SOME REMARKS ON POHOZAEV-TYPE IDENTITIES

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Abstract. In this note we present Pohozaev-type identities that have been recently established in [4] in the framework of half-harmonic maps defined either on $\mathbb{R}$ or on the sphere S^1 with values into a closed manifold $\mathcal{N}^n \subset \mathbb{R}^m$. Weak half-harmonic maps are critical points of the following nonlocal energy

$$\mathcal{L}^{1/2}_{\mathbb{R}}(u) := \int_{\mathbb{R}} |(-\Delta)^{1/4} u|^2 \, dx \quad \text{or} \quad \mathcal{L}^{1/2}_{S^1}(u) := \int_{S^1} |(-\Delta)^{1/4} u|^2 \, d\sigma(z). \tag{1}$$

By using the invariance of (1) in 1-D with respect to the trace of the Möbius transformations we derive a countable family of relations involving the Fourier coefficients of weak half-harmonic maps $u\colon S^1 \to \mathcal{N}^n$. We also present a short overview of Pohozaev formulas in 2-D in connection with Noether's theorem.

Sunto. In questa nota presentiamo in maggior dettaglio alcune formule di tipo Pohozaev trovate recentemente in [4] nell'ambito dello studio della mappe semi-armoniche definite o sulla retta reale o su la sfera S^1 e a valori in una varietà chiusa $\mathcal{N}^n \subset \mathbb{R}^m$. Le mappe semi-armoniche sono punti critici del funzionale non locale (1). Usando l'invarianza del funzionale (1) in dimensione 1 rispetto alla traccia delle trasformazioni di Möbius deriviamo una famiglia numerabile di relazioni tra i coefficienti di Fourier delle mappe semi-armoniche $u\colon S^1 \to \mathcal{N}^n$. Presentiamo inoltre una breve panoramica sul legame tra formule di Pohozaev in 2-D e il teorema di Noether.

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1. Introduction

The notion of weak 1/2-harmonic maps $u\colon \mathbb{R}^k \to \mathcal{N}^n$, where $\mathcal{N}^n \subset \mathbb{R}^m$ is a smooth n-dimensional closed (compact without boundary) manifold, has been introduced by Tristan Rivière and the author in [7, 8]. Since then the theory of fractional harmonic maps has received a lot of attention in view of their application to important geometrical problems (see e.g [9] for an overview of the theory). The L^2-regularity theory has been extended to higher dimension [3, 11, 18], and to L^p-energies [10, 16, 17].

In the sequel we focus our attention to the 1-D case ($k=1$).

We first introduce some notations and definitions. We denote by $\pi_{\mathcal{N}^n}$ the orthogonal projection onto $\mathcal{N}^n$ which happens to be a C^l map in a sufficiently small tubular neighborhood of $\mathcal{N}^n$ if $\mathcal{N}^n$ is assumed to be C^{l+1}.

We define the homogeneous fractional Sobolev space $\dot H^{1/2}(\mathbb{R},\mathbb{R}^m)$ as follows

$$\dot H^{1/2}(\mathbb{R},\mathbb{R}^m) := \left\{u \in L^2_{loc}(\mathbb{R},\mathbb{R}^m) : \|u\|^2_{\dot H^{1/2}(\mathbb{R})} := \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy < \infty\right\}.$$

We also define

$$\dot H^{1/2}(\mathbb{R},\mathcal{N}^n) := \left\{u \in \dot H^{1/2}(\mathbb{R},\mathbb{R}^m)\ ;\ u(x) \in \mathcal{N}^n \ \text{ for a.e. } x \in \mathbb{R}\right\}.$$

We introduce the following nonlocal energy:

$$\mathcal{L}^{1/2}(u) := \int_{\mathbb{R}} |(-\Delta)^{1/4}u|^2\, dx \tag{2}$$

where for $u \in \mathcal{S}(\mathbb{R})$ (1) the fractional Laplacian $(-\Delta)^{1/4}u$ can be defined by means of the the Fourier transform as follows

$$\widehat{(-\Delta)^{1/4}}u(\xi) = |\xi|^{1/2}\hat u(\xi)\,.$$

(2)

We observe that if $u \in \dot H^{1/2}(\mathbb{R},\mathbb{R}^m)$, then $(-\Delta)^{1/4}u$ is well defined and lies in $L^2(\mathbb{R})$, (see for instance Lemma B.5 in [6] and the references therein).

We now give the definition of a weak 1/2-harmonic map:

Definition 1.1. *A map $u \in \dot H^{1/2}(\mathbb{R},\mathcal{N}^n)$ is called a weak $1/2$-harmonic map into $\mathcal{N}^n$ if for any $\phi \in \dot H^{1/2}(\mathbb{R},\mathbb{R}^m)\cap L^\infty(\mathbb{R},\mathbb{R}^m)$ there holds*

$$\frac{d}{dt}\mathcal{L}^{1/2}(\pi_{\mathcal{N}^n}(u+t\phi))_{|t=0} = 0. \quad \Box$$

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(1)We denote respectively by $\mathcal{S}(\mathbb{R})$ the space of (real or complex) Schwartz functions.

(2)Given a function $\varphi \in \mathcal{S}(\mathbb{R})$ we denote either by $\hat\varphi$ or by $\mathcal{F}\varphi$ the Fourier transform of φ, i.e.

$$\hat\varphi(\xi) = \mathcal{F}\varphi(\xi) = \int_{\mathbb{R}} v(x)e^{-i\xi x}\,dx.$$

In short we say that a weak 1/2-harmonic map is a critical point of $\mathcal{L}^{1/2}$ in $\dot{H}^{1/2}(\mathbb{R},\mathcal{N}^n)$ for *perturbations in the target.*

Weak 1/2-harmonic maps satisfy the Euler-Lagrange equation

$$\nu(u)\wedge(-\Delta)^{1/2}u=0 \qquad \text{in } \mathcal{D}'(\mathbb{R}), \tag{3}$$

where $\nu(z)$ is the Gauss Map at $z\in\mathcal{N}^n$ taking values into the Grassmannian $\tilde{G}r_{m-n}(\mathbb{R}^m)$ of oriented $m-n$ planes in $\mathbb{R}^m$ which is given by the oriented normal $m-n$-plane to $T_z\mathcal{N}^n$. We denote by the symbol $\wedge$ the *exterior or wedge product* defined on the exterior algebra (or Grassmann Algebra) of $\mathbb{R}^m$, $\bigwedge(\mathbb{R}^m)$.

Equation (3) says roughly speaking that the vector $(-\Delta)^{1/2}u(x)$ is perpendicular to the tangent plane $T_{u(x)}\mathcal{N}^n$ at the point $u(x)$.

One of the main result in [8] is the local Hölder continuity of weak 1/2-harmonic maps:

Theorem 1.1. *Let $\mathcal{N}^n$ be a C^2 closed submanifold of $\mathbb{R}^m$ and let $u\in\dot{H}^{1/2}(\mathbb{R},\mathcal{N}^n)$ be a weak $1/2-$harmonic map into $\mathcal{N}^n$. Then $u\in\bigcap_{0<\delta<1}C^{0,\delta}_{loc}(\mathbb{R},\mathcal{N}^n)$.* □

Finally a bootstrap argument leads to the following result (see [6] for the details of this argument).

