Proof of Remark 2.2.8

Jeremy Feusi
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Remark 1. Let $\Omega \subseteq \mathbb{R}^n$ be μ-measurable, $f : \Omega \to [0, \infty]$ a μ-measurable function. Assume that f is bounded, $f(x) \leq M \in \mathbb{R}_{\geq 0}$ for all $x \in \Omega$. Moreover, let $A_1 := \{x \in \Omega : f(x) \geq 1\}$ and

$$A_k := \left\{ x \in \Omega : f(x) \geq \frac{1}{k} + \sum_{j=1}^{k-1} \frac{1}{j} \chi_{A_j}(x) \right\}$$

for all $k \geq 2$. Set

$$f_k = \sum_{j=1}^{k} \frac{1}{j} \chi_{A_j}(x).$$

Then

$$\sup_{x \in \Omega} |f(x) - f_k(x)| \to 0$$

as $k \to \infty$.

Proof. Define $n_1 := \min\{n \in \mathbb{N} : \sum_{j=1}^{n} \frac{1}{j} > M\}$ and inductively for all $k \in \mathbb{N}$ with $k > 1$, define

$$n_k := \min \left\{ n \in \mathbb{N} : n > n_{k-1} \text{ and } \sum_{j=n_{k-1}}^{n} \frac{1}{j} > M \right\}.$$

This is well-defined since the sum diverges and thus for all $n \in \mathbb{N}$, $\sum_{j=n}^{\infty} \frac{1}{j} = \infty > M$. Now, consider $x \in \Omega$ and $k \in \mathbb{N}$ arbitrary. As shown in the lecture, we have that $f_{n_k}(x) \leq f(x) \leq M$ and by definition of n_k:

$$\sum_{j=n_{k-1}}^{n_k} \frac{1}{j} > M \geq f(x) \geq f_{n_k}(x) \geq \sum_{j=n_{k-1}}^{n_k} \frac{1}{j} \chi_{A_j}(x).$$

Since the inequality is strict, there exists a $j_0 \in \mathbb{N}$ with $n_{k-1} \leq j_0 \leq n_k$ such that $\chi_{A_{j_0}}(x) = 0$. By definition of A_{j_0}, this implies that

$$f(x) < \frac{1}{j_0} + \sum_{j=1}^{j_0 - 1} \frac{1}{j} \chi_{A_j}(x) = \frac{1}{j_0} + f_{j_0 - 1}(x)$$

and since $j_0 \geq n_{k-1}$:

$$f(x) < \frac{1}{n_{k-1}} + f_{j_0-1}(x)$$

or

$$f(x) - f_{j_0-1}(x) < \frac{1}{n_{k-1}}.$$

Also, $f(x) \geq f_{j_0-1}(x)$ and the sequence $f_k(x)$ is monotonically increasing. Thus the sequence $f(x) - f_k(x)$ is monotonically decreasing and $f(x) - f_{n_k}(x) \leq f(x) - f_{j_0-1}(x) \leq \frac{1}{n_{k-1}}$. Since $x \in \Omega$ was arbitrary and n_k independent of x, we conclude

$$\begin{aligned}
0 &\leq \lim_{k\to\infty}(\sup_{x\in\Omega}|f(x) - f_k(x)|) = \lim_{k\to\infty}(\sup_{x\in\Omega} f(x) - f_k(x)) \\
&= \inf_{k\in\mathbb{N}}\{\sup_{x\in\Omega} f(x) - f_k(x)\} \text{ (By monotonicity)} \\
&\leq \inf_{k\in\mathbb{N}}\{\sup_{x\in\Omega} f(x) - f_{n_k}(x)\} \\
&\leq \inf_{k\in\mathbb{N}_{>1}} \frac{1}{n_{k-1}} = 0
\end{aligned}$$

since $n_k \to \infty$ for $k \to \infty$. □