Remarks on Neumann boundary problems involving Jacobians

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Abstract

In this note we explore the validity of Wente-type estimates for Neumann boundary problems involving Jacobians. We show in particular that such estimates do not in general hold under the same hypotheses on the data for Dirichlet boundary problems.

Key words. Neumann boundary conditions, Jacobians, integrability by compensation, conformal invariant problems

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1 Introduction

Integrability by compensation has played a central role in the last decades in the geometric analysis of conformally invariant problems. At the center of this theory there is Wente's discovery [10] that the distribution:

$$\varphi(x) = \log|x| \star [\partial_{x_1} b \partial_{x_2} a - \partial_{x_2} b \partial_{x_1} a]$$

with $\nabla a, \nabla b \in L^2(\mathbb{R}^2)$ is in $(L^\infty \cap W^{1,2})(\mathbb{R}^2)$ and the following estimate holds true:

$$\|\varphi\|_{L^\infty(\mathbb{R}^2)} \leq C \|\nabla a\|_{L^2(\mathbb{R}^2)} \|\nabla b\|_{L^2(\mathbb{R}^2)}.$$

It was observed by Brezis and Coron in [1] that a similar estimate holds also if we consider the following Dirichlet problem:

Theorem 1.1. *([10]) Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : \; x_1^2 + x_2^2 < 1\}$ and let $a, b \in H^1(\Omega)$. Then the solution $u \in W_0^{1,1}(\Omega)$ to the problem:*

$$\begin{cases} -\Delta u = \partial_{x_1} b \, \partial_{x_2} a - \partial_{x_2} b \, \partial_{x_1} a & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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is a continuous function in $\overline{\Omega}$ *and its gradient belongs to* $L^2(\Omega)$. *Moreover there exists a constant* $c_0 = c_0(\Omega)$ *such that*

$$\|u\|_{L^\infty(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \le c_0 \|\nabla a\|_{L^2(\Omega)} \|\nabla b\|_{L^2(\Omega)}. \tag{2}$$

Extensive investigation on the problem (1) and various generalizations has been conducted, remarkably in [2].

The goal of the present work is to explore to which extent an inequality like (2) holds or not if we replace the Dirichlet boundary condition with a Neumann boundary condition.

Our first main result gives a negative answer for general a and b. We consider for simplicity the unit disk $D^2 := \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1^2 + x_2^2 < 1\}$ and we denote by $\nu = \nu(x_1, x_2) = (x_1, x_2)$ the unit outward normal vector to ∂D^2 at $(x_1, x_2) \in \partial D^2$. Then:

Theorem 1.2. *There are* $a, b \in (L^\infty \cap H^1)(D^2)$ *with* $\int_{D^2} \nabla^\perp b \cdot \nabla a \, dy = 0$ *such that every solution* φ *of:*

$$\begin{cases} -\Delta\varphi = \nabla^\perp b \cdot \nabla a & \text{in } D^2, \\ \partial_\nu \varphi = 0 & \text{on } \partial D^2 \end{cases} \tag{3}$$

is neither in $H^1(D^2)$ *nor in* $L^\infty(D^2)$ *and in particular (2) cannot hold.*

We would like to mention that we came at the counter-example to Wente-type estimates for the solutions to (3) in the following way. First of all by the conformal invariance of the problem (3) we can transform it into an analogous problem in $\mathbb{R}^2_+ := \{(y_1, y_2) \in \mathbb{R}^2 : \ \ y_2 > 0\}$:

$$\begin{cases} -\Delta w = \nabla^\perp b \cdot \nabla a & \text{in } \mathbb{R}^2_+ \\ \partial_\nu w = 0 & \text{in } \partial\mathbb{R}^2_+. \end{cases} \tag{4}$$

If we extend w, a, b by even reflection with respect to $\partial\mathbb{R}^2_+ = \{y_2 = 0\}$ and if we denote by $\tilde{w}, \tilde{a}, \tilde{b}$ the respective extensions, we realize that

$$-\Delta\tilde{w} = \nabla^\perp\tilde{b}\nabla\tilde{a} - 2(\nabla^\perp\tilde{b}\nabla\tilde{a})1\!\!1_{\{y_2<0\}} \quad \text{in } \mathcal{D}'(\mathbb{R}^2). \tag{5}$$

For every $\varphi \in C_0^\infty(\mathbb{R}^2)$ we have

$$\begin{aligned} \iint_{\mathbb{R}^2} (\nabla^\perp\tilde{b}\nabla\tilde{a})1\!\!1_{\{y_2<0\}}\varphi(y)dy &= \int_{\mathbb{R}} [\mathrm{Trace}(\tilde{a})\partial_\tau \mathrm{Trace}(\tilde{b})]\varphi(y_1, 0)dy_1 \\ &- \iint_{\mathbb{R}^2} (\nabla^\perp\tilde{b}\nabla\varphi)\tilde{a}1\!\!1_{\{y_2<0\}}dy \end{aligned}$$

where $\tau = \tau(y_1, 0) = (1, 0)$ for every $(y_1, 0) \in \partial\mathbb{R}^2_+$ and $\partial_\tau \mathrm{Trace}(\tilde{b})(y_1, 0)$ denotes the tangential derivative to $\partial\mathbb{R}^2_+$ at $(y_1, 0)$. It is straightforward that for arbitrary $\tilde{a}, \tilde{b} \in \dot{H}^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ the boundary term $[\mathrm{Trace}(\tilde{a})\partial_\tau \mathrm{Trace}(\tilde{b})]$ is not in $H^{-1/2}(\mathbb{R})$ and therefore one can guess that an estimate like (2) cannot in general hold.