Theorem 1.2. *Let $\mathcal{N}^n\subset\mathbb{R}^m$ be a C^l closed submanifold of $\mathbb{R}^m$, with $l\geq 2$, and let $u\in\dot{H}^{1/2}(\mathbb{R},\mathcal{N}^n)$ be a weak $\frac{1}{2}$-harmonic. Then*

$$u\in\bigcap_{0<\delta<1}C^{l-1,\delta}_{loc}(\mathbb{R},\mathcal{N}^n).$$

In particular, if $\mathcal{N}^n$ is C^∞ then $u\in C^\infty(\mathbb{R},\mathcal{N}^n)$. □

Next we would like to clarify the connections between 1/2-harmonic maps defined in $\mathbb{R}$ and 1/2-harmonic maps defined in S^1 which are defined as critical points of the energy

$$\mathcal{L}^{1/2}_{S^1}(u):=\int_{S^1}|(-\Delta)^{1/4}u|^2\,d\sigma(z). \tag{4}$$

For $u\in L^1(S^1)$ we define its Fourier coefficients as

$$\hat{u}(n)=\frac{1}{2\pi}\int_0^{2\pi}u(e^{i\theta})e^{-in\theta}\,d\sigma(z),\quad n\in\mathbb{Z}.$$

If u is smooth we define for $s \in \mathbb{R}$

$$(-\Delta)^s u(\theta) = \sum_{n\in\mathbb{Z}} |n|^{2s} \hat{u}(n) e^{in\theta}. \tag{5}$$

If $u \in L^1(S^1)$ we can define $(-\Delta)^s u \in \mathcal{D}'(S^1)$ in a distributional sense as follows:

$$\langle (-\Delta)^s u, \varphi\rangle := \int_{S^1} u\,(-\Delta)^s \varphi\, d\sigma(z), \quad \varphi \in C^\infty(S^1). \tag{6}$$

Notice that $\varphi \in C^\infty(S^1)$ implies that $(-\Delta)^s\varphi \in C^\infty(S^1)$ (here $(-\Delta)^s\varphi$ is defined as in (5)).

We define in S^1 the Sobolev space:

$$H^{1/2}(S^1,\mathbb{R}^m) := \left\{ u \in L^2(S^1,\mathbb{R}^m): \quad \int_{S^1}\int_{S^1} \frac{|u(e^{i\theta}) - u(e^{i\tau})|^2}{|e^{i\theta} - e^{i\tau}|^2}\, d\theta\, d\tau < \infty \right\}.$$

If $u \in H^{1/2}(S^1,\mathbb{R}^m)$ then

$$\int_{S^1}\int_{S^1} \frac{|u(e^{i\theta}) - u(e^{i\tau})|^2}{|e^{i\theta} - e^{i\tau}|^2}\, d\theta\, d\tau = 4\pi \sum_k |k||\hat{v}(k)|^2 < +\infty.$$

We next consider the classical stereographic projection from $S^1 \setminus \{-i\}$ onto $\mathbb{R}$:

$$\mathcal{P}_{-i}\colon S^1 \setminus \{-i\} \to \mathbb{R}, \quad \mathcal{P}_{-i}(\cos(\theta) + i\sin(\theta)) = \frac{\cos(\theta)}{1+\sin(\theta)}. \tag{7}$$

Its inverse is given by

$$\mathcal{P}_{-i}^{-1}(x) = \frac{2x}{1+x^2} + i\left(-1 + \frac{2}{1+x^2}\right), \tag{8}$$

then the following relation between the 1/2-Laplacian in $\mathbb{R}$ and in S^1 holds:

Proposition 1.1. *Given $u: \mathbb{R} \to \mathbb{R}^m$, we set $v := u \circ \mathcal{P}_{-i} : S^1 \to \mathbb{R}^m$. Then $u \in L_{\frac{1}{2}}(\mathbb{R})$*(3) *if and only if $v \in L^1(S^1)$. In this case*

$$(-\Delta)^{\frac{1}{2}}_{S^1} v(e^{i\theta}) = \frac{((-\Delta)^{\frac{1}{2}}_{\mathbb{R}} u)(\mathcal{P}_{-i}(e^{i\theta}))}{1+\sin\theta} \quad \text{in } \mathcal{D}'(S^1 \setminus \{-i\}). \tag{9}$$

Observe that $(1+\sin(\theta))^{-1} = |\mathcal{P}_{-i}'(\theta)|$, hence we have

$$\int_0^{2\pi} (-\Delta)^{\frac{1}{2}} v(e^{i\theta})\, \varphi(e^{i\theta})\, d\sigma(z) = \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u(x)\, \varphi(\mathcal{P}_{-i}^{-1}(x))\, dx \quad \text{for every } \varphi \in C_0^\infty(S^1 \setminus \{-i\}).$$

(3) We recall that $L_{\frac{1}{2}}(\mathbb{R}) := \left\{ u \in L^1_{loc}(\mathbb{R}) : \int_{\mathbb{R}} \frac{|u(x)|}{1+x^2} dx < \infty \right\}$

For the proof of Proposition 1.1 we refer for instance to [5].

A key property of the Lagrangian (2) is its invariance under the trace of conformal maps that keep invariant the half space $\mathbb{R}^2_+$ (the Möebius group). From the conformal invariance and Proposition 1.1 it follows that $u \in \dot{H}^{1/2}(\mathbb{R})$ is a 1/2-harmonic map in $\mathbb{R}$ if and only if $v := u \circ \mathcal{P}_{-i} \in H^{1/2}(S^1)$ is a 1/2 harmonic map in S^1, (see i.e. [2]).

In this note we are going to describe some **Pohozaev-type identities** for the half Laplacian and the Laplacian respectively in one and two dimension.

We first consider the fundamental solution G of the fractional heat equation:

$$\begin{cases} \partial_t G + (-\Delta)^{1/2} G = 0 & x \in \mathbb{R},\, t > 0 \\ G(0,x) = \delta_0 & t = 0\,. \end{cases} \tag{10}$$

It is given by

$$G(t,x) = \frac{1}{\pi} \frac{t}{x^2+t^2}.$$

The following equalities hold

$$\partial_t G = \frac{1}{\pi} \frac{x^2-t^2}{(t^2+x^2)^2}, \quad \partial_x G = -\frac{1}{\pi} \frac{2xt}{(t^2+x^2)^2}.$$

Theorem 1.3. [Pohozaev Identity in R] *Let $u \in W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^m)$ be such that*

$$\frac{du}{dx} \cdot (-\Delta)^{1/2} u = 0 \quad a.e. \ in\ \mathbb{R}. \tag{11}$$

Assume that for some $u_0 \in R$

$$\int_{\mathbb{R}} |u-u_0| dx < +\infty, \quad \int_{\mathbb{R}} \left| \frac{du}{dx}(x) \right| dx < +\infty. \tag{12}$$

