2,\infty}(A_{h',0})} \\
&\lesssim \alpha^{1/2} \|g\|_{L^{2,1}} \|A\|_{L^{2,1}} \sum_{h=0}^{+\infty} 2^{-h/2} \|v\|_{L^{2,\infty}(A_{h,0}(r))} \, .
\end{aligned}
$$

This concludes the proof of Lemma A.5 . □

3. Localization of a term of the type Ωv with $\Omega \in L^2$ and $v \in L^q$, $q \geq 2$.

Lemma A.6 *Let $\Omega \in L^2(I\!R, \mathcal{M}_{m\times m}(I\!R)$ be such that supp $\Omega \subset B^c(0,\rho)$, $v \in L^q(I\!R)$, $q \geq 2$, $x_0 \in B(0,\rho/8)$, $g \in L^{q'}(I\!R)$, supp $g \subset B(x_0, r\alpha)$, with $0 < \alpha < 1/4$, $0 < r < \rho/16$.*
Then we have

$$
\int_{I\!R} \Omega v \Delta^{-1/4} g dx \lesssim \left(\frac{r}{\rho}\right)^{1/q} \alpha^{1/q} \|g\|_{L^{q'}} \|\Omega\|_{L^2} \|v\|_{L^q} \, . \tag{94}
$$

Proof of Lemma A.6. We give the proof for the case $q > 2$ (the case $q = 2$ is similar and even simpler). We use the fact that Ω and g have disjoint supports.

$$
\begin{aligned}
\int_{I\!R} \Omega v \Delta^{-1/4} g dx &= \int_{I\!R} \mathcal{F}^{-1}(|\cdot|^{-1/2})(\xi)(g * \Omega v)\, d\xi \\
&\lesssim \||x|^{-1/2}\|_{L^{\frac{2q}{q-2}}(B^c(0,\rho/4))} \|g * \Omega v\|_{L^{\frac{2q}{q+2}}} \\
&\lesssim \rho^{-1/q} \|g\|_{L^1} \|\Omega v\|_{L^{\frac{2q}{q+2}}} \\
&\lesssim \left(\frac{r}{\rho}\right)^{1/q} \alpha^{1/q} \|g\|_{L^{q'}} \|\Omega\|_{L^2} \|v\|_{L^q} \, .
\end{aligned}
$$

This concludes the proof of Lemma A.6 . □

Lemma A.7 *Let $\Omega \in L^2(I\!R, \mathcal{M}_{m\times m}(I\!R))$, $v \in L^q(I\!R)$, $q > 2$, $x_0 \in I\!R$, $g \in L^{q'}(I\!R)$ and $r > 0$. Then we have*

$$
\int_{I\!R} \Omega 1\!\!1_{B(x_0,r/2)} v \, \Delta^{-1/4} g dx \lesssim \|g\|_{L^{q'}} \|\Omega\|_{L^2} \|v\|_{L^q(B(x_0,r/2))} \, . \tag{95}
$$

Proof of Lemma A.7. We observe that $1 < q' < 2$ and $W^{1/2,q'}(I\!R) \hookrightarrow L^{\frac{2q}{q-2}}(I\!R)$. Thus we have $\Delta^{-1/4} g \in L^{\frac{2q}{q-2}}(I\!R)$.

Moreover one has

$$\frac{q-2}{2q} + \frac{1}{2} + \frac{1}{q} = 1\,.$$

Thus by applying the generalized Hölder inequality we get

$$\int_{I\!R} \Omega(1\!\!1_{B(x_0,r/2)} v)\, \Delta^{-1/4} g dx \lesssim \|g\|_{L^{q'}} \|\Omega\|_{L^2} \|v\|_{L^q(B(x_0,r/2)}\,.$$

This concludes the proof of Lemma A.7. □

An analogous result of Lemma A.7 for $q = 2$ still holds provided $g \in L^{2,1}(I\!R)$. Indeed in this case we use the fact that $\Delta^{-1/4} g \in L^\infty$.

Lemma A.8 *Let $\Omega \in L^2(I\!R, \mathcal{M}_{m\times m}(I\!R))$, $v \in L^2(I\!R)$, $q > 2$, $x_0 \in I\!R$, $g \in L^{2,1}(I\!R)$ and $r > 0$. Then we have*

$$\int_{I\!R} \Omega 1\!\!1_{B(x_0,r/2)} v\, \Delta^{-1/4} g dx \lesssim \|g\|_{L^{2,1}} \|\Omega\|_{L^2} \|v\|_{L^2(B(x_0,r/2))}\,. \quad \square \tag{96}$$

The proof of Lemma A.8 is similar to that of Lemma A.7 and we omit it.

Lemma A.9 *Let $\Omega \in L^2(I\!R, \mathcal{M}_{m\times m}(I\!R)$, $v \in L^q(I\!R)$, $q > 2$, $x_0 \in I\!R$, $g \in L^{q'}(I\!R)$, supp $g \subset B(x_0, r\alpha)$, with , $0 < \alpha < 1/4$, and $r > 0$.*

Then we have

$$\sum_{h=0}^{\infty} \int_{I\!R} \Omega 1\!\!1_{A_{h,x_0}(r)} v \Delta^{-1/4} g dx \lesssim \alpha^{1/q} \sum_{h=0}^{\infty} 2^{-h/q} \|g\|_{L^{q'}} \|\Omega\|_{L^2} \|v\|_{L^q(A'_{h,x_0}(r))}\,. \tag{97}$$