A Wente-type estimate holds for (3) if $\mathrm{Trace}(a) = 0$ or $\mathrm{Trace}(b) = 0$. We call this the case *vanishing Jacobian at the boundary.* If for instance $\mathrm{Trace}(a) = 0$, we can extend w, b by even reflection and a by odd reflection with respect to $\partial\mathbb{R}^2_+$. In this case we obtain

$$-\Delta\tilde{w} = \nabla^\perp\tilde{b}\nabla\tilde{a} \quad \text{in } \mathcal{D}'(\mathbb{R}^2), \tag{6}$$

namely the right hand side is a Jacobian in $\mathbb{R}^2$. Precisely, the following result holds.

Theorem 1.3. *Let Ω be a smooth bounded domain of $\mathbb{R}^2$. Let $a \in W_0^{1,2}(\Omega)$ and $b \in W^{1,2}(\Omega)$ and let u be a solution of (3). Then $\nabla^2 u \in L^1(\Omega)$ and one has:*

$$\|\nabla^2 u\|_{L^1(\Omega)} \leq c_\Omega \ \|\nabla a\|_{L^2(\Omega)} \ \|\nabla b\|_{L^2(\Omega)}. \tag{7}$$

From estimate (7), by means of improved Sobolev embeddings (see e.g. [5]) we deduce that the estimate (1) holds. Theorem 1.3 has been used by Rivière in [8] and the proof will be given in [3]. We also refer to Theorem A.4 in [9].

In some applications such as for instance in the analysis of the *Poisson problem for elastic plates* ([3]) the following Neumann boundary problem appears in a natural way:

$$\begin{cases} -\Delta w = \nabla^\perp b \cdot \nabla a & \text{in } D^2, \\ \partial_\nu w = -\ \partial_\tau b \cdot a & \text{on } \partial D^2 \end{cases} \tag{8}$$

where for every $(x_1, x_2) \in \partial D^2$, $\tau(x_1, x_2) = (-x_2, x_1)$ is the unit tangent vector to ∂D^2 at the point $(x_1, x_2) \in \partial D^2$. We observe that $H^1(D^2)$-solutions of the problem (8) are critical points of the following Lagrangian:

$$\mathcal{L}(u; a, b) = \frac{1}{2} \iint_{D^2} |\nabla u + (\nabla^\perp b) a|^2 dy_1 dy_2 \tag{9}$$

We will refer to the problem (8) as the case of *compatible Neumann boundary conditions.* Also in the case of (8) the assumption $a, b \in (L^\infty \cap H^1)(D^2)$ is not enough to guarantee the boundedness of the solution in D^2.

Theorem 1.4. *There are $a, b \in L^\infty(D^2) \cap H^1(D^2)$ such that every solution φ of:*

$$\begin{cases} -\Delta \varphi = \nabla^\perp b \cdot \nabla a & \text{in } D^2 \\ \partial_\nu \varphi = -\ \partial_\tau b \cdot a & \text{on } \partial D^2 \end{cases} \tag{10}$$

is not in $L^\infty(D^2)$ and in particular the estimate (2) cannot hold.

The boundedness of the solution is however obtained if we assume a bit more on the data a, b. More precisely we get the following result.

Theorem 1.5 ($L^{2,1}$-case)**.** *Let $\nabla a, \nabla b \in L^{(2,1)}(D^2)$,*[1] *with $\bar{a} = ⨍_{D^2} a(y)dy = 0$ and let $w \in W^{1,1}(D^2)$ be the solution with zero mean value to (8) . Then $\nabla w \in L^{(2,1)}(D^2)$ with:*

$$\|\nabla w\|_{L^{(2,1)}(D^2)} \quad \leq \quad C\|\nabla a\|_{L^{(2,1)}}\|\nabla b\|_{L^{(2,1)}}. \tag{11}$$

In particular:

$$\|w\|_{L^\infty(D^2)} \leq C\|\nabla a\|_{L^{(2,1)}}|\nabla b\|_{L^{(2,1)}}. \tag{12}$$

[1]We denote by $L^{(2,1)}(\mathbb{R}^n)$ the *Lorentz space of indices* 2 *and* 1 as the space of measurable functions satisfying:

$$\int_0^{+\infty} |\{x \in \mathbb{R}^n \ : |f(x)| \geq \lambda\}|^{1/2} d\lambda < +\infty \ .$$

See [4] for properties on Lorentz spaces.