Then the following identity holds

$$\left| \int_{\mathbb{R}} \partial_t G(t,x)(u(x)-u_0) dx \right|^2 = \left| \int_{\mathbb{R}} \partial_x G(t,x)(u(x)-u_0) dx \right|^2 \ for\ all\ t \in \mathbb{R}. \ \square \tag{13}$$

We get an analogous formula in S^1. By identifying S^1 with $[-\pi,\pi)$ we consider the following problem

$$\begin{cases} \partial_t F + (-\Delta)^{1/2} F = 0 & \theta \in [-\pi,\pi),\ t > 0 \\ F(0,\theta) = \delta_0(x) & \theta \in [-\pi,\pi]. \end{cases} \tag{14}$$

The solution of (14) is given by

$$F(\theta,t)=\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}e^{-t|n|}e^{in\theta}=\frac{e^{2t}-1}{e^{2t}-2e^t\cos(\theta)+1}.$$

In this case we have

$$\partial_t F(t,\theta)=-2e^t\frac{e^{2t}\cos(\theta)-2e^t+\cos(\theta)}{(e^{2t}-2e^t\cos(\theta)+1)^2}$$

and

$$\partial_\theta F(t,\theta)=-2e^t\frac{\sin(\theta)(e^{2t}-1)}{(e^{2t}-2e^t\cos(\theta)+1)^2}\,.$$

Then the following holds

Theorem 1.4. [Pohozaev Identity on $\mathbf{S}^1$] *Let $u\in W^{1,2}(S^1,\mathbb{R}^m)$ be such that*

$$\frac{\partial u}{\partial\theta}\cdot(-\Delta)^{1/2}u=0\quad a.e.\ S^1. \tag{15}$$

Then the following identity holds

$$\left|\int_{S^1}u(z)\partial_t F(z)\,d\sigma(z)\right|^2=\left|\int_{S^1}u(z)\partial_\theta F(z)\,d\sigma(z)\right|^2. \tag{16}$$

From (16) *one deduces in particular (by letting $t\to+\infty$) that*

$$\left|\int_0^{2\pi}u(e^{i\theta})\cos(\theta)\,d\theta\right|^2=\left|\int_0^{2\pi}u(e^{i\theta})\sin(\theta)\,d\theta\right|^2.\quad\square \tag{17}$$

For the proof of Theorem 1.3 and Theorem 1.4 and the derivation of the fundamental solution of the nonlocal heat equation we refer the reader to [4].

We could have solved (10) by requiring $G(0,x)=\delta_{x_0}$, with $x_0\in\mathbb{R}$ and we would have obtained infinitely many corresponding Pohozaev-type formulas.

Next we explain the connection between 1/2-harmonic maps and the formulas (13) and (16).

We observe that if u is a smooth critical point of (2) in $\mathbb{R}$ then it is *stationary* as well, namely it is critical with respect to the variation of the domain:

$$\left(\frac{d}{da}\int_{\mathbb{R}}|(-\Delta)^{1/4}(u(x+aX(x))|^2dx\right)\Big|_{a=0}=0 \tag{18}$$

where $X\colon\mathbb{R}\to\mathbb{R}$ is a $C^1_c(R)$ vector field.

Actually any variation the form $u(x+aX(x)) = u(x) + a\dfrac{du(x)}{dx}X(x) + o(a)$ can be interpreted as being a variation in the target with $\varphi(x) = \dfrac{du(x)}{dx}X(x)$.

From (18) we get the so-called equation of stationarity:

$$0 = \int_{\mathbb{R}} [(-\Delta)^{1/2}(u(x+aX(x))\cdot\frac{d}{da}(u(x+aX(x)))]\Big|_{a=0} dx = \int_{\mathbb{R}} (-\Delta)^{1/2}(u(x))\cdot\frac{du(x)}{dx}X(x)dx.$$

By the arbitrariness of X and the smoothness of u from (19) we deduce that

$$(-\Delta)^{1/2}u(x)\cdot\frac{du}{dx}(x) = 0 \quad x \in \mathbb{R}. \tag{19}$$

In an analogous way if u is a smooth critical point of the fractional energy (2) in S^1, it also satisfies

$$\left(\frac{d}{da}\int_{S^1} |(-\Delta)^{1/4}(u(z+aX(z)))|^2 d\sigma(z)\right)\Big|_{a=0} = 0 \tag{20}$$

where $X\colon S^1 \to \mathbb{R}^2$ is a $C^1(S^1)$ vector field. From (20) it follows that

$$(-\Delta)^{1/2}(u(z))\cdot\partial_\theta u(z) = 0 \quad z \in S^1. \tag{21}$$

Therefore the assumptions of Theorem 1.3 and Theorem 1.4 are satisfied by sufficiently smooth 1/2-harmonic maps.

We recall that one can derive the stationary equation for a certain Lagrangian before knowing any regularity assumption of the critical point. For instance if the critical point of is a local minimizer then weak solutions of the Euler Lagrange equation satisfies the stationary equation as well. On the other hand there are examples in which solutions of the Euler Lagrange equation are not solution of the stationary equation, (see [14]).

We have now to give some explanations why these identities belong to the *Pohozaev identities* family. These identities are produced by the conformal invariance of the highest order derivative term in the Lagrangian from which the Euler Lagrange is issued. For instance the Dirichlet energy

$$\mathcal{L}(u) = \int_{\mathbb{R}^2} |\nabla u|^2 dx^2 \tag{22}$$

is conformal invariant in 2-D. We recall that a map $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is conformal if it satisfies

$$\left\{\begin{array}{l} |\frac{\partial\phi}{\partial x}| = |\frac{\partial\phi}{\partial y}| \\ \langle\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\rangle = 0 \\ \det\nabla\phi \ge 0 \ \ and \ \ \nabla\phi \neq 0\,. \end{array}\right. \tag{23}$$

Then for every $u \in W^{1,2}(\mathbb{R}^2, \mathbb{R})$ and every conformal map ϕ, $\deg(\phi) = 1$, the following holds

$$\mathcal{L}(u) = \mathcal{L}(u \circ \phi) = \int_{\phi^{-1}(\mathbb{R}^2)} |\nabla(u \circ \phi)(x)|^2 dx^2\,.$$

Whereas the following fractional energy

$$\mathcal{L}^{1/2}_{\mathbb{R}}(u) = \int_{\mathbb{R}} |(-\Delta)^{1/4} u|^2 \ dx \tag{24}$$

is conformal invariant in 1-D with respect to the trace of conformal maps that keep invariant $\mathbb{R}^2_+$. The infinitesimal perturbations issued from the dilations produce in (22) and (24) respectively the following infinitesimal variations of these highest order terms

$$\sum_{i=1}^{2} x_i \frac{\partial u}{\partial x_i} \cdot \Delta u \quad \text{in 2-D} \quad \text{and} \quad x\frac{du}{dx} \cdot (-\Delta)^{1/2} u \quad \text{in 1-D}$$