Proof of Lemma A.9. We assume $x_0 = 0$. We have

$$\begin{aligned}
&\sum_{h=0}^{\infty} \int_{I\!R} \Omega(1\!\!1_{A'_{h,0}(r)} v) \Delta^{-1/4} g dx = \sum_{h=0}^{\infty} \int_{I\!R} \mathcal{F}^{-1}[|\cdot|^{-1/2}](x) g * \Omega 1\!\!1_{A'_{h,0}(r)} v dx \\
&\lesssim \sum_{h=0}^{\infty} \||x|^{-1/2}\|_{L^{\frac{2q}{q-2}}(B(0,2^{h+2}r))} \|g * \Omega 1\!\!1_{A'_{h,0}(r)} v\|_{L^{\frac{2q}{q+2}}(I\!R)} \\
&\lesssim \sum_{h=0}^{\infty} 2^{-h/q} r^{-1/q} \|g\|_{L^1(B(0,r\alpha))} \|\Omega 1\!\!1_{A'_{h,0}(r)} v\|_{L^{\frac{2q}{q+2}}} \\
&\lesssim \sum_{h=0}^{\infty} 2^{-h/q} r^{-1/q} (r\alpha)^{1/q} \|g\|_{L^{q'}} \|\Omega\|_{L^2} \|v\|_{L^q(A'_{h,0}(r))} \\
&\lesssim \alpha^{1/q} \sum_{h=0}^{\infty} 2^{-h/q} \|g\|_{L^{q'}} \|\Omega\|_{L^2} \|v\|_{L^q(A'_{h,0}(r))}\,.
\end{aligned}$$

This concludes the proof of Lemma A.9 . $\square$

4. Localization of the operator $F(Q,v) := \mathcal{R}(Q)\mathcal{R}(v) - Qv$.

Lemma A.10 *Let $Q \in L^2(I\!R) \cap L^\infty(I\!R)$, $\|Q\|_{L^2(I\!R)} \le \varepsilon_0$, $v \in L^2(I\!R)$, $g \in L^{2,1}(I\!R)$, supp $g \subset B(x_0, r\alpha)$, with $x_0 \in I\!R$, $0 < \alpha < \frac{1}{4}$, $r > 0$.*

Then we have

$$\begin{aligned} \int_{I\!R} F(Q,v)\Delta^{-1/4} g dx &\lesssim \varepsilon_0 \|g\|_{L^{2,1}} \|v\|_{L^{2,\infty}(B_r(x_0))} \\ &+ \alpha^{1/2}(\|Q\|_{L^2(I\!R)} + \|Q\|_{L^\infty})\|g\|_{L^{2,1}} \sum_{h=1}^{+\infty} 2^{-h/2} \|v\|_{L^{2,\infty}(A_{h,x_0}(r))} . \end{aligned} \tag{98}$$

Proof of Lemma A.10. We assume $x_0 = 0$. We take the scalar product of $F(Q,v)$ with $\Delta^{-1/4} g$ and we integrate. We get

$$\begin{aligned} \int_{I\!R} F(Q,v)\Delta^{-1/4} g dx &= \underbrace{\int_{I\!R} F(Q,\chi_r v)\Delta^{-1/4} g dx}_{(11)} \\ &+ \underbrace{\int_{I\!R} \sum_{k=1}^{+\infty} F(Q,\varphi_k v)\Delta^{-1/4} g dx}_{(12)} . \end{aligned}$$

To estimate (11) we use the fact that $F(Q,v) \in \dot{H}^{-1/2}(I\!R)$ and

$$\|F(Q,v)\|_{\dot{H}^{1/2}(I\!R)} \lesssim \|Q\|_{L^2(I\!R)} \|v\|_{L^{2,\infty}} .$$

$$\begin{aligned} (11) &\le \|\Delta^{-1/4} g\|_{\dot{H}^{1/2}(I\!R)} \|Q\|_{L^2(I\!R)} \|v\|_{L^{2,\infty}(B(0,r))} \\ &\lesssim \|g\|_{L^{2,1}} \|Q\|_{L^2(I\!R)} \|v\|_{L^{2,\infty}(B(0,r))} \\ &\lesssim \varepsilon_0 \|g\|_{L^{2,1}} \|v\|_{L^{2,\infty}(B(0,r))} . \end{aligned}$$

Next we spilt (12) in two parts:

$$\begin{aligned} (12) &= \underbrace{\sum_{k=1}^{\infty} \int_{I\!R} F(Q,\varphi_k v) 1\!\!1_{B(0,r/4)} \Delta^{-1/4} g dx}_{(13)} \\ &+ \underbrace{\sum_{k=1}^{\infty} \sum_{h=-1}^{\infty} \int_{I\!R} F(Q,\varphi_k v) 1\!\!1_{A'_{h,0}(r)} \Delta^{-1/4} g dx}_{(14)} . \end{aligned}$$

Estimate of (13):

$$
\begin{aligned}
(13) \quad &= \sum_{k=1}^{+\infty} \int_{I\!R} F(Q, \varphi_k v) 1\!\!1_{B(0,r/4)} \Delta^{-1/4} g dx \\
&= \sum_{k=1}^{+\infty} \int_{I\!R} \mathcal{R}(Q) \mathcal{R}(\varphi_k v) 1\!\!1_{B(0,r/4)} \Delta^{-1/4} g dx \\
&\simeq \sum_{k=1}^{+\infty} \int_{I\!R} \mathcal{F}^{-1} \left[\frac{\cdot}{|\cdot|} \right] (\xi) [(\varphi_k v) * (Q 1\!\!1_{B(0,r/4)} \Delta^{-1/4} g)] \, d\xi \\
&\lesssim \sum_{k=1}^{+\infty} \| \frac{1}{\xi} \|_{L^\infty(B^c(0,2^{k-1}r))} \|\varphi_k v\|_{L^1(I\!R)} \|Q 1\!\!1_{B(0,r/4)} \Delta^{-1/4} g\|_{L^1(I\!R)} \\
&\lesssim \sum_{k=1}^{+\infty} 2^{-k} r^{-1} 2^{k/2} r^{1/2} r \alpha^{1/2} \|v\|_{L^{2,\infty}(A_{h,0}(r))} \|Q\|_{L^\infty} \|g\|_{L^{2,1}} \\
&\lesssim (r\alpha)^{1/2} \|Q\|_{L^\infty} \|g\|_{L^{2,1}} \sum_{k=1}^{+\infty} 2^{-k/2} \|v\|_{L^{2,\infty}(A_{k,0}(r))} \,.
\end{aligned}
$$

The estimate of (14) is analogous of (4) in the proof of Lemma A.4 and we omit it. □

Lemma A.11 *Let* $Q \in L^2(I\!R) \cap L^\infty(I\!R)$*, supp* $Q \subset B^c(0,\rho)$ *for some* $\rho > 0$*,* $v \in L^2(I\!R)$*,* $x_0 \in B(0, \rho/8)$*,* $g \in L^{2,1}(I\!R)$*, supp* $g \subset B(x_0, r\alpha)$*, with* $0 < \alpha < 1/4$*,* $0 < r < \rho/16$*.*

Then we have

$$
\begin{aligned}
\int_{I\!R} F(Q, v) \Delta^{-1/4} g dx \;&\lesssim\; \left[\alpha^{1/2} + \left(\frac{r}{\rho} \right)^{1/2} \right] \|Q\|_{L^2} \|g\|_{L^{2,1}} \|v\|_{L^{2,\infty}(B(x_0,r))} \\
&+\; \alpha^{1/2} (\|Q\|_{L^2} + \|Q\|_{L^\infty}) \|g\|_{L^{2,1}} \sum_{h=1}^{+\infty} 2^{-h/2} \|v\|_{L^{2,\infty}(A_{h,x_0}(r))} \,.
\end{aligned} \tag{99}
$$

Proof of Lemma A.11. We just give a sketch of proof.