We observe that the assumption $\nabla b \in L^{(2,1)}(D^2)$ is in particular satisfied if $b \in W^{2,1}(D^2)$, see e.g [5]. We remark that the assumptions $\nabla a \in L^{(2,1)}(D^2)$ and $\bar{a} = \fint_{D^2} a(y)dy = 0$ imply $a \in L^\infty(D^2)$ with $\|a\|_{L^\infty} \le C\|\nabla a\|_{L^{(2,1)}}$.

The paper is organized as follows. In Section 2 we prove Theorem 1.5 and in Section 3 we prove Theorems 1.2 and 1.4.

2 Proof of Theorem 1.5.

Step 1. We start by observing that we can formulate problem (8) as follows:

$$\begin{cases} \operatorname{div}[\nabla w + (\nabla^\perp b)a] = 0 & \text{in } D^2, \\ \\ \partial_\nu w = -\partial_\tau b \cdot a & \text{in } \partial D^2. \end{cases} \tag{13}$$

Therefore there exists $C \in W^{1,2}(D^2)$ such that:

$$\nabla^\perp C = \nabla w + (\nabla^\perp b)a.$$

Therefore C solves:

$$\begin{cases} -\Delta C = -\operatorname{div}(\nabla b \cdot a) & \text{in } D^2, \\ \\ \partial_\tau C = 0 & \text{in } \partial D^2. \end{cases} \tag{14}$$

Since C is determined up to a constant, we can reduce to study the following Dirichlet problem:

$$\begin{cases} -\Delta C = -\operatorname{div}(\nabla b \cdot a) & \text{in } D^2, \\ \\ C = 0 & \text{in } \partial D^2. \end{cases} \tag{15}$$

Step 2. In this step and in the following we use basic facts about the theory of Calderón-Zygmund operators and interpolation theory, for which we refer to [5, 7]. We first assume $b \in W^{1,p}(D^2)$ for a fixed but arbitrary $1 < p < \infty$. Let us set $f = -\nabla b \cdot a \in L^p(D^2)$, we have:

$$\|f\|_{L^p(D^2)} \le K_p \|\nabla b\|_{L^p(D^2)} \|a\|_{L^\infty(D^2)}.$$

We denote by $\tilde{f} = f\chi_{D^2}$ its extension by 0 to $\mathbb{R}^2$. We write $C = C_1 + C_2$ where:

$$C_1(x) = \left(-\frac{1}{2\pi}\log|\cdot| * \operatorname{div}\tilde{f}\right)(x), \quad x \in \mathbb{R}^2,$$

and $C_2 = C - C_1$ which is the solution to:

$$\begin{cases} -\Delta C_2 = 0 & \text{in } D^2, \\ \\ C_2 = -C_1 & \text{in } \partial D^2. \end{cases} \tag{16}$$

We have:

$$\nabla C_1(x_1,x_2) \quad = \quad \frac{1}{2\pi}\int_{\mathbb{R}^2} \tilde{f}(y)\left[\frac{y-x}{|y-x|^3}\right] \mathrm{d}y.$$

The function

$$\mathcal{K}(x,y) = \frac{y-x}{|y-x|^3}$$

is a C-Z operator. Since $\tilde{f} \in L^p(\mathbb{R})$ for every $p>1$ we have

$$T[\tilde{f}](x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \tilde{f}(y)\left[\frac{y-x}{|y-x|^3}\right] dy \in L^p(\mathbb{R}^2)$$

and

$$\|T[\tilde{f}]\|_{L^p} \leq K_p \|\tilde{f}\|_{L^p}.$$

(see e.g. [5, 7]).

As far as C_2 is concerned, since $C_1 \in W^{1-1/p,p}(\partial D^2)$, then $C_2 \in W^{1,p}(D^2)$ and:

$$\|\nabla C_2\|_{L^p(D^2)} \leq K_p \|C_1\|_{W^{1-1/p,p}} \leq K_p \|f\|_{L^p(D^2)}.$$

In particular we get:

$$\|\nabla C\|_{L^p(D^2)} \leq K_p \|f\|_{L^p(D^2)},$$

and therefore:

$$\|\nabla w\|_{L^p(D^2)} \leq K_p \|f\|_{L^p(D^2)} \leq K_p \|\nabla b\|_{L^p(D^2)} \|a\|_{L^\infty(D^2)}.$$

We remind that if p belongs to a compact interval $I \subset (0,\infty)$, the constant K_p is uniformly bounded.

Now we define

$$\mathfrak{G}_p(D^2) := \{X \in L^p(D^2,R^2): \quad \mathrm{curl}(X) = 0\ \}.$$

[2] Note that since D^2 is simply connected, Poincaré's lemma ensures that every $X \in \mathfrak{G}_p(D^2)$ is of the form $X = \nabla f$ for some $f \in W^{1,p}(D^2)$. By step 1, if we fix $a \in L^\infty(D^2)$, the linear operator $\tilde{T}\colon \mathfrak{G}_p(D^2) \to L^p(D^2)$, which maps $X = \nabla b$ to ∇w, where w is the zero-mean solution to (8), is continuous for each $p > 1$.