Such kind of perturbations play an important role in establishing Pohozaev-type identities. We will explain in more detail in section 2 the link between Pohozaev formulas and the conformal invariance of some specific Lagrangians in 2-D. If u is a smooth critical point of (22) then it satisfies the following stationary equation

$$\sum_{i=1}^{2} \frac{\partial u}{\partial x_i} \cdot \Delta u(x) = 0, \quad x \in \mathbb{R}^2. \tag{25}$$

Integrating the identity (25) on a ball $B(x_0, r)$ ($x_0 \in R^2, r > 0$) gives a *balancing law* between the *radial* part and the *angular* part of the energy classically known as **Pohozaev identity**. Precesely it holds:[(4)]

Theorem 1.5. *Let $u \in W^{2,2}_{loc}(B(0,1), \mathbb{R}^m)$ such that*

$$\frac{\partial u}{\partial x_i}(x) \cdot \Delta u(x) = 0 \quad a.e.\ in\ B(0,1) \tag{26}$$

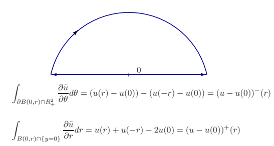
[(4)]In section 3 we will prove a more general version of Theorem 1.5.

for $i = 1, 2$. Then it holds

$$\int_{\partial B(x_0,r)} \left|\frac{1}{r}\frac{\partial u}{\partial \theta}\right|^2 d\theta = \int_{\partial B(x_0,r)} \left|\frac{\partial u}{\partial r}\right|^2 d\theta \tag{27}$$

for all $r \in [0, 1]$.

In 1 dimension one might wonder what corresponds to the 2 dimensional dichotomy between *radial* and *angular* parts. Figure 1 is intended to illustrate the following correspondence of dichotomies respectively in 1 and 2 dimensions.



$$\begin{array}{cccc}
\text{2-D} & \longleftrightarrow & \text{1-D} & \\
\textbf{radial}: \ \dfrac{\partial u}{\partial r} & \longleftrightarrow & \textbf{symmetric} \text{ part of } u & : \ u^+(x) := \frac{u(x)+u(-x)}{2} \\
\textbf{angular}: \ \dfrac{\partial u}{\partial \theta} & \longleftrightarrow & \textbf{antisymmetric} \text{ part of } u & : \ u^-(x) := \frac{u(x)-u(-x)}{2}
\end{array}$$

In this note we make the observation that by exploiting the invariance of the equation (15) with respect to the trace of Möbius transformations of the disk in $\mathbb{R}^2$ of the form $M_{\alpha,a}(z) := e^{i\alpha}\frac{z-a}{1-az}$, $\alpha \in \mathbb{R}, a \in (-1, 1)$ [5] we can derive from (17) a countable family of

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[5] We recall that since $M_{\alpha,a}(z)$ is conformal with $M'_{\alpha,a}(z) \neq 0$ we have

$$(-\Delta)^{1/2}(u \circ M_{\alpha,a}(z)) = e^{\lambda_{\alpha,a}}((-\Delta)^{1/2}u) \circ M_{\alpha,a}(z), \tag{28}$$

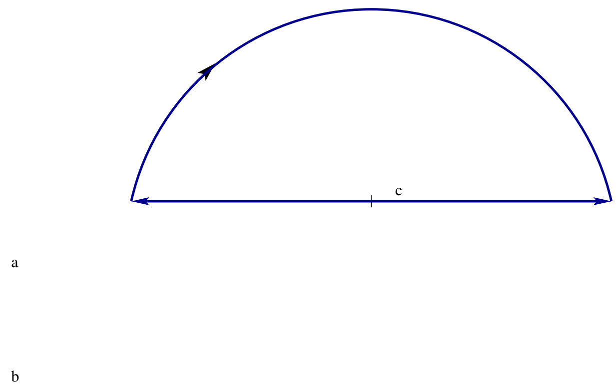

FIGURE 1. Link between the symmetric and antisymmetric part of u and the integral of the radial and tangential derivative of any extension $\tilde{u}$ of u on upper half plane R^2_+

relations involving the Fourier coefficients of solutions of (15). This fact has been already announced in the paper [4]. We heard that the proof of this property has been recently obtained also in the work of preparation [1] by using a different approach.

Given $u\colon S^1 \to \mathbb{R}^m$ we define its Fourier coefficients for every $k \geq 0$:

$$\begin{cases} a_k := \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} u(e^{i\theta})\,\cos k\theta\; d\theta \\ b_k = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} u(e^{i\theta})\,\sin k\theta\; d\theta. \end{cases}$$

The following result holds.

Proposition 1.2. [Relations of the Fourier coefficients on $\mathbf{S}^1$] *Let $u \in W^{1,2}(S^1, \mathbb{R}^m)$ satisfy* (15). *Then for every $n \geq 2$ it holds*

$$\sum_{k=1}^{n-1}(n-k)k(a_k a_{n-k} - b_k b_{n-k}) = 0 \tag{29}$$

and

$$\sum_{k=1}^{n-1}(n-k)k(a_k b_{n-k} + b_k a_{n-k}) = 0. \qquad \Box \tag{30}$$

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where $\lambda_{\alpha,a}(z) = \log(|\frac{\partial M_{\alpha,a}}{\partial \theta}(z)|), z \in S^1$

We conclude this introduction by mentioning that in the paper [15] the authors obtains a different Pohozaev identity for bounded weak solutions to the following problem

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{31}$$

where $s \in (0,1)$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain. As a consequence of their Pohozaev identity they get nonexistence results for problem (31) with supercritical nonlinearitis in star-shaped domains.

This paper is organized as follows. In section 2 we present a short overview of the connection between Pohozaev formulas in 2-D and the existence of *conservation laws.* In section 3 we obtain infinite many Pohozaev formulas for stationary harmonic maps in 2-D in correspondence to conformal vector fields in $\mathbb{C}$ generated by holomorphic functions.The strategy consistes in multiplying the stationary equation associated to Dirichlet energy by a conformal vector field and the fundamental solution. This method avoids to use suitable cut-off functions and it turns out to be useful also in the nonlocal case to get formula 13. In section 4 we prove Proposition 1.2.