We write

$$\begin{aligned}\int_{I\!R} F(Q,v)\Delta^{-1/4} g dx &= \underbrace{\int_{I\!R} F(Q,\chi_r v)1\!\!1_{B(x_0,r/4)}\Delta^{-1/4} g dx}_{(15)} \\ &+ \underbrace{\int_{I\!R} F(Q,\chi_r v)1\!\!1_{A'_{h,x_0}(r)}\Delta^{-1/4} g dx}_{(16)} \\ &+ \underbrace{\int_{I\!R} F(Q,(1-\chi_r)v)\Delta^{-1/4} g dx}_{(17)}\ .\end{aligned}$$

To estimate (15) we write $Q = \sum_{h=-2} \tilde{\varphi}_h (Q - Q_\rho)$ with supp $\tilde{\varphi}_h \subseteq B(0, 2^{h+1}\rho \setminus B(0, 2^{h-1}\rho)$ and $\tilde{\varphi}_h$ partition of unity.

$$\begin{aligned}(15) &= \sum_{h=-2}^{\infty} \int_{I\!R} \mathcal{R}(\tilde{\varphi}_h(Q-Q_\rho))\mathcal{R}(\chi_r v)1\!\!1_{B(x_0,r/4)}\Delta^{-1/4} g dx \\ &= \sum_{h=-2}^{\infty} \int_{I\!R} \mathcal{F}^{-1}\left[\frac{\cdot}{|\cdot|}\right](x)[\tilde{\varphi}_h(Q-Q_\rho)] * [\mathcal{R}(\chi_r v)1\!\!1_{B(x_0,r/4)}\Delta^{-1/4} g] dx \\ &\lesssim \sum_{h=-2}^{\infty} \|x^{-1}\|_{L^\infty(B^c(0,2^h\rho))}\|\tilde{\varphi}_h(Q-Q_\rho)\|_{L^1}\|\mathcal{R}(\chi_r v)\|_{L^1(B(x_0,r/4))}\|\Delta^{-1/4} g\|_{L^\infty(I\!R)} \\ &\lesssim \|g\|_{L^{2,1}}\|\mathcal{R}(\chi_r v)\|_{L^{2,\infty}(B(x_0,r/4))}\left(\frac{r}{\rho}\right)^{1/2}\sum_{h=-2}^{\infty} 2^{-h/2}\|Q\|_{L^2(A_{h,0}(\rho))} \\ &\lesssim \left(\frac{r}{\rho}\right)^{1/2}\|g\|_{L^{2,1}}\|Q\|_{L^2}\|v\|_{L^{2,\infty}(B(x_0,r))}\ .\end{aligned}$$

Now we write

$$(16) = \sum_{h=-2}^{+\infty}\sum_{k=-2}^{+\infty}\int_{I\!R} F(\tilde{\varphi}_k Q, \chi_r v)1\!\!1_{A'_{h,x_0}(r)}\Delta^{-1/4} g dx$$

$$= \sum_{h=-2}^{+\infty}\sum_{|k-h|\le 5}\int_{I\!R} F(\tilde{\varphi}_k Q, \chi_r v)1\!\!1_{A'_{h,x_0}(r)}\Delta^{-1/4} g dx$$

$$+ \sum_{h=-2}^{+\infty}\sum_{|k-h|> 5}\int_{I\!R} F(\tilde{\varphi}_k Q, \chi_r v)1\!\!1_{A'_{h,x_0}(r)}\Delta^{-1/4} g dx$$

by arguing as in (5) and (6)

$$\lesssim \|g\|_{L^{2,1}}\|Q\|_{L^2}\left[\left(\frac{r}{\rho}\right)^{1/2} + \alpha^{1/2}\right]\|v\|_{L^{2,\infty}(B(x_0,r))}\,.$$

The estimate of (17) is analogous to (2) in the proof of Lemma A.3 and we omit it. □

B Commutator Estimates

We consider the Littlewood-Paley decomposition of unity introduced in Section 2. For every $j \in Z\!\!\!Z$ and $f \in \mathcal{S}'(I\!R)$ we define the Littlewood-Paley projection operators P_j and $P_{\le j}$ by

$$\widehat{P_j f} = \psi_j \hat{f} \quad \widehat{P_{\le j} f} = \phi_j \hat{f}\,.$$

Informally P_j is a frequency projection to the annulus $\{2^{j-1} \le |\xi| \le 2^j\}$, while $P_{\le j}$ is a frequency projection to the ball $\{|\xi| \le 2^j\}$. We will set $f_j = P_j f$ and $f^j = P_{\le j} f$.

We observe that $f^j = \sum_{k=-\infty}^{j} f_k$ and $f = \sum_{k=-\infty}^{+\infty} f_k$ (where the convergence is in $\mathcal{S}'(I\!R)$).