Step 3. If $a \in L^\infty(D^2)$ and $\nabla b \in L^{(2,1)}(D^2)$ then $f \in L^{(2,1)}(D^2)$ with

$$\|f\|_{L^{(2,1)}(D^2)} \quad \leq \quad K \|\nabla b\|_{L^{(2,1)}(D^2)} \|a\|_{L^\infty(D^2)}.$$

By interpolation and the previous step, we get that $\nabla w \in L^{(2,1)}(D^2)$ with:

$$\begin{aligned} \|\nabla w\|_{L^{(2,1)}(D^2)} &\leq K\|f\|_{L^{(2,1)}(D^2)} \leq K\|a\|_{L^\infty(D^2)}\|\nabla b\|_{L^{(2,1)}(D^2)} \\ &\leq K\|\nabla a\|_{L^{(2,1)}(D^2)}\|\nabla b\|_{L^{(2,1)}(D^2)} \end{aligned} \tag{17}$$

for some $K > 0$.

We can conclude. □

[2]For $X \in L^p(D^2,\mathbb{R}^2)$, $\mathrm{curl}(X) = -\frac{X^1}{\partial x_2} + \frac{X^2}{\partial x_1}$.

3 Proof of Theorems 1.2 and 1.4

In this Section we provides counter-examples to Wente-type estimates for solutions to (8) and (3) even in the case $a, b \in (H^1 \cap L^\infty)(D^2)$.

3.1 A representation formula with estimates

Because of the conformal invariance of the problem (8) we can reduce to consider the analogous problem in $\mathbb{R}^2_+$:

$$\begin{cases} -\Delta w = \nabla^\perp b \cdot \nabla a & \text{in } \mathbb{R}^2_+ \\ \partial_\nu w = -\ \partial_\tau b \cdot a & \text{in } \partial\mathbb{R}^2_+. \end{cases} \tag{18}$$

In this case $\nu = (0, -1)$ and $\tau = (1, 0)$. Therefore $\partial_\nu w = -\partial_{y_2} w$ and $\partial_\tau w = \partial_{y_1} w$.

The Green function associated to the Neumann problem in the half-plane $\mathcal{G} : \mathbb{R}^2_+ \times \mathbb{R}^2_+ \to \mathbb{R}$ is the solution, for every $x \in \mathbb{R}^2_+$ of the problem:

$$\begin{cases} -\Delta_y \mathcal{G}(x, \cdot) = \delta_x & \text{in } \mathbb{R}^2_+, \\ \quad \partial_{\nu_y} \mathcal{G}(x, \cdot) = 0 & \text{in } \partial\mathbb{R}^2_+, \end{cases}$$

given by:

$$\mathcal{G}(x, y) = -\frac{1}{2\pi} \left\{ \log(|x - y|) + \log(|y - \tilde{x}|) \right\},$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$, $\tilde{x} = (x_1, -x_2)$.

We are going to consider the solution w to (18) obtained through the representation formula:

$$\begin{aligned} w(x) &= \int\int_{R^2_+} \mathcal{G}(x, y)(-\Delta w)\, dy + \int_{\partial\mathbb{R}^2_+} \mathcal{G}(x, y) \partial_\nu w\, d\sigma(y) \\ &= -\frac{1}{2\pi} \int\int_{R^2_+} \{\log(|x - y|) + \log(|y - \tilde{x}|)\} \nabla^\perp b \cdot \nabla a\, dy \\ &\quad -\frac{1}{\pi} \int_{-\infty}^{+\infty} \log((y_1 - x_1)^2 + x_2^2)^{1/2} \partial_{y_1} b \cdot a\, dy_1 \end{aligned} \tag{19}$$

and deduce a representation formula for *its trace* at the boundary $\partial\mathbb{R}^2_+$.

Step 1: We assume for the moment that a, b are in $C^\infty_c(\mathbb{R}^2)$. We integrate by parts (19) and get:

$$
\begin{aligned}
w(x) &= -\frac{1}{2\pi}\int\int_{\mathbb{R}^2_+} \mathrm{div}\left(\{\log(|x-y|)+\log(|y-\tilde{x}|)\}\nabla^\perp b\cdot a\right)dy\\
&+ \frac{1}{2\pi}\int\int_{\mathbb{R}^2_+}\nabla(\{\log(|x-y|)+\log(|y-\tilde{x}|)\})(\nabla^\perp b\cdot a)dy\\
&- \frac{1}{\pi}\int_{-\infty}^{+\infty}\log((y_1-x_1)^2+x_2^2)^{1/2}[\partial_{y_1}b\cdot a]dy_1\\
&= \frac{1}{2\pi}\int_{\partial\mathbb{R}^2_+}\{\log(|x-y|)+\log(|y-\tilde{x}|)\}\,\partial_{y_1}b\cdot a\,d\sigma(y) \qquad (20)\\
&+ \frac{1}{2\pi}\int\int_{\mathbb{R}^2_+}\nabla(\{\log(|x-y|)+\log(|y-\tilde{x}|)\})\nabla^\perp b\cdot a)dy\\
&- \frac{1}{\pi}\int_{-\infty}^{+\infty}\log((y_1-x_1)^2+x_2^2)^{1/2}\partial_{y_1}b\cdot a]dy_1\\
&= \frac{1}{2\pi}\int\int_{\mathbb{R}^2_+}\nabla(\{\log(|x-y|)+\log(|y-\tilde{x}|)\})\nabla^\perp b\cdot a dy.
\end{aligned}
$$