2. Pohozaev Identity in the light of Noether Theorem

In this section we would like to describe the relation between Pohozaev identities with Noether's theorem in 2-D. Noether's theorem is a very general result in the calculus of variations. It enables to construct a divergence-free vector field on the domain space, from a solution of a variational problem, provided we are in the presence of a continuous symmetry. Here we will consider the case of symmetries in the domain and Lagrangians of the type:

$$E(u) = \int_{B(0,1)} f(u, \nabla u)(x)dx \tag{32}$$

where $f \in C^1(\mathbb{R}^m, \mathbb{R}^m \times \mathbb{R}^2)$, $|f(z,p)| \le C(1+|p|^2)$ and $u \in W^{1,2}(B(0,1), R^m$. Given $X \in C^1_c(B(0,1), \mathbb{R}^2)$ we compute the stationary equation for the Lagrangian (32):

$$\frac{d}{dt} E(u(x+tX(x)))|_{t=0} = \delta E(u) \cdot X = 0.$$

We observe that for t small and for $k=1,2$ we have

$$\partial_{x_k}(u(x+tX(x))) = \partial_{x_k}u(x+tX(x)) + t\sum_{\ell=1}^{2}\partial_{x_\ell}u(x)\partial_{x_k}X^\ell(x) + o(t). \tag{33}$$

Therefore:

$$\begin{aligned} E(u(x+tX(x))) &= \int_{B(0,1)} f(u,\nabla u)(x+tX(x))dx \\ &+ t\int_{B(0,1)}\sum_{j=1}^{m}\sum_{k,\ell=1}^{2}\partial_{p_j^k}f(u,\nabla u)(x)\partial_{x_\ell}u^j(x)\partial_{x_k}X^\ell(x) + o(t) \end{aligned} \tag{34}$$

We derive with respect to t and get

$$\begin{aligned} 0 &= \frac{d}{dt}E(u(x+tX(x)))|_{t=0} = \int_{B(0,1)}\sum_{\ell=1}^{2}\partial_{x_\ell}f(u,\nabla u)X^\ell dx \\ &+ \sum_{j=1}^{m}\sum_{k,\ell=1}^{2}\underbrace{\int_{B(0,1)}\partial_{x_k}[\partial_{p_j^k}f(u,\nabla u)(x)\partial_{x_\ell}u^j(x)X^\ell(x)]dx}_{(1)} \\ &- \sum_{j=1}^{m}\sum_{k,\ell=1}^{2}\int_{B(0,1)}\partial_{x_k}[\partial_{p_j^k}f(u,\nabla u)(x)\partial_{x_\ell}u^j(x)]X^\ell(x)dx. \end{aligned} \tag{35}$$

Since X has compact support the term (1) is zero. Hence the system of stationary equations for the Lagrangian (32) is given by:

$$\begin{cases} \partial_{x_1}f(u,\nabla u) - \sum_{j=1}^{m}\sum_{k=1}^{2}\partial_{x_k}[\partial_{p_j^k}f(u,\nabla u)(x)\partial_{x_1}u^j(x)] = 0 \\ \partial_{x_1}f(u,\nabla u) - \sum_{j=1}^{m}\sum_{k=1}^{2}\partial_{x_k}[\partial_{p_j^k}f(u,\nabla u)(x)\partial_{x_2}u^j(x)] = 0. \end{cases} \tag{36}$$

Next we assume that for every conformal diffeomorphism $\phi\colon B(0,1)\to\mathbb{R}^2$ we have for a.e $x\in B(0,1)$:

$$f(u\circ\phi,\nabla(u\circ\phi))(x) = f(u\circ\phi,(\nabla u)\circ\phi)(x)\frac{|\nabla\phi(x)|^2}{2}. \tag{37}$$

The relation (37) implies that E is conformal invariant. Let ϕ_t be a family of conformal diffeomorphisms which is C^1 with respect to t. Set $Y(x) = \frac{d\phi_t(x)}{dt}_{|t=0}$ We derive (37) with respect to t:

$$\begin{aligned} \frac{d}{dt}\left(f(u\circ\phi_t,\nabla(u\circ\phi_t))(x)\right)|_{t=0} &= \frac{d}{dt}[f(u\circ\phi,(\nabla u)\circ\phi)]|_{t=0} + f(u,\nabla u)\partial_{x_k}Y^k(x) \\ &= \partial_{x_k}[f(u,\nabla u)]Y^k(x) + f(u,\nabla u)\mathrm{div}Y = \mathrm{div}[f(u,\nabla u)Y]. \end{aligned} \tag{38}$$

By combining (35) and (38) we get

$$\sum_{\ell=1}^{2} \partial_{x_\ell} f(u,\nabla u) Y^\ell - \sum_{j=1}^{m} \sum_{k,\ell=1}^{2} \partial_{x_k} [\partial_{p_j^k} f(u,\nabla u)(x) \partial_{x_\ell} u^j(x)] Y^\ell =$$

$$\sum_{k=1}^{2} \partial_{x_k} \left[\sum_{j=1}^{m} \sum_{\ell=1}^{2} [-\partial_{p_j^k} f(u,\nabla u)(x) \partial_{x_\ell} u^j(x) Y^\ell(x)] + f(u,\nabla u) Y^k \right]$$

From above it follows that:

Theorem 2.1 (Noether ('18)). *Let $u \in W^{1,2}(B(0,1),\mathbb{R}^m)$ be a stationary point of the Lagrangian* (32)*, namely it satisfies* (36) *in $\mathcal{D}'(B(0,1))$. If f satisfies* (37) *then the following vector field (Noether's current):*

$$J_Y[u] = \left(\sum_{j=1}^{m} \sum_{\ell=1}^{2} [\partial_{p_j^k} f(u,\nabla u)(x) \partial_{x_\ell} u^j(x) Y^\ell(x)] - f(u,\nabla u) Y^k \right)_{k=1,2}$$

is divergence free*, where Y is the infinitesimal generator of conformal transformations.*

We apply theorem 2.1 to $f(z,p) = \frac{|p|^2}{2}$. In this case we have $\partial_{p_j^k} f(u,\nabla u)(x) = p_j^k$ and

$$J_Y(x) = \left(\sum_{j=1}^{m} \sum_{\ell=1}^{2} [\partial_{x_k} u^j(x) \partial_{x_\ell} u^j(x) Y^\ell(x)] - \frac{|\nabla u|^2}{2} Y^k \right)_{k=1,2}$$

The stationary system of equations is:

$$\left\{ \begin{array}{l} \partial_{x_1} \left[\frac{u_{x_1}^2}{2} - \frac{u_{x_2}^2}{2} \right] + \partial_{x_2} [u_{x_1} u_{x_2}] = 0, \\ \partial_{x_2} \left[\frac{u_{x_1}^2}{2} - \frac{u_{x_2}^2}{2} \right] - \partial_{x_1} [u_{x_1} u_{x_2}] = 0. \end{array} \right. \tag{39}$$

If we choose $Y_1(x) = (x_1, x_2)$ (the infinitesimal generator of the dilations) and $Y_2(x) = (-x_2, x_1)$ (the infinitesimal generator of the rotations) we respectively get

$$J_{Y_1}(x) = \left(\left[\frac{u_{x_1}^2}{2} - \frac{u_{x_2}^2}{2} \right] x_1 + u_{x_1} u_{x_2} x_2, \left[\frac{u_{x_2}^2}{2} - \frac{u_{x_1}^2}{2} \right] x_2 + u_{x_1} u_{x_2} x_1 \right),$$

$$J_{Y_2}(x) = \left(\left[\frac{u_{x_2}^2}{2} - \frac{u_{x_1}^2}{2} \right] x_2 + u_{x_1} u_{x_2} x_1, \left[\frac{u_{x_2}^2}{2} - \frac{u_{x_1}^2}{2} \right] x_1 - u_{x_1} u_{x_2} x_2 \right).$$