Given $f, g \in \mathcal{S}'(I\!R)$ we can split the product in the following way

$$fg = \Pi_1(f,g) + \Pi_2(f,g) + \Pi_3(f,g), \tag{100}$$

where

$$\Pi_1(f,g) = \sum_{-\infty}^{+\infty} f_j \sum_{k\le j-4} g_k = \sum_{-\infty}^{+\infty} f_j g^{j-4}\,;$$

$$\Pi_2(f,g) = \sum_{-\infty}^{+\infty} f_j \sum_{k\ge j+4} g_k = \sum_{-\infty}^{+\infty} g_j f^{j-4}\,;$$

$$\Pi_3(f,g) = \sum_{-\infty}^{+\infty} f_j \sum_{|k-j|<4} g_k\,.$$

We observe that for every j we have

$$\mathrm{supp}\mathcal{F}[f^{j-4}g_j] \subset \{2^{j-2} \le |\xi| \le 2^{j+2}\};$$

$$\mathrm{supp}\mathcal{F}[\textstyle\sum_{k=j-3}^{j+3} f_j g_k] \subset \{|\xi| \le 2^{j+5}\}\,.$$

The three pieces of the decomposition (100) are examples of paraproducts. Informally the first paraproduct Π_1 is an operator which allows high frequences of f ($\sim 2^j$) multiplied by low frequences of g ($\ll 2^j$) to produce high frequences in the output. The second paraproduct Π_2 multiplies low fequences of f with high frequences of g to produce high fequences in the output. The third paraproduct Π_3 multiply high frequences of f with high frequences of g to produce comparable or lower frequences in the output. For a presentation of these paraproducts we refer to the reader for instance to the book [8]. The following two Lemmas will be often used in the sequel. For the proof of it we refer the reader to [4].

Lemma B.1 *For every $f \in \mathcal{S}'$ we have*

$$\sup_{j\in Z} |f^j| \le M(f)\,. \ \square$$

In the sequel we will often use the following property: for every vector field $X \in \dot{W}^{s,r}(I\!R)$ with $s<0$ we have

$$\int_{I\!R}\left(\sum_{j=-\infty}^{+\infty} 2^{2js}(X^j)^2\right)^{r/2} dx = \int_{I\!R}\left(\sum_{k,\ell} X_k X_\ell \sum_{j-4\ge k, j-4\ge \ell} 2^{2js}\right)^{r/2} dx$$

$$\simeq \int_{I\!R}\left(\sum_{k} X_k \left(\sum_{|k-\ell|\le 2} X_\ell\right) 2^{2(k+2)s}\right)^{r/2} dx$$

by Cauchy-Schwarz Inequality

$$\lesssim \int_{I\!R}\left(\sum_{k} 2^{2ks}X_k^2\right)^{r/4}\left(\sum_{k} 2^{2ks}X_k^2\right)^{r/4} dx \tag{101}$$

$$= \int_{I\!R}\left(\sum_{j=-\infty}^{+\infty} 2^{2ks}(X_k)^2\right)^{r/2} dx = \|X\|^r_{\dot{W}^{s,r}(I\!R)}\,,$$

(see also Section 4.4.2 in [14], page 165).

Now we start with a series of preliminary Lemmas which will be crucial for the construction of the gauge P in the Section 4.

Lemma B.2 *Let* $1 < r < 2$, $a \in \dot{W}^{1/2,r}(I\!R)$ *and* $b \in \dot{H}^{1/2}(I\!R) \cap L^\infty(I\!R)$. *Then*

$$\|ab\|_{\dot{W}^{1/2,r}} \le C\|a\|_{\dot{W}^{1/2,r}}(\|b\|_{\dot{H}^{1/2}} + \|b\|_{L^\infty}) .$$

Proof of Lemma B.2 .

- **Estimate of** $\|\Pi_1(\Delta^{1/4}(ab))\|_{L^r}$.

$$\|\sum_j \Delta^{1/4}(a_j b^{j-4})\|^r_{L^r} \lesssim \int_{I\!R}(\sum_j 2^j|a_j|^2|b^{j-4}|^2)^{r/2}dx$$
$$\lesssim \int_{I\!R} \sup_j |b^{j-4}|^r \left(\sum_j 2^j|a_j|^2\right)^{r/2} dx$$
$$\lesssim \int_{I\!R} |M(b)|^r \left(\sum_j 2^j|a_j|^2\right)^{r/2} dx \le \|b\|^r_{L^\infty} \int_{I\!R}\left(\sum_j 2^j|a_j|^2\right)^{r/2} dx$$
$$\lesssim \|b\|^r_{L^\infty}\|a\|^r_{\dot{W}^{1/2,r}} .$$

- **Estimate of** $\|\Pi_2\Delta^{1/4}(ab)\|_{L^r}$.

$$\|\sum_j \Delta^{1/4}(a^{j-4}b_j)\|_{L^r} \simeq \sup_{\|h\|_{L^{r'}}\le 1}\int_{I\!R}\sum_j a^{j-1}b_j\Delta^{1/4}h_j$$
$$\lesssim \sup_{\|h\|_{L^{r'}}\le 1}\int_{I\!R}\sup_j |a^{j-4}|\left(\sum_j 2^j|b_j|^2\right)^{1/2}\left(\sum_j |h_j|\right)^{1/2} dx$$
$$\lesssim \sup_{\|h\|_{L^{r'}}\le 1}\int_{I\!R}|M(a)|\left(\sum_j 2^j|b_j|^2\right)^{1/2}\left(\sum_j |h_j|\right)^{1/2} dx$$

by the generalized Hölder Inequality: $\frac{1}{r'} + \frac{1}{2} + \frac{2-r}{2r} = 1$

$$\lesssim \|b\|_{\dot{H}^{1/2}}\|a\|_{\dot{W}^{1/2,r}} .$$

- **Estimate of** $\|\Pi_3(\Delta^{1/4}(ab)\|_{L^r}$.

$$
\begin{aligned}
&\|\sum_j \Delta^{1/4}(a_j b_j)\|_{L^r} \\
&\simeq \sup_{\|h\|_{L^{r'}}\le 1}\left[\int_{I\!R}\sum_j\sum_{|k-j|\le 3}\Delta^{1/4}(a_j b_j)h_k dx + \int_{I\!R}\sum_j \Delta^{1/4}(a_j b_j)h^{j-4}dx\right] \\
&= \sup_{\|h\|_{L^{r'}}\le 1}\left[\underbrace{\int_{I\!R}\sum_j\sum_{|k-j|\le 3}(a_j b_j)\Delta^{1/4}h_k dx}_{(1)} + \underbrace{\int_{I\!R}\sum_j (a_j b_j)\Delta^{1/4}h^{j-4}dx}_{(2)}\right]
\end{aligned}
$$

We estimate the term (2) .