If $x=(x_1,0)\in\partial\mathbb{R}^2_+$, then:

$$
w(x_1,0) = \frac{1}{\pi}\int\int_{\mathbb{R}^2_+}\nabla(\left\{\log((x_1-y_1)^2+y_2^2)^{1/2}\right\})\cdot[\nabla^\perp b\cdot a]dy. \qquad (21)
$$

By translation invariance we evaluate (21) at $(0,0)$ and use polar coordinates. For every $r>0$ we set

$$
\bar{a}_r = \frac{a(r,\pi)+a(r,0)}{2} = \frac{a(x_1,0)+a(-x_1,0)}{2}.
$$

We have

$$
\begin{aligned}
w(0,0) &= \frac{1}{\pi}\int\int_{R^2_+}\nabla(\log|y|)\cdot[\nabla^\perp b\cdot a]dy\\
&= -\frac{1}{\pi}\int_0^\infty\int_0^\pi\partial_\theta b\left(\frac{1}{r}(a-\bar{a}_r)\right)d\theta dr-\frac{1}{\pi}\int_0^\infty\int_0^\pi\frac{1}{r}\bar{a}_r\partial_\theta b d\theta dr \qquad (22)
\end{aligned}
$$

We analyse the last two terms of (22).

Estimate of $\frac{1}{\pi}\int_0^\infty\int_0^\pi\partial_\theta b\left(\frac{1}{r}(a-\bar{a}_r)\right)d\theta dr$.
Note first of all that in Cartesian coordinates it reads as:

$$
\begin{aligned}
\frac{1}{\pi}\int_0^\infty\int_0^\pi\partial_\theta b\left(\frac{1}{r}(a-\bar{a}_r)\right)d\theta dr &= \frac{1}{\pi}\int\int_{R^2_+}\nabla\left(\log((y_1-x_1)^2+y_2^2)^{1/2}\right)\\
&\quad\cdot[\nabla^\perp b\cdot(a-\frac{a(y_1,0)+a(-y_1,0)}{2})]dy.
\end{aligned}
$$

Moreover we have the estimate:

$$\begin{aligned}\frac{1}{\pi}\int_0^{+\infty}\int_0^{\pi}\partial_\theta b\left(\frac{1}{r}(a-\bar a_r)\right)dr\,d\theta &\le \frac{1}{\pi}\left(\int_0^{+\infty}\int_0^{\pi}\frac{1}{r^2}|\partial_\theta a|^2 r dr\,d\theta\right)^{1/2}\left(\int_0^{+\infty}\int_0^{\pi}\frac{1}{r^2}|\partial_\theta b|^2 r dr\,d\theta\right)^{1/2}\\ &\le C\|\nabla a\|_{L^2(\mathbb{R}^2_+)}\|\nabla b\|_{L^2(\mathbb{R}^2_+)}.\end{aligned}$$

Estimate of $\frac{1}{\pi}\int_0^\infty\int_0^\pi \partial_\theta b\left(\frac{1}{r}\bar a_r\right)d\theta dr$.
The expression in Cartesian coordinates of the integral is:

$$\begin{aligned}\frac{1}{\pi}\int_0^{\infty}\int_0^{\pi}\partial_\theta b\left(\frac{1}{r}\bar a_r\right)d\theta dr &= \frac{1}{\pi}\int_0^\infty b(r,\pi)-b(r,0))\left(\frac{1}{r}\bar a_r\right)dr\\ &= \frac{1}{\pi}\int_0^\infty \frac{1}{y_1}\left((b(-y_1,0)-b(y_1,0))\frac{a(y_1,0)+a(-y_1,0)}{2}\right)dy_1.\end{aligned}$$

We deduce that the desired representation formula in $(0,0)$ is:

$$\begin{aligned}w(0,0) &= \frac{1}{\pi}\int\int_{\mathbb{R}^2_+}\nabla(\log(|y|))\cdot\left[\nabla^\perp b\cdot\left(a-\frac{a(y_1,0)+a(-y_1,0)}{2}\right)\right]dy\\ &+ \frac{1}{\pi}\int_{-\infty}^\infty\frac{1}{y_1}\left((b(-y_1,0)-b(y_1,0)\frac{(a(y_1,0)+a(-y_1,0))}{2}\right)dy_1 \qquad (23)\end{aligned}$$

By translation invariance, we then deduce the following representation formula for a generic point $(x_1,0)\in\partial\mathbb{R}^2_+$:

$$w(x_1,0)=\frac{1}{\pi}\int\int_{\mathbb{R}^2_+}\nabla(\left\{\log((y_1-x_1)^2+y_2^2)^{1/2})\right\})\cdot[\nabla^\perp b\cdot(a-\frac{a(x_1+y_1,0)+a(x_1-y_1,0)}{2})]dy$$
$$+\quad \frac{1}{\pi}\int_{-\infty}^\infty\frac{1}{y_1}\left[(b(x_1-y_1,0)-b(x_1+y_1,0))\frac{a(x_1+y_1,0)+a(x_1-y_1,0)}{2}\right]dy_1. \qquad (24)$$

Step 2. If $a,b\in(H^1\cap L^\infty)(\mathbb{R}^2_+)$ then we get the previous formula by approximation arguments.