Theorem 2.1 yields:

$$\begin{aligned} 0 = \mathrm{div} J_{Y_1}(x) \;\; &= \;\; \partial_{x_1} \left(\frac{u_{x_1}^2}{2} - \frac{u_{x_2}^2}{2} \right) x_1 - \partial_{x_2} \left(\frac{u_{x_1}^2}{2} - \frac{u_{x_2}^2}{2} \right) x_2 \\ &\quad + x_2 \partial_{x_1} [u_{x_1} u_{x_2}] + x_1 \partial_{x_2} [u_{x_1} u_{x_2}]. \end{aligned} \tag{40}$$

and

$$0 = \mathrm{div} J_{Y_2}(x) \;=\; \partial_{x_1}(\frac{u_{x_2}^2}{2} - \frac{u_{x_1}^2}{2})x_2 + \partial_{x_2}(\frac{u_{x_2}^2}{2} - \frac{u_{x_1}^2}{2})x_2$$
$$+x_1\partial_{x_1}[u_{x_1}u_{x_2}] - x_2\partial_{x_2}[u_{x_1}u_{x_2}]. \tag{41}$$

By multiplying (40) and (41) respectively by x_1 and x_2 and then by subtracting (41) to (40) we obtain

$$\left(\partial_{x_1}\left[\frac{u_{x_1}^2}{2} - \frac{u_{x_2}^2}{2}\right] + \partial_{x_2}[u_{x_1}u_{x_2}]\right)(x_1^2 + x_2^2) = 0 \tag{42}$$

By multiplying (40) and (41) respectively by x_2 and x_1 and then summing (41) and (40) we obtain

$$\left(\partial_{x_2}\left[\frac{u_{x_2}^2}{2} - \frac{u_{x_1}^2}{2}\right] + \partial_{x_1}[u_{x_1}u_{x_2}]\right)(x_1^2 + x_2^2) = 0. \tag{43}$$

Equations (42) and (43) are exactly the equations (39). In the particular case of the Dirichlet energy Noether theorem implies the stationary equation and therefore the Pohozaev formulas that we describe in section 3.

3. Pohozaev Identities for the Laplacian in $\mathbb{R}^2$

In this section we derive Pohozaev identities in 2-D (Theorem 3.1) by combining ideas from [13] and [19]. Precisely we multiply the stationary equation (39) which is satisfied for instance by sufficiently smooth harmonic maps by the fundamental solution of the heat equation and a holomorphic vector field $X\colon \mathbb{C} \to \mathbb{C}$.

We mention that the use of the fundamental solution to get Pohozaev-type identities and monotonicity formulas has been performed in [19] to study the heat flow. In Chapter 9 of [13] the authors derived in the context of Ginzurg-Landau equation generalized Pohozaev identities for the so-called *ρ-conformal vector fields* $X = (X^1, \dots, X^n)$, where ρ is a given function defined in a 2 dimensional domain. In the case $\rho \equiv 1$ then the ρ-conformal vector fields are exactly conformal vector fields generated by holomorphic functions.

We recall that the fundamental solution of the heat equation

$$\begin{cases} \partial_t G - \Delta G = 0 & t > 0 \\ G(0,x) = \delta_{x_0} & t = 0\,. \end{cases} \tag{44}$$

is given by $G(x,t) = (4\pi t)^{-1/2} e^{-\frac{|x-x_0|^2}{4t}}$.

Theorem 3.1. *[Pohozev in $\mathbb{R}^2$] Let $u \in W^{1,2}_{loc}(\mathbb{R}^2, \mathbb{R}^m)$ be a solution of*

$$\partial_{x_\ell}\left(\frac{|\nabla u|^2}{2}\right) - \sum_{k=1}^{2} \partial_{x_k}\left[\partial_{x_k} u \partial_{x_\ell} u\right] = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2), \tag{45}$$

$\ell = 1, 2$. Assume that

$$\int_{\mathbb{R}^2} |\nabla u(x)|^2 dx < +\infty. \tag{46}$$

Then for all $x_0 \in \mathbb{R}^2$, $t > 0$ and every $X = X_1 + iX_2 \colon \mathbb{C} \to \mathbb{C}$ holomorphic function the following identity holds

$$2 \iint_{R^2} e^{-\frac{|x-x_0|^2}{4t}} |x - x_0| \left(\frac{\partial u}{\partial \nu} \cdot \frac{\partial u}{\partial X}\right) dx = \iint_{R^2} e^{-\frac{|x-x_0|^2}{4t}} \left((x - x_0) \cdot X\right) |\nabla u|^2 dx. \tag{47}$$

If $X = x - x_0$ with $x_0 \in \mathbb{R}^2$ then for all $t > 0$ the following identity holds

$$\iint_{R^2} e^{-\frac{|x-x_0|^2}{4t}} |x - x_0|^2 \left|\frac{\partial u}{\partial \nu}\right|^2 dx^2 = \iint_{R^2} e^{-\frac{|x-x_0|^2}{4t}} \left|\frac{\partial u}{\partial \theta}\right|^2 dx^2. \tag{48}$$

Proof. We multiply the equation (45) by $X^\ell(x)e^{-\frac{|x-x_0|^2}{4t}}$ and we integrate[(6)]

$$\begin{aligned}
0 &= \iint_{\mathbb{R}^2}\left[\partial_{x_\ell}\left(\frac{|\nabla u|^2}{2}\right)-\partial_{x_k}\left(\partial_{x_k}u\partial_{x_\ell}u\right)\right]X^\ell e^{-\frac{|x-x_0|^2}{4t}}\,dx\\
&= \underbrace{\iint_{\mathbb{R}^2}\partial_{x_\ell}\left[\frac{|\nabla u|^2}{2}X^\ell e^{-\frac{|x-x_0|^2}{4t}}\right]dx}_{=0}-\iint_{\mathbb{R}^2}\frac{|\nabla u|^2}{2}\partial_{x_\ell}\left(X^\ell e^{-\frac{|x-x_0|^2}{4t}}\right)dx\\
&- \underbrace{\iint_{\mathbb{R}^2}\partial_{x_k}\left[\partial_{x_k}u\partial_{x_\ell}uX^\ell e^{-\frac{|x-x_0|^2}{4t}}\right]dx}_{=0}+\iint_{\mathbb{R}^2}\partial_{x_k}u\partial_{x_\ell}u\partial_{x_k}[X^\ell e^{-\frac{|x-x_0|^2}{4t}}]\,dx\\
&= -\iint_{\mathbb{R}^2}|\nabla u|^2\underbrace{\frac{\partial X^1}{\partial x_1}}_{\frac{\partial X^1}{\partial x_1}=\frac{\partial X^2}{\partial x_2}}e^{-\frac{|x-x_0|^2}{4t}}\,dx+\frac{1}{4t}\iint_{\mathbb{R}^2}|\nabla u|^2X\cdot(x-x_0)e^{-\frac{|x-x_0|^2}{4t}}\,dx\\
&+ \iint_{\mathbb{R}^2}e^{-\frac{|x-x_0|^2}{4t}}\left[\underbrace{\frac{\partial X_1}{\partial x_1}}_{\frac{\partial X_1}{\partial x_1}=\frac{\partial X_2}{\partial x_2}}|\nabla u|^2+\underbrace{(\frac{\partial X_1}{\partial x_2}+\frac{\partial X_2}{\partial x_1})}_{=0}(\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2})\right]\\
&- \frac{1}{2t}\iint_{\mathbb{R}^2}e^{-\frac{|x-x_0|^2}{4t}}(X\cdot\nabla u)\frac{\partial u}{\partial \nu}|x-x_0|dx.
\end{aligned}\tag{49}$$