$$
(2) \lesssim \|b\|_{B^0_{\infty,\infty}} \int_{I\!R}\left(\sum_j 2^{-j}|\Delta^{1/4}h^{j-4}|^2\right)^{1/2}\left(\sum_j 2^j a_j^2\right)^{1/2} dx
$$

$$
\lesssim \|b\|_{B^0_{\infty,\infty}}\left(\int_{I\!R}\left(\sum_j 2^{-j}|\Delta^{1/4}h^{j-4}|^2\right)^{r'/2}dx\right)^{1/r'}\left(\int_{I\!R}\left(\sum_j 2^j a_j^2\right)^{r/2}dx\right)^{1/r}
$$

by applying (101) to $\Delta^{1/4}h$

$$
\lesssim \|b\|_{B^0_{\infty,\infty}}\|h\|_{L^{r'}}\|a\|_{\dot W^{1/2,r}}\,.
$$

The term $(1s)$ is estimated in a similar way. Thus we get

$$
\|\sum_j \Delta^{1/4}(a_j b_j)\|_{L^r} \lesssim \|b\|_{\dot H^{1/2}}\|a\|_{\dot W^{1/2,r}}\,.
$$

This concludes the proof of Lemma B.2 . □

Lemma B.3 *Let* $1<r<2\le q$*,* $a\in \dot W^{1/2,r}(I\!R)$ *and* $b\in \dot W^{1/2,q}(I\!R)$ *and* $t=\frac{2rq}{2r+q(2-r)}$ *. Then*

$$
\|\Delta^{1/4}(ab)-(\Delta^{1/4}a)b\|_{L^t(I\!R)} \le C\|a\|_{\dot W^{1/2,r}(I\!R)}\|b\|_{\dot W^{1/2,q}(I\!R)}\,.
$$

Proof of Lemma B.3 .

• **Estimate of** $\|\Pi_2(\Delta^{1/4}(ab))\|_{L^t}$.

$$\|\sum_j \Delta^{1/4}(a^{j-4}b_j)\|_{L^t}^t \lesssim \int_{I\!R}\left(\sum_j 2^j|a^{j-4}|^2|b_j|^2\right)^{t/2} dx$$

$$\lesssim \int_{I\!R}\sup_j |a^{j-4}|^t\left(\sum_j 2^j|b_j|^2\right)^{t/2}$$

$$\lesssim \left(\int_{I\!R} M(a)^{\frac{tq}{q-t}}dx\right)^{1-\frac{t}{q}}\left(\int_{I\!R}(\sum_j 2^j|b_j|^2)^{q/2}dx\right)^{t/q}$$

$$\lesssim \|a\|^t_{L^{\frac{2r}{2-r}}}\|b\|^t_{\dot W^{1/2,q}} \lesssim \|a\|^t_{\dot W^{1/2,r}(I\!R)}\|b\|^t_{\dot W^{1/2,q}(I\!R)} .$$

In the above expression we use the fact that $\frac{tq}{q-t}=\frac{2r}{2-r}$.

• **Estimate of** $\|\Pi_2((\Delta^{1/4}a)b)\|_{L^t}$.

$$\|\sum_j(\Delta^{1/4}a^{j-4})b_j\|_{L^t}^t$$

$$\lesssim \int_{I\!R}\left(\sup_j 2^{-j/2}|\Delta^{1/4}a^{j-4}|\right)^t\left(\sum_j 2^j|b_j|^2\right)^{t/2} dx$$

$$\lesssim \int_{I\!R}\left(\sum_j 2^{-j}|\Delta^{1/4}a^{j-4}|^2\right)^{t/2}\left(\sum_j 2^j|b_j|^2\right)^{t/2} dx$$

$$\lesssim \left(\int_{I\!R}\left(\sum_j 2^{-j}|\Delta^{1/4}a^{j-4}|^2\right)^{tq/2(q-t)}\right)^{1-t/q}\left(\left(\sum_j 2^j|b_j|^2\right)^{q/2} dx\right)^{t/q}$$

$$\lesssim \|a\|^t_{L^{tq/q-t}}\|b\|^t_{\dot W^{1/2,q}} \lesssim \|a\|^t_{\dot W^{1/2,r}}\|b\|^t_{\dot W^{1/2,q}} .$$

• **Estimate of** $\|\Pi_3(\Delta^{1/4}(ab))\|_{L^t}$.

$$\|\sum_j \Delta^{1/4}(a_j b_j)\|_{L^t} \simeq \sup_{\|h\|_{L^{t'}}\le 1} \int_{I\!R} \Delta^{1/4} h \sum_j a_j b_j dx$$

$$\lesssim \sup_{\|h\|_{L^{t'}}\le 1} \left[\underbrace{\int_{I\!R} \sum_j \sum_{|j-k|\le 4} (\Delta^{1/4} h_k) a_j b_j dx}_{(3)} + \underbrace{\int_{I\!R} \sum_j (\Delta^{1/4} h^{j-4}) a_j b_j dx}_{(4)} \right] .$$

We estimate the term (4).

$$(4) \lesssim \int_{I\!R} \sup_j \left(2^{-j/2} |\Delta^{1/4} h^{j-4}|\right) \sum_j 2^{j/2} |a_j| |b_j| dx$$

$$\lesssim \int_{I\!R} \left(\sum_j |\Delta^{1/4} h^{j-4}|^2\right)^{1/2} \left(\sum_j |a_j|^2\right)^{1/2} \left(\sum_j 2^j |b_j|^2\right)^{1/2} dx$$

$$\lesssim \left[\int_{I\!R} \left(\sum_j 2^{-j} |\Delta^{1/4} h^{j-4}|^2\right)^{t'/2}\right]^{1/t'} \left[\int_{I\!R} \left(\sum_j |a_j|^2\right)^{t/2} \left(\sum_j 2^j |b_j|^2\right)^{t/2}\right]^{1/t}$$

$$\lesssim \|h\|_{L^{t'}} \left[\int_{I\!R} \left(\sum_j |a_j|^2\right)^{tq/2(q-t)} dx\right]^{\frac{q-t}{qt}} \left[\int_{I\!R} \left(\sum_j 2^j |b_j|^2\right)^{q/2} dx\right]^{1/q}$$

$$\lesssim \|h\|_{L^{t'}} \|a\|_{tq/q-t} \|b\|_{W^{1/2,q}}$$

$$\lesssim \|h\|_{L^{t'}} \|a\|_{W^{1/2,r}} \|b\|_{W^{1/2,q}} .$$

The estimate of (3) is similar.