3.2 A Counter-Example to L^∞-Estimates

In this Section we will provide a counter-example to Wente type estimates for the problem (8). Precisely we will show that even in the case $a,b\in(H^1\cap L^\infty)(\mathbb{R}^2_+)$ the solution given by (19) needs not to be bounded.

Let $\psi\colon\mathbb{R}^2\to[0,+\infty)$ be a radial smooth function such that:

$$\psi(x,y)=\begin{cases}1 & (x,y)\in B(0,1/4),\\ 0 & (x,y)\in B^c(0,1/2),\end{cases} \qquad (25)$$

Let $\chi\colon \mathbb{R}\to\mathbb{R}$ be a Lipschitz continuous function such that:

$$\chi(x) = \begin{cases} 1 & \text{if } x \geq 1, \\ 0 & \text{if } x \leq -1. \end{cases}$$

Take for instance

$$\chi(x) = \begin{cases} \frac{2}{\pi}[\arctan(x) + \frac{\pi}{4}] & \text{if } -1 \leq x \leq 1, \\ 1 & \text{if } x \geq 1, \\ 0 & \text{if } x \leq -1, \end{cases}$$

We observe that $\chi(\frac{x}{\varepsilon})$ converges as $\varepsilon \to 0$ to the Heaviside function:

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Proposition 3.1. *Let $\beta \in \mathbb{R}$ and consider the function:*

$$f(x) = (-\log|x|)^{-\beta}\psi(x). \tag{26}$$

Then

- *i) if $\beta \geq 0$, $f(x) \in (H^{1/2} \cap L^\infty)(\mathbb{R})$;*
- *ii) if $1/2 < \beta$, $f(x)H(x) \in (H^{1/2} \cap L^\infty)(\mathbb{R})$.*

Proof of Proposition 3.1. We prove only *ii*). The proof of *i*) is similar and even simpler.
It is clear that $f(x)H(x) \in L^\infty(\mathbb{R})$.
$f(x)H(x)$ can be seen as the trace of the following function:

$$\tilde{f}(x,y) = (-1/2\log(x^2+y^2))^{-\beta}\psi(\sqrt{x^2+y^2}))\chi\left(\frac{x}{y}\right)$$

Claim: $\tilde{f}(x,y) \in H^1(\mathbb{R}^2_+)$, (this implies that $f(x)H(x) \in H^{1/2}(\mathbb{R})$).
Proof of the Claim. We estimate the L^2 norm of its partial derivatives.

Derivatives of $\tilde{f}$:

$$\begin{aligned} \partial_y \tilde{f}(x,y) &= \chi'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)(-1/2\log(x^2+y^2))^{-\beta}\psi(\sqrt{x^2+y^2})) \\ &+ \chi\left(\frac{x}{y}\right)\partial_y\left((-1/2\log(x^2+y^2))^{-\beta}\psi(\sqrt{x^2+y^2})\right) \\ \partial_x \tilde{f}(x,y) &= \chi'\left(\frac{x}{y}\right)\frac{1}{y}(-1/2\log(x^2+y^2))^{-\beta}\psi(\sqrt{x^2+y^2})) \\ &+ \chi\left(\frac{x}{y}\right)\partial_x\left((-1/2\log(x^2+y^2))^{-\beta}\psi(\sqrt{x^2+y^2})\right). \end{aligned}$$

L^2-estimate of $\partial_y \tilde{f}(x,y)$:

$$\begin{aligned}
&\int\int_{\mathbb{R}^2_+} |\partial_y \tilde{f}(x,y)(x,y)|^2 dxdy \\
&\lessapprox \underbrace{\int_{-1/2}^{1/2}\int_{|x|}^{1/2} (\chi'\left(\frac{x}{y}\right)^2 \left(\frac{x^2}{y^4}\right)(-1/2\log(x^2+y^2))^{-(2\beta)}\psi^2(\sqrt{x^2+y^2}))\,dxdy}_{(1)} \\
&+\underbrace{\int\int_{(x^2+y^2)^{1/2}<1/2} |\partial_y\left((-1/2\log(x^2+y^2))^{-\beta}\psi(\sqrt{x^2+y^2})\right)|^2\,dxdy}_{(2)}.
\end{aligned}$$

Les us prove that (1) and (2) are convergent integrals.