From (49) we obtain that

$$2\iint_{\mathbb{R}^2}e^{-\frac{|x-x_0|^2}{4t}}(X\cdot\nabla u)\frac{\partial u}{\partial\nu}|x-x_0|dx=\iint_{\mathbb{R}^2}e^{-\frac{|x-x_0|^2}{4t}}(x-x_0)\cdot X|\nabla u|^2dx.\tag{50}$$

In particular if $X=(x-x_0)$ by using that $\nabla u=(\frac{\partial u}{\partial\nu},|x-x_0|^{-1}\frac{\partial u}{\partial\theta})$, from (50) we get the identity

$$\iint_{R^2}e^{-\frac{|x-x_0|^2}{4t}}|x-x_0|^2\left|\frac{\partial u}{\partial\nu}\right|^2dx=\iint_{R^2}e^{-\frac{|x-x_0|^2}{4t}}\left|\frac{\partial u}{\partial\theta}\right|^2dx\tag{51}$$

and we conclude. □

We observe that if u is smooth then equation (45) is equivalent to the equations

$$\frac{\partial u}{\partial x_i}\cdot\Delta u=0,\quad x\in\mathbb{R}^2,\quad i=1,2.$$

[(6)]We use the Einstein summation convention

In Theorem 3.2 we get infinite many Pohozaev identities over balls in correspondence to holomorphic vector fields $X = X_1 + iX_2\colon \mathbb{C} \to \mathbb{C}$ for maps $u \in W^{1,2}_{loc}(\mathbb{R}^2, \mathbb{R}^m)$ satisfying (45)

Theorem 3.2. *[Pohozev in $\mathbb{R}^2$- Ball Case] Let $u \in W^{1,2}_{loc}(\mathbb{R}^2, \mathbb{R}^m)$ be a solution of*

$$\partial_{x_\ell}\left(\frac{|\nabla u|^2}{2}\right) - \sum_{k=1}^{2} \partial_{x_k}\left[\partial_{x_k} u \partial_{x_\ell} u\right] = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2), \quad \ell = 1, 2 \tag{52}$$

Then for all $x_0 \in \mathbb{R}^2$, $r > 0$ and every $X = X_1 + iX_2\colon \mathbb{C} \to \mathbb{C}$ holomorphic function the following identity holds

$$2\int_{\partial B(x_0,r)} \frac{\partial u}{\partial \nu}\, \nabla u \cdot X dx = \int_{\partial B(x_0,r)} X \cdot \nu |\nabla u|^2 dx \tag{53}$$

In the particular case $X = x - x_0$ with $x_0 \in \mathbb{R}^2$, then for all $r > 0$ the following identity holds

$$2\int_{\partial B(x_0,r)} \left|\frac{\partial u}{\partial \nu}\right|^2 d\sigma = \int_{\partial B(x_0,r)} |\nabla u|^2\, d\sigma. \tag{54}$$

or

$$\int_{\partial B(x_0,r)} \left|\frac{\partial u}{\partial \nu}\right|^2 d\sigma = \frac{1}{r^2}\int_{\partial B(x_0,r)} \left|\frac{\partial u}{\partial \theta}\right|^2 d\sigma \tag{55}$$

Proof. We multiply the equation (52) by X^ℓ and we integrate over $B(x_0, r)$. By using the Cauchy Riemann equations we get

$$\begin{aligned}
0 &= \int_{B(x_0,r)} X^\ell \left[\frac{\partial}{\partial x_\ell}\left(\frac{|\nabla u|^2}{2}\right) - \partial_{x_k}\left[\partial_{x_k} u \partial_{x_\ell} u\right]\right] dx \\
&= \int_{B(x_0,r)} \frac{\partial}{\partial x_\ell}\left[X^\ell \frac{|\nabla u|^2}{2}\right] dx - \int_{B(x_0,r)} \frac{|\nabla u|^2}{2}\left[\frac{\partial X^1}{\partial x_1} + \frac{\partial X^2}{\partial x_2}\right] dx \\
&- \int_{B(x_0,r)} \partial_{x_k}\left[X^\ell \partial_{x_k} u \partial_{x_\ell} u\right] dx + \int_{B(x_0,r)} \frac{\partial X^\ell}{\partial x_k}\left[\partial_{x_k} u \partial_{x_\ell} u\right] dx \\
&= -\frac{1}{2r}\int_{\partial B(x_0,r)} X \cdot (x - x_0)|\nabla u|^2 d\sigma - \int_{B(x_0,r)} |\nabla u|^2 dx \\
&+ \frac{1}{r}\int_{\partial B(x_0,r)} (X \cdot \nabla u)\left(\frac{\partial u}{\partial \nu}\right) d\sigma + \int_{B(x_0,r)} |\nabla u|^2 dx
\end{aligned} \tag{56}$$

It follows that

$$\int_{\partial B(x_0,r)}(X\cdot\nabla u)(\nabla u\cdot(x-x_0))d\sigma=\frac{1}{2}\int_{\partial B(x_0,r)}X\cdot(x-x_0)|\nabla u|^2d\sigma.$$

and we conclude. □

4. Proof of Proposition 1.2.

From Theorem 1.4 it follows that u satisfies in particular

$$\left|\int_0^{2\pi}u(e^{i\theta})\cos(\theta)\,d\theta\right|^2=\left|\int_0^{2\pi}u(e^{i\theta})\sin(\theta)\,d\theta\right|^2 \tag{57}$$

We can rewrite (57) as follows

$$\left|\int_0^{2\pi}u(e^{i\theta})\,\Re(de^{i\theta})\right|^2=\left|\int_0^{2\pi}u(e^{i\theta})\,\Im(de^{i\theta})\right|^2. \tag{58}$$

Given $a\in\mathbb{R}$ with $|a|<1$ and $\alpha\in\mathbb{R}$ we consider the Möbius map $M_{\alpha,a}(z):=e^{i\alpha}\frac{z-a}{1-az}$ and we define

$$u_{a,\alpha}(e^{i\theta}):=u\circ M_{\alpha,a}(z).$$

Since the condition (15) is invariant with respect to Möbius transformations for every $\alpha\in\mathbb{R}$ and for every $a\in(-1,1)$ we get

$$\left|\int_0^{2\pi}u\left(e^{i\alpha}\frac{z-a}{1-az}\right)\,\Re(de^{i\theta})\right|^2=\left|\int_0^{2\pi}u\left(e^{i\alpha}\frac{z-a}{1-az}\right)\,\Im(de^{i\theta})\right|^2. \tag{59}$$

or equivalently

$$\left|\Re\left(\int_0^{2\pi}u\left(e^{i\alpha}\frac{e^{i\theta}-a}{1-ae^{i\theta}}\right)\,de^{i\theta}\right)\right|^2=\left|\Im\left(\int_0^{2\pi}u\left(e^{i\alpha}\frac{e^{i\theta}-a}{1-ae^{i\theta}}\right)\,de^{i\theta}\right)\right|^2. \tag{60}$$