- **Estimate of** $\|\Pi_3((\Delta^{1/4} a) b)\|_{L^t}$.

$$\| \sum_j (\Delta^{1/4} a_j) b_j) \|_{L^t}^t \lesssim \int_{I\!R} | \sum_j \Delta^{1/4} a_j b_j |^t$$

$$\lesssim \int_{I\!R} \left(\sum_j 2^{-j} |\Delta^{1/4} a_j|^2 \right)^{t/2} \left(\sum_j 2^j b_j|^2 \right)^{t/2} dx$$

$$\lesssim \|a\|_{tq/q-t}^t \|b\|_{W^{1/2,q}}^t \lesssim \|a\|_{W^{1/2,r}}^t \|b\|_{W^{1/2,q}}^t .$$

- **Estimate of** $\|\Pi_2(\Delta^{1/4}(ab) - (\Delta^{1/4} a) b)\|_{L^t}$.

$$\| \sum_j (\Delta^{1/4}(ab) - (\Delta^{1/4} a) b) \| \|_{L^t} \tag{102}$$

$$= \sup_{\|h\|_{L^{t'}} \le 1} \int_{I\!R} \sum_j h_j [\Delta^{1/4}(a_j b^{j-4}) - (\Delta^{1/4} a_j) b^{j-4}] dx$$

$$= \sup_{\|h\|_{L^{t'}} \le 1} \int_{I\!R} \sum_j b^{j-4} [(\Delta^{1/4} h_j) a_j - h_j (\Delta^{1/4} a_j)] dx$$

$$= \sup_{\|h\|_{L^{t'}} \le 1} \int_{I\!R} \sum_j \mathcal{F}[b]^{j-4}(\eta) (\int_{I\!R} \mathcal{F}[h]_j(\xi) \mathcal{F}[a]_j(\eta - \xi) [|\xi|^{1/2} - |\eta - \xi|^{1/2}] d\xi) d\eta .$$

Now we observe that in (102) we have $|\eta| \le 2^{j-3}$ and $2^{j-2} \le |\xi| \le 2^{j+2}$. Thus $|\frac{\xi}{\eta}| \le \frac{1}{2}$. Hence

$$\begin{aligned} |\xi|^{1/2} - |\eta - \xi|^{1/2} &= |\xi|^{1/2} [1 - |1 - \frac{\eta}{\xi}|^{1/2}] \\ &= |\xi|^{1/2} \frac{\eta}{\xi} [1 + |1 - \frac{\eta}{\xi}|^{1/2}]^{-1} \\ &= |\xi|^{1/2} \sum_{k=0}^{\infty} \frac{c_k}{k!} (\frac{\eta}{\xi})^{k+1} . \end{aligned} \tag{103}$$

We may suppose that $\sum_{k=0}^{\infty} \frac{c_k}{k!} (\frac{\eta}{\xi})^{k+1}$ is convergent if $|\frac{\xi}{\eta}| \le \frac{1}{2}$, otherwise one may consider a different Littlewood-Paley decomposition by replacing the exponent $j-4$ with $j-s$, $s > 0$ large enough. We introduce the following notation: for every $k \ge 0$ we set

$$S_k g = \mathcal{F}^{-1} [\xi^{-(k+1)} |\xi|^{1/2} \mathcal{F} g].$$

We note that if $g \in L^{t'}$ then $S_k g \in \dot{W}^{1/2+k,t'}$.

We have

$$(102) = \sup_{\|h\|_{L^{t'}} \leq 1} \sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} \int_{I\!R} \mathcal{F}[b^{j-4}](\xi)$$
$$\left[\int_{I\!R} \mathcal{F}[h_j](\eta)\mathcal{F}[a_j](\xi-\eta)[|\eta|^{1/2}(\frac{\xi}{\eta})^{\ell+1}]d\eta\right]$$

$$= \sup_{\|h\|_{L^{t'}} \leq 1} \sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} \int_{I\!R} (\xi^{\ell+1}\mathcal{F}[b^{j-4}])(\xi)$$
$$\left[\int_{I\!R} (\eta^{-(\ell+1)}|\eta|^{1/2}\mathcal{F}[h_j])(\eta)\mathcal{F}[a_j](\xi-\eta)d\eta\right]$$

$$\lesssim \sup_{\|h\|_{L^{t'}} \leq 1} \sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} \int_{I\!R} \sum_j \sum_{|k-j|\leq 3} [\nabla^{\ell+1} b^{j-4}]\,[(S_\ell h_k)a_j](x)dx$$

$$\lesssim \sup_{\|h\|_{L^{t'}} \leq 1} \sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} \int_{I\!R} \sum_j \sum_{|k-j|\leq 3} [\nabla^{\ell+1} b^{j-4}][(S_\ell h_k)a_j)](x)dx$$

$$\lesssim \sup_{\|h\|_{L^{t'}} \leq 1} \sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} \int_{I\!R} \sum_j [2^{-(\ell+1/2)j}\nabla^{\ell+1} b^{j-4}][2^{(\ell+1/2)j}(S_\ell h_j)a_j)](x)dx$$

$$\lesssim \sup_{\|h\|_{L^{t'}} \leq 1} \sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} \int_{I\!R} \sup_j [|2^{(\ell+1/2)j}(S_\ell h_j)|]$$
$$\left(\sum_j |a_j|^2\right)^{1/2} \left(\sum_j 2^{-2(\ell+1/2)j}|\nabla^{\ell+1} b^{j-4}|^2\right)^{1/2} dx$$

$$\lesssim \sup_{\|h\|_{L^{t'}}\le 1} \sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} 2^{-2\ell} \int_{I\!R} \left(\sum_j 2^{-2(\ell+1/2)j} |S_\ell h_j|^2\right)^{1/2}$$

$$\left(\sum_j |a_j|^2\right)^{1/2} \left(\sum_j 2^{-2(\ell+1/2)j} |\nabla^{\ell+1} b^{j-4}|^2\right)^{1/2} dx$$

$$\lesssim \sup_{\|h\|_{L^{t'}}\le 1} (\int_{I\!R} (\sum_j 2^j |\Delta^{-1/4} h_j|^2)^{t'/2})^{t'}$$

$$\left[\int_{I\!R} \left(\sum_j |a_j|^2\right)^{qt/2(q-t)}\right]^{q-t/qt} \left[\int_{I\!R} \left(\sum_j |b_j|^2\right)^{q/2}\right]^{1/q}$$

$$\lesssim \|a\|_{L^{qt/q-t}} \|b\|_{W^{1/2,q}} \lesssim \|a\|_{W^{1/2,r}} \|b\|_{W^{1/2,q}} .$$