- We estimate (2).

$$\begin{aligned}
(2) \quad &= \int\int_{(x^2+y^2)^{1/2}<1/2} \Bigg| \psi'(\sqrt{x^2+y^2})\frac{y}{(x^2+y^2)^{1/2}}(-1/2\log(x^2+y^2))^{-\beta} \\
&\quad + \psi(\sqrt{x^2+y^2})\beta(-1/2\log(x^2+y^2))^{-(\beta+1)}\frac{2y}{x^2+y^2}\Bigg|^2 dxdy \\
&\leq C\int\int_{(x^2+y^2)^{1/2}<1/2}(-1/2\log(x^2+y^2))^{-2\beta} \\
&\quad + (-1/2\log(x^2+y^2))^{-2(\beta+1)}\frac{4y^2}{(x^2+y^2)^2}dxdy < +\infty.
\end{aligned}$$

- We estimate (1), by recalling that $\chi'\left(\frac{x}{y}\right) \neq 0$ iff $\frac{|x|}{|y|} \leq 1$ and that

$$\begin{aligned}
&\partial_y(-\frac{1}{y}(-1/2\log(x^2+y^2))^{-\beta}) \\
&= \frac{1}{y^2}(-1/2\log(x^2+y^2))^{-\beta}) - \frac{1}{y}\partial_y((-1/2\log(x^2+y^2))^{-\beta}) \\
= \quad & \frac{1}{y^2}(-1/2\log(x^2+y^2))^{-\beta}) - \beta\frac{1}{y}\frac{y}{x^2+y^2}(-1/2\log(x^2+y^2))^{-(1+\beta)}).
\end{aligned}$$

We observe that

$$\beta\frac{1}{y}\frac{y}{x^2+y^2}(-1/2\log(x^2+y^2))^{-(1+\beta)}) = o(\frac{1}{y^2}(-1/2\log(x^2+y^2))^{-\beta})) \quad \text{as } (x,y)\to(0,0).$$

Therefore if $(x,y) \in B(0,1/2)$ we have

$$\frac{1}{y^2}(-1/2\log(x^2+y^2))^{-\beta}) \leq C\partial_y(-\frac{1}{y}(-1/2\log(x^2+y^2))^{-\beta}).$$

Hence:

$$\begin{aligned}
(1) &= \int_{-1/2}^{1/2}\int_{|x|}^{1/2} (\chi'\left(\frac{x}{y}\right)^2 \left(\frac{x^2}{y^4}\right) (-1/2\log(x^2+y^2))^{-2\beta}\psi^2(\sqrt{x^2+y^2})) \\
&\lessapprox \int_{-1/2}^{1/2}\int_{|x|}^{1/2} (\chi'\left(\frac{x}{y}\right)^2 \left(\frac{x^2}{y^4}\right) \left((-1/2\log(x^2+y^2))^{-\beta}\right) dxdy \\
&\leq \int_{-1/2}^{1/2}\int_{|x|}^{1/2} \partial_y(-\frac{1}{y}(-1/2\log(x^2+y^2))^{-\beta})\, dydx \\
&\leq \int_{-1/2}^{1/2} \left[\frac{1}{|x|}(-1/2\log(x^2+x^2))^{-\beta}\right] - 2(-1/2\log(x^2+\frac{1}{4}))^{-\beta} dx < +\infty.
\end{aligned}$$

Observe that since $\beta > 1/2$, the last integral is convergent.

L^2-estimate of $\partial_x\tilde{f}(x,y)$.

$$\begin{aligned}
&\int\int_{\mathbb{R}^2_+} |\partial_x\tilde{f}(x,y)|^2 dxdy \\
&\lessapprox \underbrace{\int_{-1/2}^{1/2}\int_{|x|}^{1/2} (\chi'\left(\frac{x}{y}\right)^2 \left(\frac{1}{y^2}\right) (-1/2\log(x^2+y^2))^{-\beta}\psi^2(\sqrt{x^2+y^2}))\, dxdy}_{(3)} \\
&+ \underbrace{\int\int_{(x^2+y^2)^{1/2}<1/2} |\partial_x\left((-1/2\log(x^2+y^2))^{-\beta}\psi(\sqrt{x^2+y^2})\right)|^2\, dxdy}_{(4)}.
\end{aligned}$$

The estimate of (3) is similar to (1) and the estimate of (4) is similar to the estimate of (2).

We can conclude the proof of Proposition 3.1. □

Estimate of $w(0,0)$.

Let us come back to the situation of subsection 3.1 and consider:

$$a(x) = \psi(x) \quad \text{and} \quad b(x) = (-\log|x|)^{-\beta}\psi(x)H(x).$$

where $1/2 < \beta < 1$ and ψ is defined in (25).