We set

$$e^{i\varphi}:=e^{i\alpha}\frac{e^{i\theta}-a}{1-ae^{i\theta}},$$

which implies that

$$e^{i\theta}=\frac{e^{i(\varphi-\alpha)}+a}{1+ae^{i(\varphi-\alpha)}} \tag{61}$$

$$d(e^{i\theta})=\frac{1-a^2}{(1+ae^{i(\varphi-\alpha)})^2}d(e^{i(\varphi-\alpha)}) \tag{62}$$

By plugging (61) and (62) into (60) and dividing by $(1-a^2)$ we get

$$\left|\Re\left(\int_0^{2\pi} u(e^{i\varphi})\,\frac{e^{-i\alpha}}{(1+ae^{i(\varphi-\alpha)})^2}d(e^{i\varphi})\right)\right|^2 = \left|\Im\left(\int_0^{2\pi} u(e^{i\varphi})\,\frac{e^{-i\alpha}}{(1+ae^{i(\varphi-\alpha)})^2}d(e^{i\varphi}\right)\right|^2. \tag{63}$$

Observe that for all $|z|<1$ we have

$$\frac{z}{(1+z)^2} = \sum_{n=1}^{\infty} n(-1)^{n-1}z^n$$

In particular

$$\frac{e^{i(\varphi-\alpha)}}{(1+ae^{i(\varphi-\alpha)})^2} = \sum_{n=1}^{\infty} n(-1)^{n-1}a^{n-1}e^{in(\varphi-\alpha)} \tag{64}$$

and

$$\Re\left(\frac{e^{i(\varphi-\alpha)}}{(1+ae^{i(\varphi-\alpha)})^2}\right) = \sum_{n=1}^{\infty} n(-1)^{n-1}a^{n-1}\cos(n(\varphi-\alpha)) \tag{65}$$

$$\Im\left(\frac{e^{i(\varphi-\alpha)}}{(1+ae^{i(\varphi-\alpha)})^2}\right) = \sum_{n=1}^{\infty} n(-1)^{n-1}a^{n-1}\sin(n(\varphi-\alpha)) \tag{66}$$

We can write

(67)
$$\left|\Re\left(\int_0^{2\pi} u(e^{i\varphi})\,\frac{e^{i(\varphi-\alpha)}}{(1+ae^{i(\varphi-\alpha)})^2}d\varphi\right)\right|^2$$
$$= \sum_{n=1}^{\infty}(-1)^{n-1}a^{n-1}\sum_{k=1}^{n-1}(n-k)k\left(\int_0^{2\pi} u(e^{i\varphi})\cos(k(\varphi-\alpha))d\varphi\right)\left(\int_0^{2\pi} u(e^{i\varphi})\cos((n-k)(\varphi-\alpha))d\varphi\right)$$

and

(68)
$$\left|\Im\left(\int_0^{2\pi} u(e^{i\varphi})\,\frac{e^{i(\varphi-\alpha)}}{(1+ae^{i(\varphi-\alpha)})^2}d\varphi\right)\right|^2$$
$$= \sum_{n=1}^{\infty}(-1)^{n}a^{n-1}\sum_{k=1}^{n-1}(n-k)k\left(\int_0^{2\pi} u(e^{i\varphi})\sin(k(\varphi-\alpha))d\varphi\right)\left(\int_0^{2\pi} u(e^{i\varphi})\sin((n-k)(\varphi-\alpha))d\varphi\right)$$

The identity (63) and the relations (67), (68) imply that for every $n \geq 2$ we obtain the following identities

(69)
$$\sum_{k=1}^{n-1}(n-k)k\left(\int_0^{2\pi} u(e^{i\varphi})\cos(k(\varphi-\alpha))d\varphi\right)\left(\int_0^{2\pi} u(e^{i\varphi})\cos((n-k)(\varphi-\alpha))d\varphi\right)$$
$$=\sum_{k=1}^{n-1}(n-k)k\left(\int_0^{2\pi} u(e^{i\varphi})\sin(k(\varphi-\alpha))d\varphi\right)\left(\int_0^{2\pi} u(e^{i\varphi})\sin((n-k)(\varphi-\alpha))d\varphi\right).$$

From (69) we can deduce a countable family of relations between the Fourier coefficients of the map u. Precisely if we set for every $n \geq 1$

$$\begin{cases} a_n := \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} u(e^{\theta})\ \cos n\theta\ d\theta \\ b_n = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} u(e^{\theta})\ \sin n\theta\ d\theta, \end{cases}$$

we get

$$\sum_{k=1}^{n-1}(n-k)k\left[(\cos(k\alpha)a_k+\sin(k\alpha)b_k)\left(\cos((n-k)\alpha)a_{n-k}+\sin((n-k)\alpha)b_{n-k}\right)\right.$$
$$\left. -\ (\cos(k\alpha)b_k-\sin(k\alpha)a_k)\left(\cos((n-k)\alpha)b_{n-k}-\sin((n-k)\alpha)a_{n-k}\right)\right]=0 \tag{70}$$

The identity (70) can be rewritten as follows

(71)
$$\cos(n\alpha)(\sum_{k=1}^{n-1}(n-k)k(a_ka_{n-k}-b_kb_{n-k}))+\sin(n\alpha)(\sum_{k=1}^{n-1}(n-k)k(a_kb_{n-k}+b_ka_{n-k}))=0.$$

The relation (71) yields (29) and (30) because of the linear dependence of $\cos(n\alpha)$ and $\sin(n\alpha)$.

We observe that for $n = 2$ we obtain:

$$(|a_1|^2-|b_1|^2)\cos(2\alpha)-2a_1\cdot b_1\sin(2\alpha)=0. \tag{72}$$

Since $\alpha \in \mathbb{R}$ is arbitrary we get

$$\begin{cases} |a_1| = |b_1| \\ a_1\cdot b_1 = 0 \end{cases}$$

If $n=3$ we get

$$4(a_1\cdot a_2-b_1\cdot b_2)\cos(3\alpha)-4(a_1\cdot b_2+b_1\cdot a_2)\sin(3\alpha)=0. \tag{73}$$

The relation (73) gives

$$\begin{cases} a_1\cdot a_2=b_1\cdot b_2 \\ \\ a_1\cdot b_2=-a_2\cdot b_1. \end{cases}$$

If $n=4$ we get

$$\begin{cases} |a_2|^2-|b_2|^2=\dfrac{3}{2}(b_1\cdot b_3-a_1\cdot a_3) \\ a_2\cdot b_2=-\dfrac{3}{4}(a_1\cdot b_3+b_1\cdot a_3). \end{cases}$$

We can conclude the proof. □

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