We observe that $\sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} 2^{-2\ell} < +\infty$ since we have supposed that $\sum_{\ell=0}^{\infty} \frac{c_\ell}{\ell!} x^\ell$ is convergent for $|x| \le 1/2$. This concludes the proof of Lemma B.3. □

Lemma B.4 *Let* $a \in L^\infty(I\!R)$, $b \in W^{1/2,q}(I\!R)$, $1 < q < +\infty$. *Then*

$$\|\Delta^{1/4}(ab) - (\Delta^{1/4} a) b\|_{L^q(I\!R)} \le C \|b\|_{\dot{W}^{1/2,q}(I\!R)} \|a\|_{L^\infty(I\!R)} .$$

Proof of Lemma B.4.

- **Estimate of** $\|\Pi_1(\Delta^{1/4}(ab))\|_{L^q}^q$.

$$\|\sum_j \Delta^{1/4}(a^{j-4} b_j)\|_{L^q}^q \simeq \int_{I\!R} \left(\sum_j 2^j |a^{j-4}|^2 |b_j|^2\right)^{q/2}$$

$$\lesssim \|a\|_{L^\infty}^q \|b\|_{W^{1/2,q}}^q .$$

- **Estimate of** $\|\Pi_1((\Delta^{1/4} a) b)\|_{L^q}^q$.

$$\|\sum_j (\Delta^{1/4} a^{j-4}) b_j\|_{L^q}^q \lesssim \int_{I\!R} \left(\sum_j |\Delta^{1/4} a^{j-4}|^2 |b_j|^2\right)^{q/2}$$

$$\lesssim \sup_j \|2^{-j/2} |\Delta^{1/4} a^{j-4}\|_{L^\infty}^q \int_{I\!R} \left(\sum_j 2^j |b_j|^2\right)^{q/2} dx$$

$$\lesssim \|b\|_{W^{1/2,q}}^q \|a\|_{B^0_{\infty,\infty}}^q \lesssim \|a\|_{L^\infty}^q \|b\|_{W^{1/2,q}}^q .$$

• **Estimate of** $\|\Pi_3(\Delta^{1/4}(ab))\|_{L^q}$.

$$\|\sum_j \Delta^{1/4}(a_j b_j)\|_{L^q} = \sup_{\|h\|_{L^{q'}}\le 1}\int_{I\!R}(\Delta^{1/4}h)\sum_j a_j b_j dx$$

$$= \sup_{\|h\|_{L^{q'}}\le 1}\left[\underbrace{\int_{I\!R}\sum_j\sum_{|k-j|\le 4}(\Delta^{1/4}h_k)a_j b_j dx}_{(1)} + \underbrace{\int_{I\!R}\sum_j(\Delta^{1/4}h^{j-4})a_j b_j dx}_{(2)}\right] .$$

We estimate (2):

$$(2) \lesssim \|a\|_{B^0_{\infty,\infty}}\int_{I\!R}\left|\sum_j 2^{-j}|\Delta^{1/4}h^{j-4}|^2\right|^{1/2}\left|\sum_j 2^j|b_j|^2\right|^{1/2} dx$$

$$\lesssim \|a\|_{B^0_{\infty,\infty}}\left(\int_{I\!R}\left|\sum_j 2^{-j}|\Delta^{1/4}h^{j-4}|^2\right|^{q'/2}\right)^{1/q'}\left(\int_{I\!R}\left|\sum_j 2^j|b_j|^2\right|^{q/2}\right)^{1/q}$$

$$\lesssim \|b\|_{W^{1/2,q}}\|a\|_{B^0_{\infty,\infty}} \lesssim \|a\|_{L^\infty}\|b\|_{W^{1/2,q}} .$$

The estimate of (1) is similar.

• **Estimate of** $\|\Pi_3((\Delta^{1/4}a)b)\|_{L^q}$.

$$\|\sum_j \Delta^{1/4}a_j b_j\|_{L^q} = \sup_{\|h\|_{L^{q'}}\le 1}\int_{I\!R h\sum_j}\Delta^{1/4}a_j b_j dx$$

$$= \sup_{\|h\|_{L^{q'}}\le 1}\left[\int_{I\!R}\sum_j\sum_{|k-j|\le 4}h_k(\Delta^{1/4}a_j)b_j dx + \int_{I\!R}\sum_j h^{j-4}(\Delta^{1/4}a_j)b_j dx\right]$$

We estimate the last term $\int_{I\!R}\sum_j h^{j-4}\Delta^{1/4}a_j b_j dx$.

To this purpose we show that $\sum_j \Delta^{1/4}(h^{j-4}b_j) \in \mathcal{H}^1$ and the conclusion follows from the embedding $\dot H^{1/2}(I\!R) \hookrightarrow BMO(I\!R)$. We have

$$\|\sum_j \Delta^{1/4}(h^{j-4}b_j)\|_{\mathcal{H}^1} \simeq \int_{I\!R}\left(\sum_j 2^j|h^{j-4}b_j|^2\right)^{1/2} dx$$

$$\lesssim \int_{I\!R}|h^{j-4}|\left(\sum_j 2^j|b_j|^2\right)^{1/2} dx$$

$$\lesssim \left(\int_{I\!R}\sup_j |h^{j-4}|^{q'}\right)^{1/q'}\left(\int_{I\!R}(\sum_j 2^j|b_j|^2)^{q/2}\right)^{1/q}$$

$$\lesssim \|h\|_{L^{q'}}\|b\|_{W^{1/2,q}}\,.$$

- **Estimate of** $\|\Pi_2(\Delta^{1/4}(ab)-(\Delta^{1/4}a)b)\|_{L^q}$.