Since $b \equiv 0$ in $y_1 \leq 0$ and a is symmetric we have, from (23)

$$\begin{aligned}
w(0,0) &= \frac{1}{\pi}\int\int_{R^2_+} \nabla(\log(|y|))\cdot\left[\nabla^\perp b\cdot\left(a - \frac{a(y_1,0)+a(-y_1,0)}{2}\right)\right] dy \\
&- \frac{1}{\pi}\int_0^{+\infty}\frac{1}{y_1}\left(b(y_1,0)a(y_1,0)\right) dy_1 \qquad (27)
\end{aligned}$$

We already know that the first integral is finite. As for the second one, since we have chosen $\beta < 1$ we see that:

$$\frac{1}{\pi}\int_0^{+\infty}\frac{1}{y_1}\left(b(y_1,0)a(y_1,0)\right) dy_1 = \frac{1}{\pi}\int_0^{+\infty}\frac{1}{y_1}(-\log|y_1|)^{-\beta}\psi^2(y_1)dy_1$$

is divergent. Hence $w(0,0)$ is not finite.

One can also show that w is continuous in $\bar{\mathbb{R}}^2_+ \setminus \{(0,0)\}$. Therefore $\|w\|_{L^\infty(\mathbb{R}^2)} = +\infty$.

3.3 A Counter-Example to H^1-estimates

Consider now the solution of the problem with vanishing Neumann boundary conditions:

$$\begin{cases} -\Delta v_1 = \nabla^\perp b \cdot \nabla a & \text{in } \mathbb{R}^2_+ \\ \partial_\nu v_1 = 0 & \text{on } \partial\mathbb{R}^2_+ \end{cases} \tag{28}$$

given by the representation formula: $v_1(x) = \int_D \mathcal{G}(x,y)\nabla^\perp b(y)\cdot\nabla a(y)\,dy$. By the same computations in subsection 3.2 we find that:

$$\begin{aligned} v_1(x_1,0) &= \frac{1}{\pi}\int\int_{R^2_+} \nabla(\log((y_1-x_1)^2+x_2^2)^{1/2}))\cdot\left[\nabla^\perp b\cdot\left(a - \frac{a(x_1+y_1,0)+a(x_1-y_1,0)}{2}\right)\right]dy \\ &+ \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{y_1}\left[(b(x_1-y_1,0)-b(x_1+y_1,0))\frac{a(x_1+y_1,0)+a(x_1-y_1,0)}{2}\right]dy_1 \\ &- \frac{1}{\pi}\int_{-\infty}^{+\infty}\log(|y_1-x_1|)[\partial_{y_1} b\cdot a]dy_1 \end{aligned} \tag{29}$$

We take:

$$a(x) = \psi(x) \quad \text{and} \quad b(x) = (-\log|x|)^{-\beta}\psi(x), \tag{30}$$

with $0<\beta<1/2$ and ψ defined as in (25). In this case the solution v_1 is not in $H^{1/2}(\mathbb{R})$. Indeed if $0<\beta<1/2$, we have that:

$$a\partial_{x_1} b = \psi(x)[(-\beta)(-\log|x|)^{-(\beta+1)}\frac{1}{x}\psi(x) + \psi'(x)(-\log|x|)^{-\beta} \notin H^{-1/2}(\mathbb{R})$$

One can check this fact by putting it in duality with $f(x) = [(\log|x|)^-]^\beta \in H^{1/2}(\mathbb{R})$.

Now we observe that in the representation of $v(x_1,0)$ the sum of the first two terms gives a function in $H^{1/2}(\mathbb{R})$ (it is the trace of a solution of the problem Neumann problem (18) which is in $H^1(\mathbb{R}^2_+)$), the third term:

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\log(|y_1-x_1|)[\partial_{y_1} b\cdot a]dy_1$$

cannot be in $H^{1/2}(\mathbb{R})$ since $[\partial_{x_1} b\cdot a] \notin H^{-1/2}(\mathbb{R})$. Therefore $v_1(x_1,0)\notin H^{1/2}(\mathbb{R})$ and therefore $v_1 \notin H^1(\mathbb{R}^2_+)$.

Let $\Phi\colon D^2 \to \mathbb{R}^2_+$ be a Möbius transformation such that $\Phi(\partial D^2) = \mathbb{R}$ and let $\tilde{v}_1 = v_1\circ\Phi$, $\tilde{b} = b\circ\Phi$, $\tilde{a} = a\circ\Phi$.

Claim: *There cannot be an uniform L^∞ bound for the solution $\tilde{v}_1$.*

Proof of the claim. Suppose by contradiction that $\|\tilde{v}_1\|_{L^\infty(D^2)} < +\infty$. We multiply both sides of the equation $-\Delta\tilde{v}_1 = \nabla^\perp\tilde{b}\cdot\nabla\tilde{a}$ by $\tilde{v}_1$. An integration by parts gives

$$\begin{aligned} \int_{D^2}|\nabla\tilde{v}_1|^2 dy &= \int_{D^2}\tilde{v}_1(\nabla^\perp\tilde{b}\cdot\nabla\tilde{a})dy \\ &\le \|\tilde{v}_1\|_{L^\infty(D^2)}\|\nabla^\perp\tilde{b}\cdot\nabla\tilde{a}\|_{L^1(D^1)} < +\infty \end{aligned}$$

and this is in contradiction with the fact that $\tilde{v}_1 \notin H^1(D^2)$. □

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