$$\|\Pi_2(\Delta^{1/4}(ab)-(\Delta^{1/4}a)b)\|_{L^q}$$

$$\simeq \sup_{\|h\|_{L^{q'}}\le 1}\int_{I\!R}\sum_j h_j(\Delta^{1/4}(a_jb^{j-4})-\Delta a_jb^{j-4})dx$$

$$\simeq \sup_{\|h\|_{L^{q'}}\le 1}\int_{I\!R}\sum_j b^{j-4}(\Delta^{1/4}(h_j)a_j-h_j\Delta a_j)dx$$

$$\sup_{\|h\|_{L^{q'}}\le 1}\int_{I\!R}\sum_j \mathcal{F}[b]^{j-4}(\eta)\int_{I\!R}\mathcal{F}[h]_j(\xi)\mathcal{F}[a]_j(\eta-\xi)(|\xi|^{1/2}-|\eta-\xi|^{1/2})d\xi$$

by arguing as in (102)

$$\lesssim \sup_{\|h\|_{L^{q'}}\le 1}\|a\|_{B^0_{\infty,\infty}}\|b\|_{W^{1/2,q}}\|h\|_{L^{q'}}$$

$$\lesssim \|a\|_{L^\infty}\|b\|_{W^{1/2,q}}\,.$$

This conclues the proof of Lemma B.4. □

In the next Theorem we prove an estimate for the dual of the operator F introduced in (19). It is defined as follows: given $Q\in L^2(I\!R)$, $v\in \dot H^{1/2}(I\!R)$ we have

$$F^*(Q,v)=\Delta^{1/4}(Qv)-\Delta^{1/4}\mathcal{R}(\mathcal{R}(Q)v)\,.$$

Lemma B.5 *Let $Q \in L^2(I\!R)$, $v \in \dot{H}^{1/2}(I\!R)$. Then*

$$\|\Delta^{1/4}(Qv) - \Delta^{1/4}\mathcal{R}(\mathcal{R}(Q)v)\|_{\mathcal{H}^1} \lesssim \|Q\|_{L^2}\|v\|_{\dot{H}^{1/2}}\,. \tag{104}$$

Proof of Lemma B.5 .
Estimate of $\Pi_2(\Delta^{1/4}(Q,v))$.

$$\begin{aligned}
\|\Pi_2(\Delta^{1/4}(Q,v))\|_{\mathcal{H}^1} &= \int_{I\!R}\left(\sum_{i=-\infty}^{+\infty} 2^i (Q^{i-4})^2 (v_i)^2\right)^{1/2} dx \qquad (105)\\
&\lesssim \int_{I\!R} |M(Q)| \left(\sum_{i=-\infty}^{+\infty} 2^i (v_i)^2\right) dx\\
&\lesssim \|Q\|_{L^2}\|v\|_{\dot{H}^{1/2}}\,.
\end{aligned}$$

The estimate of $\Pi_2(\Delta^{1/4}\mathcal{R}((\mathcal{R}Q)v))$ is analogous to (105) .
Estimate of $\Pi_3(\Delta^{1/4}(Q,v))$.

$$\begin{aligned}
\|\Pi_3(Q,v)\|_{B^0_{1,1}} &\simeq \sup_{\|h\|_{B^0_{\infty,\infty}}\le 1} \int_R (Q_i v_i)\left[\Delta^{1/4}h^{i-6} + \sum_{t=h-5}^{i+6} \Delta^{1/4} h_t\right] dx \qquad (106)\\
&\lesssim \sup_{\|h\|_{B^0_{\infty,\infty}}\le 1} \|h\|_{B^0_{\infty,\infty}} \int_{I\!R} 2^{i/2}|Q_i v_i| dx\\
&\lesssim \left(\int_{I\!R}\sum_i 2^i v_i^2 dx\right)^{1/2}\left(\int_{I\!R}\sum_i Q_i^2 dx\right)^{1/2} = \|Q\|_{L^2}\|v\|_{\dot{H}^{1/2}}\,.
\end{aligned}$$

The estimate of $\Pi_3(\Delta^{1/4}\mathcal{R}((\mathcal{R}Q)v))$ is analogous to (106) .
Estimate of $\|\Pi_1(\Delta^{1/4}(Qv) - \Delta^{1/4}\mathcal{R}((\mathcal{R}Q)v))\|_{B^0_{1,1}(I\!R)}$.
We show that

$$\|\Pi_1(\Delta^{1/4}(Qv) - \Delta^{1/4}\mathcal{R}((\mathcal{R}Q)v))\|_{B^0_{1,1}(I\!R)} = 0\,.$$

We have

$$\|\Pi_1(\Delta^{1/4}(Qv)-\Delta^{1/4}\mathcal{R}(\mathcal{R}Q)v))\|_{B^0_{1,1}(I\!R)}$$

$$\simeq \sup_{\|h\|_{B^0_{\infty,\infty}}\le 1}\int_{I\!R}\sum_j[\Delta^{1/4}(Q_j v^{j-4})-\Delta^{1/4}\mathcal{R}((\mathcal{R}Q_j)v^{j-4})]h_j dx$$

$$= \sup_{\|h\|_{B^0_{\infty,\infty}}\le 1}\int_{I\!R}\sum_j v^{j-4}[Q_j\Delta^{1/4}h_j-(\mathcal{R}Q_j)\mathcal{R}\Delta^{1/4}h_j]dx$$

$$= \sup_{\|h\|_{B^0_{\infty,\infty}}\le 1}\int_{I\!R}\sum_j \mathcal{F}[v^{j-4}]\mathcal{F}[Q_j\Delta^{1/4}h_j-(\mathcal{R}Q_j)\mathcal{R}\Delta^{1/4}h_j]d\xi$$

$$= \sup_{\|h\|_{B^0_{\infty,\infty}}\le 1}\int_{I\!R}\sum_j \mathcal{F}[v^{j-4}]\int_{I\!R}\mathcal{F}[Q_j]\mathcal{F}[\Delta^{1/4}h_j](1+\frac{\eta}{|\eta|}\frac{\xi-\eta}{|\xi-\eta|})d\eta=0$$

This concludes the proof of Lemma B.5. □

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